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INDEPENDENCE OF THE AXIOMS AND RULES OF INFERENCE OF ONE SYSTEM OF THE EXTENDED PROPOSITIONAL CALCULUS

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In [1] A. Church introduced an extended propositional calculas **P**, built up by a logical operator, $\supset$ (implication), an universal quantifier and propositional variables. The only operator variables in **P** are propositional variables.

The axioms of **P** are the three following:

A1. $p \supset q \supset . q \supset r \supset . p \supset r$
A2. $p \supset q \supset p \supset p$
A3. $p \supset . q \supset p$

The primitive rules of inference are:

R1. $\dfrac{A \supset B, A}{B}$ (modus ponens)

R2: $\dfrac{A}{\check{\mathsf{S}}_B^p A \Big|}.$ (rule of substitution)

R3. $\dfrac{A \supset B}{A \supset (a)B}$

R4. $\dfrac{A \supset (a)B}{A \supset B}$

In R3 and R4 a is a propositional variable, which is not free in A.

The purpose of this work is to show, that the axioms and rules of **P** are independent.

1. *Theorems* Now we go on to the proof of some theorems of **P**.

1. $\vdash p \supset p$

By A1, R2, A3 and R1:
$\vdash q \supset p \supset r \supset . p \supset r$
$\vdash p \supset q \supset p \supset p \supset . p \supset p$

Hence by A2 and R1:
$\vdash p \supset p$

2. $\vdash p \supset [p \supset q] \supset . p \supset q$

By A1, R2 and R1:

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$\vdash q \supset r \supset [p \supset r] \supset s \supset . p \supset q \supset s$

By R2, A2 and R1 obtain 2.

3. $\vdash p \supset . p \supset q \supset q$

By A1, R2:

$\vdash p \supset q \supset p \supset . p \supset q \supset . p \supset q \supset q$

By A3, A1, R2 and R1:
$\vdash p \supset . p \supset q \supset . p \supset q \supset q$
Hence by 2, A1, R2 and R1 obtain 3.

4. $\vdash p \supset [q \supset r] \supset . q \supset [p \supset r]$

By 3, A1, R1 and R2:

$\vdash q \supset r \supset r \supset [p \supset r] \supset . q \supset [p \supset r]$

By A1 and R2:

$\vdash p \supset [q \supset r] \supset . q \supset r \supset r \supset [p \supset r]$

Then use A1 and R2 to obtain 4.

5. $\vdash A \supset B \supset . A \supset (a) [B \supset (s)s] \supset (s)s$

By R4, 1, R1, R2:

$\vdash (a)[B \supset (s) s] \supset . B \supset (s) s$

By A1, R2, R1, 3:

$\vdash B \supset (s) s \supset (s) s \supset . (a) [B \supset (s) s] \supset (s) s$
$\vdash B \supset . (a) [B \supset (s) s] \supset (s) s$
$\vdash A \supset B \supset . B \supset [(a) [B \supset (s) s] \supset (s) s] \supset . A \supset . (a) [B \supset (s) s] \supset (s) s$

Hence by 4, R2 and R1, 5 follows.

6. $\vdash (s) s \supset a$ and $\vdash (s) s \supset (a) a$

By 1, R4 and R1:

$\vdash (s) s \supset (s) s$
$\vdash (s) s \supset s$

Owing to R2, R3 we have 6.

7. $\vdash p \supset (s) s \supset (s) s \supset p$

By A1, 4, 6, R1, R2:

$\vdash p \supset (s) s \supset (s) s \supset . p \supset (s) s \supset p$

Hence by A1, A2, R1, R2 establish 7.

8. $\vdash A \supset [B \supset [B \supset (s) s \supset (s) s] \supset (s) s] \supset . A \supset B$

By A3, 3, R1, R2:

$\vdash B \supset (s) s \supset . B \supset [B \supset (s) s \supset (s) s]$

By A1, R2, R1:

$\vdash B \supset (s)s \supset (s)s \supset B \supset . B \supset [B \supset (s) s \supset (s) s] \supset (s) s \supset B$

Then use 7, R1 and R2.

2. *Independence of the Axioms and Rules of* $\mathbf{P}$. Let $\mathbf{P}_1$ be a propositional calculus built up by propositional variables, a logical constant f and logical operators: $\supset$ and & (conjunction). The axioms of $\mathbf{P}_1$ are A1, A2, A3 and the primitive rules of inference are R1 together with:

$$\text{R2}': \quad \frac{A}{\mathsf{S}_B^p A \,|\cdot} \qquad\qquad \text{R4}': \quad \frac{A \supset \mathrm{f}}{A \supset a}$$

$$\text{R3}': \quad \frac{A \supset B_1}{A \supset B_1 \,\&\, B_2} \qquad\qquad \text{R4}'': \quad \frac{A \supset B_1 \,\&\, B_2}{A \supset B_1}$$

where B_2 is $\mathsf{S}_p^a B_1 \,|.$ and a is a propositional variable which does not occur in A.

Every wff A from $\mathbf{P}$ corresponds to wff A^* from $\mathbf{P}_1$ which is obtained according to the following procedure. If A does not contain an universal quantifier, then A^* is A. If there are universal quantifiers in A, then all occurrences of $(s)s$ (where s is any propositional variable) are replaced by f and wfp of the form $(a)B(a)$ ($B(a)$ is not a) are replaced by $B(a)$ & $B(a \supset \mathrm{f})$ where B^* is the corresponding formula of $\bar{B}$.

Let any proof D be given of a theorem T in $\mathbf{P}$ and let $a_1, \ldots, a_n$ be the complete list of variables which are quantifier variables occurring in the proof. Choose propositional variables $c_1, \ldots, c_n$, which are distinct among themselves and distinct from all variables in D. Throughout D substitute $c_1, \ldots, c_n$ for $a_1, \ldots, a_n$ respectively. Then any wff is to be replaced by the corresponding formula. We shall show how this list of wffs can be transformed into a proof of T^* in $\mathbf{P}_1$. The proof proceeds by mathematical induction with respect to the length of D. If T is an axiom then T^* is an axiom in $\mathbf{P}_1$ too. If B is inferred by R1 from premisses $A \supset B$ and A then from $(A \supset B)^*$ and A^* by R2$'$ and R1 we may infer B^*. When R2 is applied, there are two cases: (a) A does not contain a free occurrence of p in the scope of (x), where x is a free variable of B. From R2:

$$\frac{A}{\check{\mathsf{S}}_B^p A \,|\cdot} \text{ stands for } \frac{A}{\mathsf{S}_B^p A \,|\cdot}$$

Hence by R2$'$ from A^* we can establish $(\mathsf{S}_B^p A|)^*$. (b) A contains some p in the scope of (x). By R2: $\check{\mathsf{S}}_B^p A|$ stands for A. Then the proof of T^* is that one of A^*. If $A \supset (a)B$ is inferred from premiss $A \supset B$ by R3, then A does not contain a free variable a, and so by R3$'$, R2$'$, R4$'$ it is possible to infer $A^* \supset B^*(a) \,\&\, B^*(a \supset \mathrm{f})$, which is $(A \supset (a)B)^*$. When $A \supset B$ is inferred from $A \supset (a)B$ by R4, A does not contain free a and then $(A \supset B(a))^*$ is inferred by R4$'$, R4$''$ and R2$'$.

From this it follows that if we show the independence of the rule of modus ponens and each of the axioms A1, A2, A3 for $\mathbf{P}_1$ we have, for $\mathbf{P}$, the independence of each of R1, A1, A2 and A3. Independence of the A1, A2, A3 and R1 can be established by the following truth-tables. The designated truth-values in it are 0 for A1, A2, R1; 0, 1 for A3. The theorem of $\mathbf{P}_1$, which is not a tautology according to the truth-table for R1 is $p \supset p$. 5 is assigned to the primitive constant f as a value for A1; 2 for A2 and A3; 1 for R1.

A1												A2				A3				R1			
p	q	$p\supset q$	$p\&q$	p	q	$p\supset q$	$p\&q$	p	q	$p\supset q$	$p\&q$	p	q	$p\supset q$	$p\&q$	p	q	$p\supset q$	$p\&q$	p	q	$p\supset q$	$p\&q$
0	0	0	0	2	0	0	2	4	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0
0	1	3	0	2	1	3	2	4	1	0	4	0	1	1	0	0	1	0	0	0	1	0	0
0	2	4	0	2	2	3	2	4	2	0	4	0	2	1	0	0	2	2	0	0	2	0	0
0	3	1	0	2	3	0	2	4	3	1	4	1	0	0	1	1	0	2	1	1	0	0	1
0	4	2	0	2	4	0	2	4	4	1	4	1	1	0	1	1	1	2	1	1	1	2	1
0	5	5	0	2	5	3	2	4	5	1	4	1	2	1	1	1	2	2	1	1	2	0	1
1	0	0	1	3	0	0	3	5	0	0	5	2	0	0	2	2	0	0	2	2	0	0	2
1	1	2	1	3	1	0	3	5	1	0	5	2	1	0	2	2	1	0	2	2	1	0	2
1	2	0	1	3	2	4	3	5	2	0	5	2	2	1	2	2	2	0	2	2	2	0	2
1	3	0	1	3	3	4	3	5	3	0	5												
1	4	2	1	3	4	0	3	5	4	0	5												
1	5	2	1	3	5	4	3	5	5	0	5												

For the independence of R2, consider the transformation upon the wffs of **P** which consists in omitting the universal quantifier (with its variable) whereever it occurs. This transforms every axiom into a theorem and every primitive rule except R2 into a primitive or derived rule. But $\vdash (s)s \supset a$ transforms into the non-theorem $s \supset a$.

For the independence of R3, consider the transformation upon the wffs of **P** which consists in replacing every wfp of the form $(a)A$ by $A \supset [A \supset (s)s \supset (s)s] \supset (s)s$. This transforms every axiom into a theorem and every primitive rule except R3 into a primitive or derived rule (for R4 see 8). But it transforms the theorem $\vdash p \supset (a)[a \supset a]$ into a non-theorem;

$p \supset . a \supset a \supset [a \supset a \supset (s)s \supset (s)s] \supset (s)s$

Finally, in order to establish the independence of R4, we use a transformation upon the wffs of **P** which consists in replacing $(a)A$ by $(a)[A \supset (s)s] \supset (s)s$. Except for R4, all the axioms and rules of **P** transform into axioms and primitive or derived rules (for R3 see 5). But the theorem, $\vdash (a)a \supset a$, is transformed into a non-theorem,

$(a)[a \supset (s)s] \supset (s)s \supset a$

If we assume that it is a theorem, then by R2:
$\vdash (a)[a \supset (s)s] \supset (s)s \supset p$
by R4, A1, R1, R2:
$\vdash a \supset (s)s \supset (s)s \supset p$
By 3, R2, A1, R1:
$\vdash a \supset p$

which is a non-theorem. With this the independence of the axioms and rules of **P** is proved.

REFERENCES

[1] Church, A., *Introduction to Mathematical Logic*, vol. 1, Princeton: Princeton University Press (1956).

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