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A STUDY OF SOME SYSTEMS IN THE NEIGHBORHOOD OF S4.4.

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This paper investigates S4.4 and some closely related systems both semantically and proof-theoretically. In what follows, we:

(1) Set down a characteristic matrix for S4.4.
(2) Extend the methods employed in (1) above to show that "Group II" of Lewis and Langford [3] is characteristic for the system K4 of Sobociński [9].
(3) Set down and investigate semantically and proof-theoretically a system S4.9 which is between S4.4 and S5; there is no system properly contained in S5 and properly containing S4.9.

In [10], Sobociński introduced the system S4.4, which is S4 + $\mathfrak{C}pCMLpLp$; S4.4 properly includes Prior's Diodorean system D (S4.3 + $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLppCMLpp$); this is shown in [10], where D is called S4.3.1 (for a thorough discussion of D, see [6, p. 20 ff.]). D is a modal system whose modal operators may be considered defined in a time framework. It is the logic in which "necessarily α" means "α is true now and will be at every instant in the future," and "possibly α" means "α either now is true, or will be at some instant in the future." [6] discusses also time interpretations for several systems included in D; S4.3, S4.2, and S4; also considered is a time interpretation for S5. The time sequence for S4.3 is linear and connected as is that of D, but S4.3 differs from D in having its time sequence continuous while that for D is composed of discrete instants. The sequences for S4.2 and S4 are not connected, but permit "branching" (some statements which are possible now can turn out never to be true). S4.2 differs from S4 in requiring "convergence" of its branchings—all statements which are "possibly necessary" eventually become necessary. While these systems define their modal operators in terms of the future alone, "necessarily α" for S5 is defined as "α is now true, and always will be, and always was," and "possibly α" is "either α now is true, or at some instant in the future it will be true, or at some instant in the past it was true."

Since S4.4 is "surrounded" by systems with time interpretations, one may wonder if it is possible to come up with such an interpretation of it. I

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would like to suggest such an interpretation, and to show that this interpretation provides a characteristic matrix for S4.4. Although the possible necessity of a statement in S4.4 does not imply its necessity as is the case in S5, if a statement is true right now as well as possibly necessary, then it *is* necessary in S4.4. This suggests a basic difference for this system between the present instant and all instants following it. If all that is needed for a possibly necessary statement to be fully necessary (that is, true now and forever) is that it be true right now, then all statements which are, indeed, possibly necessary must become "true forever" in the *very instant following the present instant.* Under this interpretation, then, S4.4 emerges as a "logic of the end of the world." One pictures an angel appearing, golden horn in hand, and announcing, "I am about to blow this horn, and when I do, the world will end; time will pass into eternity, and at that instant all eternal truths will be realized; all that ever is to be true 'necessarily' will then become true necessarily." The angel lifts the horn to his lips, and the instant just before he blows it is the instant for which S4.4 is expressive of the time sequence.

We will call the suggested time-sequence model the "end of the world matrix," or "eow matrix." This model has exactly one instant in what could be called "time," and a denumerable infinity of instants in what could be called "eternity." So far as the tense-defined modal operators are concerned, "eternity" behaves like the S5 time sequence; the tense-oriented definition of possibility and necessity for the whole model would be:

"α is possible" (i.e., $M\alpha$) means

"If the instant for which $M\alpha$ is being evaluated is the one instant in time, then α holds either at that instant or at some instant in eternity; if the instant for which $M\alpha$ is being evaluated is one of the instants in eternity, then α is true at some instant in eternity."

"α is necessary" (i.e., $L\alpha$) means

"If the instant for which $L\alpha$ is being evaluated is the one instant in time, then α is true at that instant and at every instant in eternity as well; if the instant for which $L\alpha$ is being evaluated is one of the instants of eternity, then α is true at every instant of eternity."

The corresponding time-sequence matrix for S5 would have $L\alpha$ true at a given instant iff α is true at every instant, and $M\alpha$ false at a given instant iff α is false at every instant. The proposed S4.4 matrix differs from the S5 matrix in its evaluation for M and L at one and only one point for each of these operators. When α is true only at the first instant of the time sequence (the instant in time) $M\alpha$ will also be true only at that point, and when α is false at the first instant and then true at all of the rest of the instants, $L\alpha$ will be false and true at those same instants. Otherwise, the eow matrix is the same as that for S5.

Slightly more formally, the proposed S4.4 matrix would be the quadruple $\langle L/R, \cap, -, \Diamond\rangle$. The set of elements L/R is a set of ordered pairs x/y, $x \in L$ and $y \in R$, where R is the set of elements for a characteristic matrix for S5 and L is a set of two elements, say 1 and 0. The designated element of this matrix would be $1/\mathbf{s}$, where $\mathbf{s}$ is the designated element of R. $-(x/y)$ would be $(L - x/R - y)$ and $(x/y) \cap (u/v) = (x \cap u)/(y \cap v)$. $\cap$ and $-$ behave as usual in an algebra of this kind. Designating -1 as 0 and $-\mathbf{s}$ as ϕ, $\Diamond(x/y) = 1/\mathbf{s}$ unless $y = \phi$, in which case $\Diamond(x/y) = x/y$. We may define an operator $\Box$; $\Box(x/y) = -\Diamond-(x/y) = 0/\phi$ unless $y = \mathbf{s}$, in which case $\Box(x/y) = x/y$. Finite versions of this matrix with a time interpretation may be constructed using, say, the digits 1 and 0 to represent truth and falsity respectively at a given instant, and using a binary number of n digits to represent a time sequence of n instants. As is usual, a statement true at the last of these instants is considered to be true forever after; the same holds of falsity. Evaluating the "two instant" time sequence as above, we get:

a = 11	10	01	00
$\Diamond a$ = 11	10	11	00
$\Box a$ = 11	00	01	00

We may translate the above table as is done with similar matrices in, say, [5] to obtain a table conforming to common usage in the literature:

p = 1*	2	3	4
Mp = 1	2	1	4
Lp = 1	4	3	4

(* indicates designated element). This is, of course, Group II of Lewis and Langford [3]; the two-instant (four-valued) version of the end of the world matrix is then the same as the two-instant version of Prior's Diodorean matrix. The three-instant versions of the Diodorean and the eow matrices differ, however; note that here we have the first instant in "time" and the last two instants in "eternity":

a = 111	110	101	100	011	010	001	000
$\Diamond a$ = 111	111	111	100	111	111	111	000
$\Box a$ = 111	000	000	000	011	000	000	000

Translating as we did with the four-valued table:

p = 1*	2	3	4	5	6	7	8
Mp = 1	1	1	4	1	1	1	8
Lp = 1	8	8	8	5	8	8	8

All such finite versions of the eow matrix verify S4.4, as does the infinite version; all, of course, fail to verify $\mathfrak{C}MLpLp$. This matrix is used by Schumm in [7].

We shall now show that the infinite end of the world matrix is characteristic for S4.4 by establishing a system of proof tableaux as in Kripke [1] and [2]; familiarity with these articles is assumed. As in these papers, we shall think of a set K of "possible worlds" as being given. In this case, each of the possible worlds will correspond to an instant of the eow matrix. Again, one element, G, will be singled out as the "real world," if you will.

G in this case will correspond to the one moment in "time" in the eow matrix; the other possible worlds will correspond to the eow instants in "eternity." We may evaluate a wff in this set of possible worlds by constructing a system of tableaux for it. So far as **PC** connectives are concerned, these tableaux will behave just as do the tableaux of [1] and [2]. We shall take L to be the primitive modal operator in this study, and we shall introduce rules for the handling of L which will make the present systems of tableaux behave like the end of the world matrix. We start off the construction of a system of tableaux by writing the wff to be evaluated on the right of the main tableau, which corresponds to the "real world" G, to the first instant in the eow matrix. Under certain conditions we will have to construct auxiliary tableaux; these will correspond to the moments of eternity in the eow matrix. Let us begin to state the rules governing L in these evaluations; the first rule tells what happens when a formula beginning with L occurs on the *left* of a *main tableau*:

L-left (main): *If $L\alpha$ occurs on the left of a main tableau, then α is to be written on the left of every tableau of the set for which the given tableau is main.*

What happens when a formula beginning with L occurs on the left of an auxiliary tableau is somewhat different:

L-left (aux): *If $L\alpha$ occurs on the left of an auxiliary tableau t, then α is to be written on the left of every auxiliary tableau in the alternative set to which t belongs.*

We see here a certain primacy of the main tableau. L's on its left affect every tableau in its set; it is unaffected by L's on the left of its auxiliary tableaux; L's on the left of an auxiliary tableau, however, affect all its "fellow" auxiliary tableaux. This is analogous to the evaluation of L in the eow matrix.

When a formula beginning with L occurs on the right of an auxiliary tableau, the rule for handling it is as usual in these systems:

L-right (aux): *If $L\alpha$ occurs on the right of an auxiliary tableau, construct a new auxiliary tableau having α on its right.*

When a formula beginning with L occurs on the right of a main tableau, we have a different situation:

L-right (main): *If $L\alpha$ occurs on the right of a main tableau, split that tableau into two alternative main tableaux. These are alike except as follows: one will have α on the right; for this tableau nothing further is done so far as the L here in question is concerned. The other tableau will have α on the left; construct a tableau auxiliary to this one and beginning with α on its right.*

If $L\alpha$ is to be false in the first instant of the eow matrix, then either α is false in the first instant of that matrix, or α is true in the first instant and false in some later instant. In the first case we need look no further; the alternative main tableau with α right explores this possibility. The alternative main tableau with α left explores the other situation by creation of an auxiliary tableau.

We wish to establish that if one of these systems of tableaux is such that every one of its alternative sets closes (has a tableau with a wff α both on its left and its right), then the formula for which the construction of the system of tableaux was begun is a theorem of S4.4. We shall do this in the manner of Kripke by defining a "characteristic formula" for each stage in the construction of the system of tableaux. The first stage is the main tableau with just the original formula written on its right. The $(n + 1)$-th stage is the system of tableaux resulting from the application of a rule for one of the operators to the n-th stage. As with Kripke, each tableau at a given stage of the construction will have its own characteristic wff, and here as in [1] and [2] the characteristic wff of a tableau at a given stage will be

$$KK\alpha_1 \dots \alpha_n KN\beta_1 \dots N\beta_m$$

where the α's are the n (≥ 0) wffs on the left of the tableau and the β's are the m (≥ 1) wffs on the right of the tableau at the given stage of construction. We will define the characteristic formula of one of the alternative sets of the system in terms of the characteristic wffs of the tableaux of that set; let μ be the characteristic wff of the main tableau of that set and let $\sigma_1, \dots, \sigma_s$ be the characteristic wffs of the s auxiliary tableaux of that set at a given stage of construction. Then the characteristic formula of the set at that stage of construction will be:

$$K\mu LMKM\sigma_1 \dots M\sigma_s\ .$$

The rest of the definition of characteristic formula is just as with Kripke; where $\delta_1, \dots, \delta_r$ are the characteristic wffs of the r alternative sets of the system of tableaux at a given stage of construction, $A\delta_1 \dots \delta_r$ is the characteristic wff of the entire system at that stage.

It will be noted that the definition of characteristic formula for the S4.4 tableaux is, as one might expect, quite similar to that for S5 in [1]; it differs in fact, only in the insertion of the LM before the conjunction of the $M\sigma_i$ in the definition of characteristic wff of alternative set. And this is expressive of the key difference between S4.4 and S5; this LM gives us the "difference in kind" between "time and eternity" which is typical of the end of the world matrix. The key step in the completeness results of [1] and [2] is showing that where α is the characteristic wff of a system of tableaux at the n-th stage and β is the characteristic wff at the $(n + 1)$-th stage, then $C\alpha\beta$ is a thesis of the system in question. Our formulation differs from Kripke's only in its rules for L and in its definition of the characteristic wff of an alternative set of tableaux. Our completeness proof need consider only these points, then; the rest of the proof will follow as in [1] for S5.

We will recall that our rules for L, both on left and right, differ for the main and for the auxiliary tableaux. Let us consider first the cases in which these rules are applied because of L's occurring in auxiliary tableaux. Let ϕ be the characteristic wff of an alternative set at stage n of construction and let ψ be the characteristic wff at stage $n + 1$, where the $(n + 1)$-th step is an application of a rule for L in an auxiliary tableau. ϕ is then the formula $K\alpha LM\beta$ and ψ is $K\alpha LM\gamma$. Now if we examine the rules for the handling of L in auxiliary tableaux, we will see that the auxiliary tableaux of a given alternative set behave toward each other just as do the tableaux of a set of S5 tableaux. Indeed, with ϕ and ψ as described above and given the work of Kripke, the formula $C\beta\gamma$ will be a thesis of S5. But it is well-known that if a wff χ is a theorem of S5, then $ML\chi$ is provable even in S4; thus, $MLC\beta\gamma$ is a thesis of S4.4. Now, it is a characteristic feature of the system S4.2 that ML distributes over C; in that system we now execute the proof of a similar principle; in S3° we have:

(1) $\mathfrak{C}LCpqCLMpLMq$
(2) $CMLCpqCLLMpMLMq$ (1), S1°
(3) $CMLCpqCLMpLMq$ (2), S4.2

With (3) and $MLC\beta\gamma$, we have $CLM\beta LM\gamma$ as an S4.4 thesis; given all the above assumptions, then, we have

(4) $CK\alpha LM\beta K\alpha LM\gamma$

provable in S4.4; (4) is the required implication between the characteristic wffs of the n-th and the $(n + 1)$-th stages; the desired result obtains, then, whenever a rule for L in an auxiliary tableau is applied. (Recall that the rules for L in auxiliary tableaux affect not at all the main tableau here.)

We now turn to these rules as applied for L's in the main tableau. For an application of L-left (main) the characteristic wff for the set involved is $KKL\alpha\beta LMK\gamma M\delta$. The characteristic formula for the $(n + 1)$-th stage is then $KKL\alpha\beta LMK\gamma MK\alpha\delta$ (except for the trivial case involving the writing of α on the left of the main tableau itself). By $CKLpLMqKLpLMKLpq$, which holds in S4°, and $\mathfrak{C}KLpMqMKpq$, which holds even in S2°, the required implication between characteristic wffs holds. Thus does the required result obtain when the rule used is L-left (main).

We now turn to the rule L-right (main). It will be recalled that the first thing done under this rule is to split the main tableau into two alternative main tableaux. The main tableau has as its characteristic wff here the formula $K\alpha NL\beta$. The characteristic wff after the split is $AKK\alpha\beta NL\beta K\alpha$-$KNL\beta N\beta$; the required implication is justified by **PC**, specifically by excluded middle. For one of the cases now involved, the alternate tableau in which β is placed on the right, no further action is required. The case of the split in which β is placed on the left, however, requires the creation of an auxiliary tableau. The characteristic wff involved is that of the alternative set having as its main tableau the alternate tableau from above with β on its left. This formula is:

(5) $KKK\alpha\beta NL\beta LM\gamma$ (or $KK\alpha\beta NL\beta$)

The parenthesized versions of (5) and (6) apply if no auxiliary tableaux had been constructed to this stage for this set. The characteristic wff of the alternative set following this construction will be:

(6) $KKK\alpha\beta NL\beta LMK\gamma MN\beta$ (or $KKK\alpha\beta NL\beta LMMN\beta$)

Now, S4.4 is S4 + $\mathfrak{C}pCMLpLp$. This last formula transforms in the field of S1° with the substitution p/β to

(7) $\mathfrak{C}\beta CNL\beta LMN\beta$

Clearly, in the presence of (7), the parenthesized version of (5) implies the parenthesized version of (6).

(8)	$\mathfrak{C}NLpCqKqMNp$	S1°
(9)	$CLNLp\mathfrak{C}qKqMNp$	(8), S1°
(10)	$CLNLpCLMqLMKqMNp$	(9), S3°
(11)	$C\beta CNL\beta CLM\gamma LMK\gamma MN\beta$	(10), (7), S1°

In the presence of (11), formula (5) (unparenthesized) implies formula (6). We see, then, the characteristic features of S4.4 coming into play in this case of the rule L-right (main). This exhausts the cases in which the present systems of tableaux differ from Kripke's systems for S5; the remainder of the completeness proof will be as in [1] or [2]. We then assert:

MTHM 1: *If each alternative set of a system of* S4.4 *tableaux* (*the tableaux described in this paper*) *closes, the formula for which the construction was begun is a thesis of* S4.4.

If we consider the instants of the end of the world matrix to be "possible worlds" in the sense of [1] or [2], it is easy to see that the same restrictions are placed upon the "accessibility relation"—Kripke's "R"—in the S4.4 tableaux and in the eow matrix. This means that Kripke's Lemmas 1 and 2 [2, pp. 76-80] apply here; we are interested in the following version of his Lemma 2:

MTHM 2: *If ϕ is valid in the* eow *matrix, then each alternative set of the system of* S4.4 *tableaux constructed with ϕ as its initial stage closes.*

MTHM 1 and **2**, along with the fact that it is easily established that every theorem of S4.4 is verified by the eow matrix lead to:

MTHM 3: *The end of the world matrix is characteristic for* S4.4.

We shall now see how the methods employed above may be adapted to achieve some further interesting results. From page 343 it will be recalled that the matrix of Group II may be looked upon as the "two instant" version of the end of the world matrix. Let us return for the time being to our proof tableaux and construct a system of such tableaux which will be analogous to Group II rather than to the infinite eow matrix. The rules for such a system will be like those for S4.4 in having different versions for main and for auxiliary tableaux; since the time sequence for Group II has two and only two instants, however, there will be *at most one auxiliary tableau* in each alternative set of tableaux in a construction. The main tableau again corresponds to the one instant in time and the one auxiliary tableau corresponds to the one instant in eternity. The L-rules for this system of tableaux are (**PC** rules remain as before):

L-left (main): *If $L\alpha$ appears on the left of the main tableau, α will be written on the left of both the main and the auxiliary tableau.*

L-left (aux): *If $L\alpha$ appears on the left of the auxiliary tableau, α will be written on the left of that tableau.*

L-right (main): *If $L\alpha$ appears on the right of a main tableau, split that tableau, with one of the alternative mains having α on its left and the other α on its right. For the main tableau having α on its left, construct an auxiliary tableau beginning with α on its right, or if an auxiliary tableau has already been begun for that main one, insert α on its right.*

L-right (aux): *If $L\alpha$ occurs on the right of an auxiliary tableau, write α on the right of that tableau.*

These tableaux clearly parallel Group II in their accessibility relations, and so Kripke's Lemmas 1 and 2 [2] apply here, and a formula will be valid in Group II iff its system of tableaux as defined above closes.

Our work with these tableaux will proceed just as it did in the case of S4.4, with the definition of characteristic formulas for the various stages of construction of a system of tableaux; in fact, the definition here of characteristic wff may be considered to be exactly the same as it was for S4.4. The only thing to be noted here is that where the characteristic wff for an alternative set in the S4.4 tableaux has the form $K\mu LMKM\sigma_1 \ldots M\sigma_s$, for Group II tableaux $s \leq 1$, then the M immediately preceding the σ will be (in S4) redundent, and the characteristic wff for an alternative set having an auxiliary tableau will always take the form $K\mu LM\sigma$, with μ characteristic for the main tableau, and σ for the auxiliary tableau. We will set out to investigate, then, the logical transformations involving the characteristic wffs and paralleling the above given rules for L.

The question we wish to answer is, "For which system is Group II the characteristic matrix?" We will do this by noting what formulas are necessary to insure that the characteristic wff at the n-th stage of construction of a system of tableaux such as we have proposed implies the characteristic wff of the $(n+1)$-th stage. For the L-left rules, this is easy, both for main and for auxiliary tableaux. The cases to be considered here are clearly subcases of the L-left cases in our work on the S4.4 tableaux. For the L-right rules, however, the situation is more complex. The split in L-right (main) is again justified by **PC**, and when L-right (main) is applied for the *first time* in an alternate set thereby *beginning* construction of the auxiliary tableau, the situation is the same as for the same case in the S4.4 tableaux; the move is from a characteristic formula of form $KK\alpha\beta NL\beta$ to one of form $KKK\alpha\beta NL\beta LMN\beta$, and is justified by S4.4. The system for which Group II is characteristic will then include S4.4.

When L-right, either (main) or (aux), is applied in any other situation than that discussed in the last paragraph, there will be a preexisting auxiliary tableau. As the rules indicate, no new tableau will be constructed in this situation, but the L-right rules will insert formulas into the already existing auxiliary tableau (this corresponds to the requirement in Group II considered as a time sequence matrix that if $L\alpha$ is false and α is true at the first instant, then α must become false at the second instant). In an application of L-right (main) subsequent to the first application, the characteristic wff of the appropriate alternate set after splitting the tableau but before insertion of a formula in the auxiliary tableau will be

(12) $KK\alpha K\beta NL\beta LM\gamma$

After completion of the application of L-right (main) the characteristic wff is

(13) $KK\alpha K\beta NL\beta LMK\gamma N\beta$

The first step in moving from (12) to (13) is to use the typical S4.4 thesis $\mathfrak{C}pCNLpLMNp$ to show that (12) implies

(14) $KK\alpha K\beta NL\beta KLM\gamma LMN\beta$

It now should be clear what formula is needed; the non-Lewis-modal

(15) $CKLMpLMqLMKpq$

will make (14) imply (13). This formula is valid in Group II and is easily derived in Sobociński's system K4 [9], which is S4.4 + $\mathfrak{C}LMpMLp$; indeed, S4.4 + formula (15) itself yields K4.

We now turn to L-right (aux); here the characteristic wff of the alternative set before application of the rule is

(16) $K\mu LMK\gamma NL\beta$

After the application, the characteristic wff is

(17) $K\mu LMK\gamma KNL\beta N\beta$

Now, even in S2° we have $\mathfrak{C}LMKpqKLMpLMq$ as a thesis, and so (16) implies

(18) $K\mu KLM\gamma LMNL\beta$

In S4 we have $\mathfrak{C}LMNLpLMNp$ provable, and so (18) implies

(19) $K\mu KLM\gamma KLMNL\beta LMN\beta$

which in the presence of (15) implies (17), the desired formula. The rest of the completeness proof will be as for S4.4, and it is clear that the system for which Group II is characteristic is K4, which is S4.4 + (15); we assert:

MTHM 4: *Lewis and Langford's* Group II *is the characteristic matrix for Sobociński's system* K4.

We shall now make use of the results we have so far achieved to discuss the situation of systems properly contained in S5 and properly containing S4.4. Let us say that such systems are "properly between" S4.4 and S5; the first such system to be found is due to Schumm [7]; this is S4.4 plus

(20) $ACMLppCLMqMLq$

Schumm gives a slightly different but deductively equivalent axiom. Sobociński [8] axiomatizes this system by adding

(21) $A\mathfrak{C}MLpLp\mathfrak{C}MLMqCqLq$

to S4; Schumm calls this system S4.7; for reasons to be discussed, we shall suggest a different designation for it.

Schumm's system was an affirmative response to a question asked by Sobociński [10]: Is there any system properly between S4.4 and S5? We shall study certain related problems. First of all, suppose that α is any wff that is:

(1) A theorem of S5, and
(2) is not a theorem of S4.4, and
(3) has only one propositional variable.

Since α is not a thesis of S4.4, it will, by our completeness result, fail in the end of the world matrix. Since it is a thesis of S5, it will take only the values $1/\mathbf{s}$ and $0/\mathbf{s}$ in that matrix; at its points of failure, then, it will take only the value $0/\mathbf{s}$. If we note the set of elements in the eow matrix *excluding* $0/\mathbf{s}$ and $1/\phi$, we will see that this set is closed under the operations associated with the **PC** connectives and L in the eow matrix, provided the formula being evaluated has only one propositional variable. For: if such a wff has no connectives, then its value in this matrix for the assignment x/y is x/y. Assume that if such a formula has k or fewer connectives, then its value will always be either x/y, $-(x/y)$, $1/\mathbf{s}$, or $0/\phi$ for the assignment of x/y to its variable. Let $K\beta\gamma$, $N\beta$, and $M\beta$ each have $k + 1$ connectives. With β and γ under the induction hypothesis, each of β and γ draws its values for this assignment from the above mentioned set (the variable of β is the same as that of γ), and so it is clear that each of the $k + 1$ connective wffs above will also draw its value from this set; it is then impossible with a one-variable formula to get a value (for the whole formula) of $0/\mathbf{s}$ unless the variable of that formula is assigned either $0/\mathbf{s}$ or $1/\phi$. The formula α we are here considering, then, takes both and only the values $1/\mathbf{s}$ and $0/\mathbf{s}$ in the eow matrix, and it takes $0/\mathbf{s}$ only at the assignment to its variable of $0/\mathbf{s}$ or $1/\phi$ or both. In the end of the world matrix, then, α will (with p as its variable) always imply one or both of the formulas $\mathfrak{C}MLpLp$ (which fails only at $p = 0/\mathbf{s}$) or $\mathfrak{C}MLpLMp$ (which fails only at $p = 1/\phi$). If, then, we add α to S4.4 as an extra axiom, either $\mathfrak{C}MLpLp$ or $\mathfrak{C}MpLMp$ becomes provable, and the system becomes S5. We have then:

MTHM 5: *If α is a wff with only one variable, and is a thesis of* S5 *but not of* S4.4, *then its addition to* S4.4 *yields* S5.

There then is no system with a single-variable proper axiom which is properly between S4.4 and S5.

Schumm's result shows that the above does not apply in the general case, for wffs which may have more variables than one. But we may rephrase Sobociński's question as follows: Is there any system (properly between S4.4 and S5, of course) which is such that there is *no* system properly between it and S5? We shall now define such a system, and show that it is as we claim. The system in question is simply the intersection of systems S5 and K4, the system whose theses are all and only the theses common to S5 and K4. We shall call this system S4.9. By our completeness result for K4, we can see that each theorem of S4.9 is both a thesis of S5 and is valid on Group II. Let us suppose that α is a wff which is a theorem of S5 but not valid in Group II. The n distinct propositional variables of α are $p_1, \dots, p_n$. Now examine the assignment in Group II for which α fails. For each i, if for this assignment the variable p_i receives:

1, make the substitution p_i/Cpp
2, make the substitution p_i/Np
3, make the substitution p_i/p
4, make the substitution $p_i/KpNp$

The formula resulting from these substitutions in α is α^*. Clearly, α^* is a thesis of S4.9 + α, and fails in Group II; specifically, it will fail at the assignment of the value 3 to its variable p. α^* is then a formula which is a thesis of S5 but not of S4.4, and which has only one variable (all S4.4 theses are verified by Group II). Since S4.4 is included in S4.9, the addition of α^* and so of α to S4.9 is sufficient to extend that system to S5, by **MTHM 5.** We then have:

MTHM 6: *There is no system properly between* S4.9 (= S5 $\cap$ K4) *and* S5.

This answers negatively another question of Sobociński [8]: Is there any proper subsystem of S5 which is also not a subsystem of K4?

We now undertake the task of finding an axiomatization for S4.9. Note the following:

(22) $\mathfrak{C}LMqCMLCpMLqCpMLq$

We wish to show that S4.4 + (22) = S4.9. (The reader may easily verify that formula (22) is both a thesis of S5 and is verified by Group II.) When we check a wff α for theoremhood in S4.9, we may think of ourselves as first applying some decision procedure for S5 to α, and then checking α in Group II. In terms of our tableaux, this means that we take the candidate formula, and if it proves to be an S5 thesis, we construct a system of tableaux for it, using the rules for L as given above for the system K4; if the system of tableaux closes, then the formula in question is a thesis of S4.9; we wish to show that it is provable in S4.4 + (22). The proof will be quite similar to that for the completeness of K4; it will differ from that proof only at the point where the K4 proof makes use of the non-Lewis-modal $CKLMpLMq$-$LMKpq$. Instead of using that formula it will employ (22) plus the fact that the wff being tested is an S5 thesis. We first establish that

MTHM 7: *If α is an S5 thesis, and if χ_n is the characteristic wff of the n-th stage of construction of a system of tableaux for α, then $MLN\chi_n$ is an* S4 *thesis; indeed, where χ_n is of form $A\delta_1 \dots \delta_m$, then each of the m wffs $MLN\delta_i$ is an* S4 *thesis.*

As we have noted above, where β is an S5 thesis, $ML\beta$ is provable in S4; thus **MTHM 7** holds for $n = 1$. For the induction step, we need only note that given the rules for tableaux as we have been considering them, it is easily seen that even in S2° we have $\mathfrak{C}\chi_{k+1}\chi_k$ as a thesis, and again, even in S2° this converts to $CMLN\chi_k MLN\chi_{k+1}$, and **MTHM 7** holds so far as $MLN\chi_n$'s being a thesis is concerned. Note that $MLNA\,\delta_1 \dots \delta_m$ is equivalent to $MLKN\delta_1 \dots N\delta_m$; distributing the ML over the m conjuncts gives us each of the $MLN\delta_i$ as an S4 thesis, and the metatheorem holds so far as its final clause is concerned.

We may now refer back to the completeness proof for K4. The critical points for our present consideration are the movements from formula (14) to formula (13), and from (19) to (17). These are the only non-Lewis-modal steps in the proof, involving formula (15) to show that a formula of form

(23) $K\alpha KLM\gamma LM\beta$

implies one of form

(24) $K\alpha LMK\gamma\beta$

We shall show that (23) implies (24) in S4.9 by showing that the negation of formula (24) implies the negation of (23); for clarity we will use the notation of deduction from hypotheses; we will show that

(25) $C\alpha MLC\gamma N\beta \vdash C\alpha CLM\gamma MLN\beta$

or equivalently

(26) $C\alpha MLC\gamma N\beta,\ \alpha,\ LM\gamma \vdash MLN\beta$

holds in S4.9. First of all, since the formula for which the construction was begun is an S5 thesis, by **MTHM 7** we have

(27)	$\vdash MLC\alpha CLM\gamma MLN\beta$	
(28)	$\vdash CLM\gamma MLC\alpha MLN\beta$	(27), S4.2
(29)	$LM\gamma$	Hyp.
(30)	$MLC\alpha MLN\beta$	(29), (28), **PC**
(31)	$C\alpha MLC\gamma N\beta$	Hyp.
(32)	α	Hyp.
(33)	$MLC\gamma N\beta$	(31), (32), **PC**
(34)	$CLM\gamma LMN\beta$	(33), S4.2
(35)	$LMN\beta$	(34), (29), **PC**

We now note that the suggested axiom for S4.9 is formula (22); with the substitutions $q/N\beta$ and p/α, it is

(36)	$\mathfrak{C}LMN\beta CMLC\alpha MLN\beta C\alpha MLN\beta$	
(37)	$C\alpha MLN\beta$	(36), (35), (30), S1°
(38)	$MLN\beta$	(37), (32), **PC**

By the deduction theorem, we now may establish that the negation of (24) implies that of (23), and so that (23) implies (24) as required. The rest of the proof follows just as does the proof that Group II is characteristic of K4; no non-Lewis-modal transformations are needed; it will then follow that

MTHM 8: S4.4 + (22) = S4.9, *that is*, S5 ∩ K4.

Formula (22) is handy for the proof above, but it is not as simple as it might be; for the following, we might appeal to the above result for S4.9 or do the following in S4.9:

(39)	$\mathfrak{C}MLCNpMLqCNpCLMqMLq$	(22) p/Np, S1°
(40)	$\mathfrak{C}MLpMLCNpr$	S2°
(41)	$\mathfrak{C}MLpCNpCLMqMLq$	(39), (40), S1°
(42)	$ACMLppCLMqMLq$	(41), S1°

Formula (42) yields (41) in the field of S4.2. It is interesting to note that (42) reveals S4.9 to be "unreasonable in the sense of Hallden" [4]; it is a thesis which is a disjunction neither of whose disjuncts are theses and whose disjuncts have no variables in common.

Since (42)'s left disjunct shares no variables with the right, and since the left disjunct would, if added to S4, complete it to S5, it will be the case that if α is an S5 thesis, then $A\alpha CLMqMLq$ will be provable in S4.4 + (42); indeed, $LA\alpha CLMqMLq$ will be provable therein. Now formula (22) is itself an S5 thesis, and so

(43) $LACLMqCMLCpMLqCpMLqCLMqMLq$

is provable in S4.4 + (42). By S1°, specifically by $CLACpCqCrsCps\mathfrak{C}pCqCrs$, formula (43) leads to formula (22), and so S4.4 + (42) = S4.9; but this system is Schumm's [7] S4.7, and our S4.9 = his S4.7. Since this system is a limiting system in the sense that there are no systems properly between it and S5, we propose that its name be standardized as S4.9.

Now note the formula

(44) $\mathfrak{C}MKpqCMKpNqCMLpLp$

This formula is a thesis of S4.9, for it is verified by Group II and is an S5 theorem. Furthermore, it is a theorem of the system V1, which is S4 + $ALpA\mathfrak{C}pq\mathfrak{C}pNq$ [10]; as shown in [10], S4 + $\mathfrak{C}MKpqCMKpNqLp$ gives V1 also.

(45)	$\mathfrak{C}MLpCLMqMKpq$	S4
(46)	$\mathfrak{C}MLpCLMqCMKpNqCMLpLp$	(44), (45), S1°

By S1° we drop the second MLp in (46) as redundent, and transpose the $MKpNq$ and the Lp, giving

(47)	$\mathfrak{C}MLpCLMqCNLp\mathfrak{C}pq$	
(48)	$\mathfrak{C}MLpCLMqCNp\mathfrak{C}pq$	(47), S1
(49)	$\mathfrak{C}\mathfrak{C}pqCMLpMLq$	S3°
(50)	$\mathfrak{C}MLpCLMqCNpCMLpMLq$	(48), (49), S1°
(51)	$\mathfrak{C}MLpCNpCLMqMLq$	(50), S1°

This last formula transforms in the field of S1° to (42), which when added to S4.4 gives S4.9; S4.4 + 44, then, gives S4.9, and S4.9 is contained in V1. We thus note that not only are all proper subsystems of S5 contained in K4, they are contained in V1 as well. That K4 contains V1 was established by Thomas [11] and follows easily from our completeness result for K4.

Another formula we might consider is:

(52) $\mathfrak{C}LMpCLMqCMKpqLMKpq$

This is clearly in S4.9 ($\mathfrak{C}MKpqLMKpq$ is an obvious S5 thesis and (52) is verified by Group II). On the other hand, it fails in all versions of the end of the world matrix of three instants or more; in the eight-valued version above: $CLM2CLM3CMK23LMK23 = CL1CL1CM4LM4 = C1C1C4L4 = C1C48 = 5$. Let us say, then, that S4.4 + (52) = S4.6.

In [10] it is shown that D (S4.3.1) is properly contained in S4.4, and in [12] it is shown that S4.3.2, which is S4 + $A\mathfrak{C}LpqCMLqp$, is also properly contained in S4.4. The matrices used to show that the inclusions are proper and that D and S4.3.2 are distinct are both displayed in [12, p. 297]. It will be noted that each of these matrices verifies the non-Lewis modal formula

(53) $\mathfrak{C}LMpMLp$

whose addition to S4.4 gives the system K4 [9]. Both K3.1 (= D + (53)) and K3.2 (= S4.3.2 + (53)) are then properly contained in K4, and are themselves distinct. Now K4, K3.1, and K3.2 clearly contain the formula

(54) $\mathfrak{C}LMpCLMqLMKpq$

as a thesis, and so all have (52) as a thesis, and K4 contains S4.6. But if the addition of (52) to D or to S4.3.2 extend either of those systems to S4.6, then the addition of (53) to D or to S4.3.2 would extend the respective system to K4, which by the above it does not. Call D + (52) S4.3.3 and S4.3.2 + (52) S4.3.4. These two systems are then properly contained in S4.6.

Now let us add to S4 either the formula

(55) $\mathfrak{C}CpLMpCCqLMqCMKpqLMKpq$

or

(56) $\mathfrak{C}CpLMqCCqLMpCMKpqLMKpq$

By S1°, ultimately by the thesis $\mathfrak{C}pCqp$, each of these is transformable to formula (52), $\mathfrak{C}LMpCLMqCMKpqLMKpq$. Again by S1°, here ultimately by $\mathfrak{C}NpCpq$, each of these formulas transforms to

(57) $\mathfrak{C}NpCNqCMKpqLMKpq$

With the substitution q/p, (57) reduces in the field of S1° to

(58) $\mathfrak{C}NpCMpLMp$

which when added to S4 gives S4.4. We have thus established that S4 plus either (55) or (56) contains S4.6.

Let us now assume S4.4 + (52). In what follows we shall make use of a deduction from hypotheses, as the proof directly from the axioms involves formulas that are fairly long, making such a proof more difficult to follow. What we wish to show is that

$$CpLMp,\ CqLMq,\ MKpq \vdash LMKpq$$

holds in S4.4 plus (52), and that it holds as well when $CpLMp$ and $CqLMq$ are replaced respectively by $CpLMq$ and $CqLMp$.

(59)	$MKpq$	Hyp.
(60)	$CMKpqCNKpqLMKpq$	S4.4
(61)	$CNKpqLMKpq$	(59), (60), PC
(62)	$CNpLMKpq$	(61), PC
(63)	$CNqLMKpq$	(61), PC
(64)	$CpLMp$	Hyp.
(65)	$CqLMq$	Hyp.

Let us note that what follows holds equally well with (64) as $CpLMq$ and (65) as $CqLMp$.

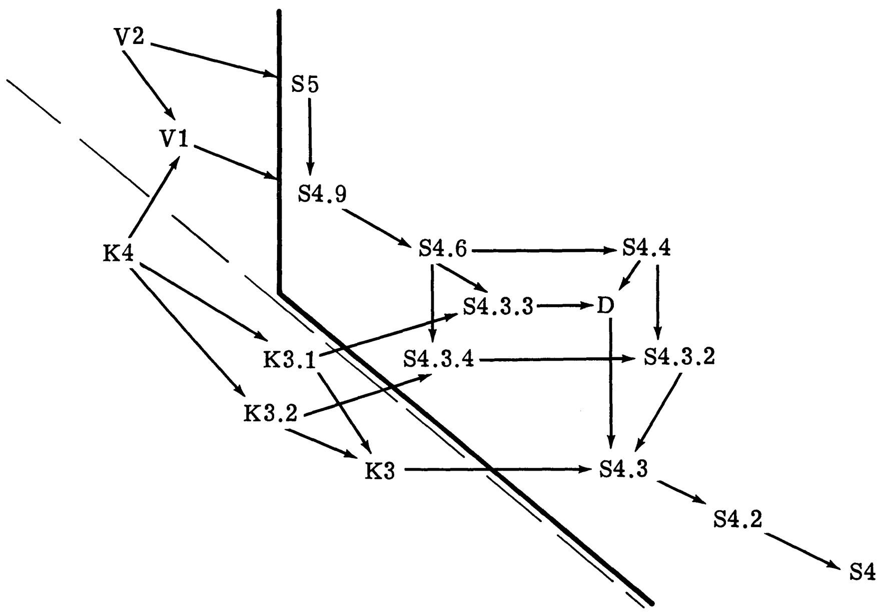
(66)	$CLMpCLMqCMKpqLMKpq$	(52), S1°
(67)	$CpCqCMKpqLMKpq$	(64), (65), (66), PC
(68)	$CpCqLMKpq$	(67), (59), PC
(69)	$CNpCqLMKpq$	(62), PC
(70)	$CqLMKpq$	(68), (69), PC
(71)	$LMKpq$	(70), (63), PC

We have thus established that $LMKpq$ follows from the hypotheses $CpLMp$, $CqLMq$, and $MKpq$ or from $CpLMq$, $CqLMp$, and $MKpq$ in the system S4.4 + (52). By the deduction theorem as it holds in S4, then, (55) and (56) are both provable here, and S4.6 = S4.4 + (52) = S4 + (55) = S4 + (56).

We now summarize the systems we have been discussing and the relationships between them. Assuming standard axiomatizations of the well-known S4, S4.2, S4.3, and S5, the other systems are as follows:

D (S4.3.1) = S4.3 + $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLppCMLpp$
S4.3.2 = S4 + $A\mathfrak{C}LpqCMLqp$
S4.3.3 = D + (52), $\mathfrak{C}LMpCLMqCMKpqLMKpq$
S4.3.4 = S4.3.2 + (52)
S4.4 = S4 + $\mathfrak{C}pCMLpLp$
S4.6 = S4.4 + (52), or
= S4 + (55), $\mathfrak{C}CpLMpCCqLMqCMKpqLMKpq$, or
= S4 + (56), $\mathfrak{C}CpLMqCCqLMpCMKpqLMKpq$
S4.9 = S4.4 + (42), $ACMLppCLMqMLq$, or
= S4.4 + (22), $\mathfrak{C}LMqCMLCpMLqCpMLq$, or
= S4.4 + (44), $\mathfrak{C}MKpqCMKpNqCMLpLp$, or
= S4 + (21), $A\mathfrak{C}MLpLp\mathfrak{C}MLMqCqLq$
V1 = S4 + $ALpA\mathfrak{C}pq\mathfrak{C}pNq$
V2 = S5 + $ALpA\mathfrak{C}pq\mathfrak{C}pNq$ [8]
K3 = S4.3 + (53), $\mathfrak{C}LMpMLp$
K3.1 = D + (53)
K3.2 = S4.3.2 + (53)
K4 = S4.4 + (53)

The following diagram gives the relationships between these systems as established in this paper. Note that the question of whether or not S4.9 properly contains S4.6 remains open; otherwise, all containments as indicated by arrows are established to be proper. Systems to the left of the solid line are non-Lewis-modal; those to the left of the broken line are incompatible with S5.



In addition to and related to the question of whether or not the system S4.9 contains S4.6 properly is the question of whether there are *any* systems properly between S4.4 and S4.9. Further, we know that there are no systems properly between S4.9 and S5. But are there any properly between S4.9 and V1? or between V1 and V2? or between V1 and K4? My immediate guess is that these are problems which will not yield too easily.

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