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K1, K2 AND RELATED MODAL SYSTEMS.

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1. Sobociński refers in [5] to two systems which he calls K1 and K2. If S4 is axiomatised with the rule to infer $\vdash L\alpha$, from $\vdash \alpha$, these systems are axiomatisable by adding *CLMpMLp* and *ELMpMLp* respectively to S4. It is obvious that K1 is a subsystem of K2, since *ELMpMLp* is equivalent to *CLMpMLp* plus its converse *CMLpLMp*; Sobociński, in conclusion, raises the question whether it is a "proper" subsystem. This question is equivalent to the question whether, given S4, *CMLpLMp* is independent of *CLMpMLp*. That it is, may be established by the following matrix:—

C	1	2	3	4	5	6	7	8	N	M	L
* 1	1	2	3	4	5	6	7	8	8	1	1
2	1	1	3	3	5	5	7	7	7	2	6
3	1	2	1	2	5	6	5	6	6	3	7
4	1	1	1	1	5	5	5	5	5	4	8
5	1	2	3	4	1	2	3	4	4	1	5
6	1	1	3	3	1	1	3	3	3	2	6
7	1	2	1	2	1	2	1	2	2	3	7
8	1	1	1	1	1	1	1	1	1	8	8

This verifies S4 and *CLMpMLp*, but falsifies *CMLpLMp* when $p = 2, 3, 6$ or 7.

The history of this matrix is worth giving, as it suggests solutions to certain connected problems.

2. In [3], [4] and other papers an interpretation is given for modal functors which may be re-stated, more in the spirit of [2], as follows:— Use p, q, r, etc. for propositional variables and a, b, c, etc. for "worlds" or total states of affairs. Let U represent a certain relation between worlds, and write *Tap* for "It is the case in world a that p". Assume, beside quantification theory and identity theory, the following:—

1. *ETANpNTap*
2. *ETaCpqCTapTaq*
3. *ETaLp*Π*bCUabTbp*

From these, given *Mp* as short for *NLNp*, it is easy to deduce

4. *ETaMp*Σ*bKUabTbp*

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It is useful also to have a "world"-constant O (the "actual" world) such that 5. $ETOpp$. In a U-system, nothing but 3 is assumed for L, but there may be added to the above basis various special axioms for U, e.g. that it is reflexive (Uaa), that it is symmetrical ($CUabUba$), that it is transitive ($CUabCUbcUac$), that it is connected ($CNUabUba$). The effects of weakening certain of these special axioms in various ways are considered in [6]; and some effects of weakening the basic axioms 1 and 2 (e.g. the invalidation of the proof of 4 from 3), in [2].

To many ordinary modal calculi there correspond U-systems such that a formula ϕ is a thesis of the modal calculus if and only if $Ta\phi$ is a thesis of the corresponding U-system. Well-known results in this field (most of them obvious by-products of Kripke's [1]) are that the U-system in which the only special assumption about U is its reflexiveness corresponds to the modal system T; that that in which reflexiveness and transitiveness alone are assumed corresponds to S4; that in which reflexiveness and symmetry alone are assumed, to the "Brouwersche" system B (i.e. T plus $CpLMp$); reflexiveness, symmetry and transitiveness give S5; and transitiveness and connexity, S4.3.

Sobociński has noted (in [5], **3.3** and **4**) that the distinctive axiom $CMLpLMp$ of S4.2 is derivable in both S4.3 and B; this means, in view of the foregoing, that its U-counterpart should be provable equally from the symmetry and from the connexity of U. The proof from connexity is given in [3]; that from symmetry we give below. $TaCMLpLMp$ expands to

$$C\Sigma bKUab\Pi cCUbcTcp\Pi bCUab\Sigma cKUbcTcp,$$

which quantification theory equates to

$$\Pi bdCKUabK\Pi cCUbcTcpUad\Sigma eKUdeTep,$$

and this may be proved, assuming the symmetry of U, as follows:—

$\Pi bdCK$	(1) Uab	
K	(2) $\Pi cCUbcTcp$	
	(3) Uad	
K	(4) Uda	[3; Symm]
K	(5) Uba	[1; Symm]
K	(6) Tap	[2; 5]
	(7) $\Sigma eKUdeTep$	[4; 6].

Sobociński's result, incidentally, shows that although the addition of $LCMLpLMp$ to S3 results in the same system (S4.2) as its addition to S4, its addition to T, or the equivalent addition to T, (axiomatized with the rule to infer $\vdash L\alpha$ from $\vdash\alpha$) of $CMLpLMp$, does not yield S4.2 but a weaker system. For this system is contained in B, which does not contain S4. Whether the addition of the S4.3 formula $ALCLpqLCLqp$ to T yields S4.3 or a weaker system will be considered below.

3. Returning to K1 and K2, what assumption about U would yield the U-counterpart of their distinctive formula $CLMpMLp$? One approach to this problem is via tense-logic. If we take the "world"-variables $a, b, c,$

etc. to represent total states of the world at given moments of time, and Uab to mean that state b either is identical with state a or is one of its temporal successors, the difference between MLp and LMp will be that $TaMLp$ means that, at a, it either is or will be the case that it is and always will be the case that p, while $TaLMp$ means that, at a, it is and always will be the case that it either is or will be the case that p. If p is something which will for ever be the case *intermittently* (being the case for a time and then not being the case for a time), the second of these will be true but not the first. If, however, there is a last moment of time, both $TaMLp$ and $TaLMp$ will be true if and only if p is the case at that last moment, and so will be equivalent. We will therefore verify $CLMpMLp$ as well as its converse if we assume time to have an end. With this intuitive background, it was shown in [3] that the U-counterpart of the system which Sobociński names K2 (or of this at least; in fact we get K3 also this way) will be obtained if we assume for U, beside reflexiveness, transitiveness and connexity, the axiom "Fin", i.e.

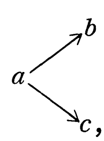
$$\Sigma b \Pi c CUbcIbc,$$

"For some moment b, if any moment c is either identical with c or after it, it is identical with it", i.e. there is a moment which has no *other* moment after it. $TaCLMpMLp$ was, however, only proved from Fin by assuming the connexity of U as well. With the time-series this is a plausible assumption but it is clear that this use of Fin will not help us to distinguish between Sobociński's K1 and K2, since connexity also gives us $TaCMLpLMp$ (and, indeed, $TaALCLpqLCLqp$). If the U-system is not connected but has diverging branches, Fin only asserts that at least one branch has an end, and to prove $TaCLMpMLp$ in a non-connected system, we must replace Fin by an assumption that will guarantee that *every* branch has an end, e.g. the assumption

$$\Pi a \Sigma b KUab \Pi c CUbcIbc,$$

"For all a, there is some b which is after a but has nothing else after it". And from this assumption, with transitiveness, we can prove $TaCLMpMLp$ without using either symmetry or connexity, and without verifying the converse.

A simple system embodying the required assumptions would be one represented by

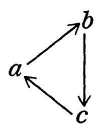


where the arrows plus identity represent the U relation (i.e. the true elementary U-propositions are Uaa, Ubb, Ucc, Uab, Uac). If p is true at b only, MLp is true at a (because Lp is true at b) and LMp false there (because Mp is false at c). To show that this system verifies everything in K1, we translate it into a matrix by representing the possible truth-values of p at a, b, and c by the eight triads

	1		*1*		*0*		*0*		*1*		*1*		*0*		*0*	
1		*1*		*1*		*1*		*0*		*0*		*0*		*0*		
	1,		*0*,		*1*,		*0*,		*1*,		*0*,		*1*,		*0*,	

and working out the values of *Mp*, *Lp*, *LMp*, etc. from these, *Mp* having *1* at a given point if and only if *p* has *1* either there or at a point to the right of it, and *Lp* having *0* at a given point if and only if *p* has *0* either there or at a point to the right of it. If we number the triads *1* to *8*, we have the matrix used at the beginning.

4. Similar methods may be used to show that the addition of *ALCLpqLCLqp* to T instead of to S4 does not yield S4.3 but a weaker system. The problem here is to show that the connexity of *U* does not prove its transitiveness. A non-transitive but connected system would be



(i.e. we have *Uab*, *Ubc*, and *Uca* beside *Uaa*, *Ubb* and *Ucc*). So we use the same 8 triads as elements of our matrix, but specify *Mp* as having a *1* at a given point if and only if *p* has a *1* either at that point or at the next point clockwise round, and similarly with *Lp* and *0*. This, when the triads are numbered in the same order as before, gives us the following valuations for *M* and *L* (*C* and *N* being as before):—

p :	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Mp :	*1*	*1*	*1*	*3*	*1*	*2*	*5*	*8*
Lp :	*1*	*4*	*7*	*8*	*6*	*8*	*8*	*8*

The matrix verifies *ALCLpqLCLqp* (and, of course, the postulates of T) but falsifies *CLpLLp* where *p* = *2*, *3* or *5*.

5. Dummett and Lemmon's original axiom for S4.3 was *ALCLpLqLCLqLp*, the *D1* of Sobociński's [5]. That *ALCLpqLCLqp*, the *D2* of [5], is equivalent given S4, to *D1*, was first pointed out, so far as I know, by P. T. Geach in the late 1950's. (See [4], p. 139). At about the same time (this is also reported in [4]) yet another axiom was considered by Hintikka and shown to be equivalent, given S4, to *D2*; namely

H1. *CKMpMqAMKpMqMKqMp*.

Though longer than *D2*, this axiom, in its tense-logical interpretation, reflects more directly the connexity of the time-series. We show now that it is not only equivalent to *D2* given S4, but also given the system T. In the first place it is equivalent, by simple transpositions and substitutions, to

H2. *CLCpLqCLCMpqCMpLq*.

From this we may prove *D2* as follows:—

1.	*LCLpLCqp*	[T]
2.	*LCKLpNqLCLqp*	[1 *q*/*Lq*; *LCKpqp*]
3.	*LCKLqNpLCLpq*	[2 *p*/*Nq*, *q*/*p*]
4.	*LCNLCLpqNKLqNp*	[3]
5.	*LCMKLpNqCLqp*	[4]
6.	*CMKLpNqLCLqp*	[*H2* *p*/*KLpNq*, *q*/*CLpq*; 2; 5]
D2.	*ALCLpqLCLqp*	[6]

Conversely, *D2* is equivalent by simple transpositions and substitutions to what we may call

D5. *ALCpMqLCqMp*,

from which *H1* is derivable as follows:—

1.	*CLCpMqLCpKpMq*	[T]
2.	*CLCpKpMqCMpMKpMq*	[T]
3.	*CLCpMqCMpMKpMq*	[1; 2]
4.	*CLCqMpCMqMKpMq*	[3 *p*/*q*, *q*/*p*]
H1.	*CKMpMqAMKpMqMKqMp*	[*D5*; 3; 4; **P.C.**]

H1 or *H2* may therefore replace *D2* not only in axiomatising S4.3 but also in axiomatising the system, which we may call T.3, which was shown in the last section to be a proper subsystem of S4.3.

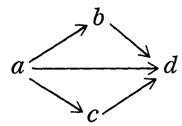
Using *H2*, it is easy to show that T.3 enriched with the "Brouwersche" axiom *CpLMp* (which with the rule of T to infer $\vdash L\alpha$ from $\vdash \alpha$ is equivalent to *LCpLMp*) is equivalent, like S4 with the same enrichment, to S5. For if in *H2* we put *q*/*Mp*, its first antecedent becomes *LCpLMp*, its second *LCMpMp*, and its consequent *CMpLMp*.

In the U-system, correspondingly, connexity and symmetry together (i.e. *CNUabUba* and *CUabUba*) turn *U* into an equivalence relation, since they turn it into the universal relation.

6. E. J. Lemmon has communicated to me a solution to a further problem raised by Sobociński, namely whether K2 contains a further system in which it is certainly contained, namely K3, in which *CLMpMLp* is added not to S4.2 but to S4.3. This amounts to the question whether the formula *ALCLpqLCLqp* is independent of the postulates of K2. Lemmon points out that a postulate for U which (added to reflexiveness and transitiveness) will verify S4.3 but not S4.2 is that U is "convergent," i.e. that even if the U-lines diverge at some point they eventually come together. The formula expressing this condition would be *CKUabUac*Σ*dKUbdUcd*. The proof of *CMLpLMp* from this (for the preliminaries, see its proof from Symm in Section 2) would be

Π*bdCK*	(1)	*Uab*	
K	(2)	Π*cCUbcTcp*	
	(3)	*Uad*	
Σ*eK*	(4)	*Ube*	[1, Conv.]
K	(5)	*Ude*	[3, Conv.]
K	(6)	*Tep*	[2, 4]
	(7)	*KUdeTep*	[5, 6].

A simple system that is reflexive, transitive, convergent and has an end-point would be



(where the true elementary U-formulae are *Uaa*, *Ubb*, *Ucc*, *Udd*, *Uab*, *Ubd*, *Ucd*, *Uad*). This suggests a 16-valued matrix in which the elements are such tetrads as

	1				*0*	
1		*1*		*1*		*1*
	0				*1*	

and *Mp* has a *1* where *p* has a *1* either there or at some point to the right, and *Lp* has a *0* where *p* has a *0* either there or at some point to the right. If *p* and *q* are respectively assigned the two values just used as illustrations, *ALCLpqLCLqp* takes the value *0* at the extreme left position.

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