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ON THE EXTENSION OF S4 WITH *CLMpMLp*

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Over the last few years various logicians have considered the modal system obtained by extending S4[1] with

$$CLMpMLp,$$

but no demonstration that the system is decidable, or description of a characteristic model for it, has been published.[2] The purpose of this paper is to fill this gap by showing the system to have the finite model property—so that it is decidable (by [2], Lemma 4) and characterized by order closure models (by [1], Lemma 1)—and obtaining a characteristic order closure model for it. I assume familiarity with closure algebras (see [3] and [4]), with the order closure models of [1], and with the finite model property (see [2]). I do not distinguish between a closure algebra and the model obtained from it by designating the unit element; a closure algebra can be regarded as a Boolean algebra with a closure operator defined on it, and this representation is the most convenient for my purposes. I use the symbol - for relative complement, instead of in its normal role of complement proper; and I use the interior operator **I** (complement of closure of complement).

(It may be of interest that the system can also be obtained by extending S4 with either of the rules

$$\vdash M\alpha \Longrightarrow \vdash ML\alpha$$
$$\vdash M\alpha,\ \vdash M\beta \Longrightarrow \vdash MK\alpha\beta.$$

To prove this I derive them in rotation:

(a) Given $CLMpMLp$,
$$\begin{aligned}\vdash M\alpha &\Longrightarrow \vdash LM\alpha\\ &\Longrightarrow \vdash LM\alpha,\ \vdash CLM\alpha ML\alpha\\ &\Longrightarrow \vdash ML\alpha.\end{aligned}$$

(b) Given $\vdash M\alpha \Longrightarrow \vdash ML\alpha$, since $\vdash_{S4} CLMLpCLMLqMKpq$,

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$$\vdash M\alpha,\ \vdash M\beta \Longrightarrow \vdash ML\alpha,\ \vdash ML\beta$$
$$\Longrightarrow \vdash LML\alpha,\ \vdash LML\beta$$
$$\Longrightarrow \vdash LML\alpha,\ \vdash LML\beta,\ \vdash CLML\alpha CLML\beta MK\alpha\beta$$
$$\Longrightarrow \vdash MK\alpha\beta.$$

(c) Given $\vdash M\alpha$, $\vdash M\beta \Longrightarrow \vdash MK\alpha\beta$, since $\vdash_{S4} CMCpqCLpMq$, $\vdash_{S4} MCMpp, \vdash_{S4} MCMpCpLp \Longrightarrow \vdash MKCMppCMpCpLp$

$$\Longrightarrow \vdash MCMpKpCpLp$$
$$\Longrightarrow \vdash MCMpLp$$
$$\Longrightarrow \vdash MCMpLp,\ \vdash CMCMpLpCLMpMLp$$
$$\Longrightarrow \vdash CLMpMLp.)$$

In what follows I shall refer to the Lindenbaum algebra of equivalence classes of words in the system as $\mathfrak{M}$ (= $<\mathfrak{B},\mathsf{C}>$). This is known to be a closure algebra characterising the system—see Theorem 3.6 of [4]. In the next three paragraphs I show how to embed a given finite fragment of $\mathfrak{M}$ in a finite closure algebra which verifies the system. For each non-thesis of the system, construct such a closure algebra on the values of the parts of the non-thesis in a rejection with $\mathfrak{M}$: clearly the system has the finite model property with these models.

Following [3], for any finite sub-set Y of the elements of $\mathfrak{M}$, I define $\mathfrak{M}_y = <\mathfrak{B}_y,\mathsf{C}_y>$ as follows:

(1) $\mathfrak{B}_y$ is the (finite) sub-Boolean algebra of $\mathfrak{B}$ generated by Y.

(2) C_y is the function on $\mathfrak{B}_y$ given by $\mathsf{C}_y x = \bigcup_{i\in I} y_i$, where $\{y_i | i \in I\}$ is the set of all elements y of $\mathfrak{B}_y$ such that $x \subseteq y = \mathsf{C}y$.

Given a finite sub-set X of the elements of $\mathfrak{M}$, I use a_x for the element of $\mathfrak{M}$ given by

$$a_x = \bigcup_{x\in\mathfrak{M}_x} (\mathsf{C}x - x)\ ,$$

take $X' = X \cup \{a_x\}$, and define an algebra $\mathfrak{M}^1_x = <\mathfrak{B}^1_x,\mathsf{C}^1_x>$ by

(3) $\mathfrak{B}^1_x = \mathfrak{B}_x$

(4) C^1_y is the function on $\mathfrak{B}^1_x$ given by $\mathsf{C}^1_x x = (x - a_x) \cup (\mathsf{C}_{x'} x \cap a_x)$.

$\mathfrak{M}_y$ is known to be a closure algebra—see Lemma 2.3 of [3]—and $\mathfrak{M}^1_x$ can be shown to be a closure algebra by straightforward applications of the properties of $\mathfrak{M}_{x'}$. (In showing that $\mathsf{C}^1_x\mathsf{C}^1_x x = \mathsf{C}^1_x x$, note that $\mathsf{C}_{x'}\mathsf{C}^1_x x = \mathsf{C}_{x'}x$, since $\mathsf{C}_{x'}x$ contains $\mathsf{C}_{x'}(x - a_x)$ and $\mathsf{C}_{x'}(\mathsf{C}_{x'}x \cap a_x)$.)

The properties of $\mathfrak{M}_{x'}$ can also be used to check that

$$\mathsf{I}^1_x x = (x - a_x) \cup (\mathsf{I}_{x'}x \cap a_x)$$

Using this we find that $\mathsf{I}^1_x a_x$ is Λ; for

$\mathsf{I}(\mathsf{C}x - x) = \Lambda$ in all closure algebras;

$\therefore \mathsf{I} \bigcup_{x\in\mathfrak{M}_x} (\mathsf{C}x - x) = \Lambda$ in $\mathfrak{M}$, using the strong verification of the rule $\vdash M\alpha$, $\vdash M\beta \Longrightarrow \vdash MK\alpha\beta$ in $\mathfrak{M}$;

$\therefore \mathsf{I}a_x = \Lambda$, by the definition of a_x;

$\therefore \mathsf{I}_{x'}a_x = \Lambda$, using the properties of $\mathfrak{M}_{x'}$;

$\therefore \mathsf{I}^1_x a = (a_x - a_x) \cup (\mathsf{I}_{x'}a_x \cap a_x) = \Lambda$.

I can now show that $\mathfrak{M}_x^1$ verifies $\vdash M\alpha \Longrightarrow \vdash ML\alpha$, and so the system; for if $\mathsf{I}_x^1 x$ is Λ then $(x - a_x)$ is Λ, and

$$\begin{aligned} \mathsf{I}_x^1 \mathsf{C}_x^1 x &= \mathsf{I}_x^1((x - a_x) \cup (\mathsf{C}_{x'} x \cap a_x)) \\ &= \mathsf{I}_x^1(\mathsf{C}_{x'} x \cap a_x) \\ &\subseteq \mathsf{I}_x^1 a_x , \end{aligned}$$

so that $\mathsf{I}_x^1 \mathsf{C}_x^1 x$ is Λ.

I must finally show that the fragment of $\mathfrak{M}$ with elements X is embedded in $\mathfrak{M}_x^1$. That it is embedded qua Boolean algebra follows immediately from definitions (1) and (3). It remains to show that if x and $\mathsf{C}x$ are in X then $\mathsf{C}_x^1 x$ is $\mathsf{C}x$. It is known that in this case $\mathsf{C}_{x'} x$ is $\mathsf{C}x$ (see Lemma 2.3 of [3]); therefore

$$\begin{aligned} \mathsf{C}_x^1 x &= (x - a_x) \cup (\mathsf{C}_{x'} x \cap a_x) \\ &= (x - a_x) \cup (\mathsf{C}x \cap a_x) \\ &= (x - a_x) \cup ((x \cup (\mathsf{C}x - x)) \cap a_x) \\ &= (x - a_x) \cup (x \cap a_x) \cup ((\mathsf{C}x - x) \cap a_x) \\ &= x \cup (\mathsf{C}x - x), \text{ since } a_x \supseteq \mathsf{C}x - x \text{ by the definition of } a_x \text{ and the hypothesis that } x \text{ is in } X, \\ &= \mathsf{C}x. \end{aligned}$$

We now know that the system is characterized by order closure models, and the rest of the paper is devoted to discussing them. For this I must add some notation to that of [1]. (Also, on one point I wish to alter that of [1]: the construction on quasi-ordered sets, given on p. 253 of [1], which is represented by a subscript 1, shall be represented by a superscript 1 here.) I use the term *bottom point* for a member a of a quasi-ordered set for which $a \geqslant x$ only if $a \leqslant x$, and use the term *strict bottom point* for a member a of a quasi-ordered set for which $a \geqslant x$ only if $a = x$. I use $\mathfrak{K}^-$ for the quasi-ordered set obtained from $\mathfrak{K}$ by deleting its strict bottom points, and use $\mathfrak{K}^b$ for the quasi-ordered set obtained from $\mathfrak{K}$ by adding a point strictly below each bottom point.

It is easily checked that a finite order closure model verifies the system if and only if each bottom point of the quasi-ordered set on which it is defined is a strict bottom point. Here, then, is a set of models characterizing the system. Following Example 1 of [1], I shall use this to show that $\mathfrak{H}^{1b+}$ (for $\mathfrak{H}^1$ see the third paragraph of p. 258 of [1]) is a characteristic model for the system. It is easily checked that this model verifies the system, so it remains to show that every word rejected by a finite model of the kind described is rejected by $\mathfrak{H}^{1b+}$. First I need the following variant of Lemma 5 of [1].

Lemma. *If $\mathfrak{K}$ is a finite quasi-ordered set such that $\mathfrak{K}^+$ is a model for the system, and if $\mathfrak{L}$ is a partially ordered set such that $\mathfrak{K}^{-e+}$ is isomorphic to a subalgebra of $\mathfrak{L}^+$, then $\mathfrak{K}^+$ is isomorphic to a subalgebra of $\mathfrak{L}^{1b+}$*

Proof. Let Φ be the isomorphism between $\mathfrak{K}^{-e+}$ and a subalgebra of $\mathfrak{L}^+$. I define a mapping Ψ of $\mathfrak{K}$ onto $\mathfrak{L}^{1b}$ in two steps:

(1) If $\{a_0, a_1, \ldots, a_n\}$ is an equivalence class of members of $\mathfrak{K}^-$ then

$$\Psi a_i = \{<b,i>|b \,\varepsilon\, \Phi\{a_0\}\}, \text{ for } 0 \leq i \leq n-1,$$
$$\Psi a_n = \{<b,j>|b \,\varepsilon\, \Phi\{a_0\},\ j \geq n\}.$$

(2) If $a_0, a_1, \ldots, a_n$ are strict bottom points immediately below a point b, and D is the set of those bottom points of $\mathfrak{L}$ which are in $\Phi\{b\}$, then

$$\Psi a_i = \{<c,i>|<c,i> \; < \; <d,i> \text{ for } d \,\varepsilon D\}, \text{ for } 0 \leq i \leq n-1,$$
$$\Psi a_n = \{<c,j>|<c,j> \; < \; <d,j> \text{ for } d \,\varepsilon D, j \geq n\}.$$

(In both cases, when n is 0 apply the second clause to a_0.) Mapping sets of points of $\mathfrak{K}$ onto the unions of their images under Ψ gives the required isomorphism.

Now suppose we are given a non-thesis of the system. We know that this is rejected by a finite order closure model, $\mathfrak{K}^+$ say, for the system. We may take $\mathfrak{K}^{-e}$ to have a greatest point, since a word rejected by an order closure model must be rejected by the order closure model on the sub-tree of points below a point in a non-empty value of the word in the original model. By Lemmas 3 and 10 of [1], $\mathfrak{K}^{-e+}$ is isomorphic to a subalgebra of $\mathfrak{H}_i^+$ (for $\mathfrak{H}_i$ see the first paragraph of p. 258 of [1]) for some i. Therefore, by the Lemma given above, $\mathfrak{K}^+$ must be isomorphic to a subalgebra of $\mathfrak{H}_i^{1b+}$; so the given non-thesis is rejected by $\mathfrak{H}_i^{1b+}$. So the given non-thesis is rejected by $\mathfrak{H}^{1b+}$, using any allocation which copies a rejecting allocation in $\mathfrak{H}_i^{1b+}$ in a sub-tree of depth $i+1$.

NOTES

1. In this paper I take S4, and extensions of it, to be given with $\vdash\alpha \implies \vdash L\alpha$ as a derivation rule.

2. While in Oxford in 1963, Kripke mentioned some very interesting results for this system with predicates and quantifiers, so I would not be surprised if he has also obtained the results given here.

   Added 18-8-65: It is pointed out to me that this system was first put forward by McKinsey, in 1945, and that systems containing *CLMpMLp* are being examined by Prior, Sobociński, and Ivo Thomas in current numbers of the *Notre Dame Journal of Formal Logic*. See the discussion in [5].

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