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GENTZEN-LIKE SYSTEMS FOR PARTIAL PROPOSITIONAL CALCULI: II

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In part I of this paper [3], we gave a generalized definition of a Gentzen-like system and then applied this definition to construct cut-free Gentzen-like systems for two of the three one-axiom subcalcului of the system of [1]. We now consider the remaining one axiom subcalculus. Throughout we will be concerned with the rules of simultaneous substitution (SS) and modus ponens (MP).

In order that we may handle the system whose sole axiom is:

$A3$. $(s \supset (p \supset q)) \supset ((s \supset p) \supset (s \supset q))$

it is convenient to first discover what the consequences of this axiom are. We further introduce the following terminology which allows us, for a given consequence of $A3$, to indicate something about the "length of a proof" of that consequent.

Definition. $(3,0)$ is the set whose only element is $A3$. If $(3,n)$ is given, then $(3, n+1)$ is the set whose elements are those of $(3,n)$ together with any well-formed formulae W of the following forms:

(i) W is a SS instance of an element of $(3,n)$
(ii) W is a MP instance of elements of $(3,n)$, that is, there are well-formed formulae A and B in $(3,n)$ and B is of the form $A \supset W$.

Then any formulae provable from $A3$ belong to some $(3,n)$ (since each has a proof of finite length), so that if we can find $(3, \infty) = \bigcup_n (3,n)$, we will have the desired set of consequences. In the following proof, we refer to the $A \supset B$ premiss of MP as *major*; the A premiss as *minor* and employ the following notation for the parts of formulae. If T is a formulae of the form $B \supset D$, we write AT for B (*antecedent* of T), and CT for D (*consequent* of T). This notation may be iterated in one expression for example, ACT, the antecedent of the consequent of T, etc.).

Proposition 6.
$(3,0) = \{A3\}$

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$(3,1) = \{T_{-1}\colon T_{-1} = (S \supset (P \supset Q)) \supset ((S \supset P) \supset (S \supset Q))\}$, for any well-formed formulae S, P, Q.

$(3,2) = (3,1) \cup \{T_0\colon T_0 = (A_0 \supset A_{-1}) \supset (A_0 \supset B)\}$ where $A_0 = (S \supset (P \supset Q))$, $A_{-1} = (S \supset P)$, $B = (S \supset Q)$ for any well-formed formulae S, P, Q.

$(3,3) = (3,2) \cup \{T_1\colon T_1 = A_2 \supset (A_1 \supset B)\}$ where in general $A_n = (A_{n-1} \supset A_{n-2})$ for $n > 0$.

$(3,n) = \{T_m\colon T_m = A_{m+1} \supset (A_m \supset B)\}$ for $m = -1, 0, 1, \ldots, n\text{-}2$.

$(3,\infty) = \{T_m\}$, for $m = -1, 0, 1, \ldots, n, \ldots$

Proof: (3,0) is $\{A3\}$ by definition. Since it contains only one element, only SS may be applied to get new members in (3,1), so (3,1) appears as described. In (3,2) and the following, it will be noted that SS does not apply, since in each case the collection of antecedents is closed under SS at the start. For (3,2), applying a substitution instance of T_{-1} (with S,P,Q as minor premiss) to a T_{-1} (with S',P',Q' as major premiss) we see that $S' = S \supset (P \supset Q)$, $P' = S \supset P$, and $Q' = S \supset Q$. Then applying MP, we get all the formulae of the form T_0 as desired. From this point on we need consider only whether the newly obtained formulae will combine with any of the older ones by MP (or with other new formulae), since all the combinations of the older ones have been tried at an earlier level.

For (3,3) we certainly get all of (3,2). Now T_{-1} is not AT_0 as a check of the elements corresponding to S in AT_0 will show. Likewise T_0 is not AT_0 as a check of the elements corresponding to P in AT_0 will show. But T_0 is AT_{-1} and application gives us T_1. Since all other MP and SS possibilities have been eliminated by the above commentary, (3,3) is as claimed.

For (3,4), we have all of (3,3) and again need consider only MP involving T_1. A check of the elements corresponding to S in AT_1 shows that none of T_{-1}, T_0, or T_1 is an AT_1. In the other direction T_1 is an AT_{-1} giving us by MP all the T_2. But T_1 is not an AT_0 as a check of the formulae corresponding to P in AT_0 shows. Hence (3,4) is as claimed.

For (3,5), the proof is identical to that for (3,4), this time studying T_2. The only new fact needed is that T_2 is not an AT_1 which follows by considering the elements corresponding to S in AT_1. For $(3,n)$, $n \geqq 5$, we are dealing with a T_n, $n \geqq 3$. Hence, $T_n = A_{n+1} \supset (A_n \supset B) = (A_n \supset A_{n-2}) \supset (A_n \supset B)$ $= ((A_{n-1} \supset A_{n-2}) \supset (A_{n-2} \supset A_{n-3})) \supset (A_n \supset (S \supset Q))$. This is also true for n=2. So, for $n \geqq 2$, $AAT_n = ACT_n = A_n$, but for any $m > 2$, $AAAT_m = A_{n-2} \supset A_{n-3}$ while $ACAT_m = A_{n-2}$ which are distinct as one is a proper subformula of the other. Hence none of these T_m and T_n is an antecedent for the other and no MP is possible in these cases. Hence we need consider only the interaction of T_{-1}, T_0 and T_1 with T_n and $n > 3$. A comparison of T_{-1} and $A\,T_n$ gives that $S = A_{n-1} = A_{n-2} \supset A_{n-3}$ and that $A_{n-2} = S \supset P$. Hence, since S is not a proper subformula of itself, T_{-1} is not an AT_n. A similar comparison of T_0 and AT_n gives that $S = A_{n-2}$ and $S = A_{n-3}$ which is not possible for $n > 3$. Also, a comparison of T_1 and AT_n gives that the S in A_1 must correspond both to the S in AT_n and to the A_{n-3} which is a formula properly containing that S. Hence, the new T_n is never the major premiss for MP.

In the other direction, trying to match $A\,T_1$ with T_n matches S with both A_n and A_{n-1}, a proper subformula of A_n. Trying to match AT_0 with T_n

matches the P of the former to the $S \supset Q$ of the latter and to the A_{n-2} of the latter which is not $S \supset Q$ but contains S and Q as elements. Hence, the only remaining possibility is to use T_n as the antecedent of T_{-1} which gives us T_{n+1} as desired. Hence, the various $(3,n)$ are as described. Hence their union is as desired. Q.E.D.

Now, knowing the elements which we want to have appear in our Gentzen-like system, we can construct the system and avoid the problem of eliminating a cut-type rule by simply never introducing one.

Our system **G3** will consist of 7-tuples from the well-formed formulae plus #, with the following axiom schema and rules.

$(S, P, Q, \#, \#, \#, \#)$

(a) $(S, P, Q, \#, \#, \#, \#) \rightarrow (\#, \#, \#, S \supset P, S \supset (P \supset Q), S \supset Q, \#)$.

(b) $(\#, \#, \#, A, B, C, \#) \rightarrow (\#, \#, \#, B, B \supset A, C, \#)$.

(c) $(\#, \#, \#, A, B, C, \#) \rightarrow (\#, \#, \#, \#, \#, \#, B \supset (A \supset C))$.

It is now easy to show that this system generates all the consequences of *A3*.

Proposition 7. *P is a theorem of A3 if and only if* $(\#, \#, \#, \#, \#, \#, P)$ *is deducible from the system* **G3**.

Proof: First let us see that, by using rules (a) and (b), we generate the triples A_n, A_{n+1}, B (in the form $(\#, \#, \#, A_n, A_{n+1}, B, \#)$) of the theorem characterizing the theorems of *A3*. For $n = -1$, this is immediate by applying rule (a) to the axiom to get the desired result. Now suppose that $(\#, \#, \#, A_n, A_{n+1}, B, \#)$ is generated. Then apply rule (b) to get $(\#, \#, \#, A_{n+1}, A_{n+1} \supset A_n, B, \#)$, that is $(\#, \#, \#, A_{n+1}, A_{n+2}, B, \#)$ as desired.

Now it is sufficient to show that we can get all the T_n. But with the 7-tuple $(\#, \#, \#, A_n, A_{n+1}, B, \#)$, which we now know we have, the application of rule (c) gives $(\#, \#, \#, \#, \#, \#, T_n)$ by the definition of T_n. Hence our system generates all the desired formulae.

Now let us suppose that a formula T is generated by the system. By its form $(\#, \#, \#, \#, \#, \#, T)$, the last step in the proof must have occurred by rule (c), and T will be a T_n if the 7-tuple from which $(\#, \#, \#, \#, \#, \#, T)$ was deduced $(\#, \#, \#, A', B', C', \#) = X$ is of the appropriate form with $A' = A_n$, $B' = A_{n+1}$, and $C' = B$ for some n and the usual A and B. If this latter 7-tuple was concluded by rule (a), it is of the right form, and in fact has $n = -1$. By the form, if the tuple in question was not deduced by rule (a), it must have been deduced from a tuple Y of the same form by rule (b). Similarly Y from which X was deduced must have itself been deduced either by (a) or (b), and so forth. Hence we will do deduction on the number of times that rule (b) was used, a number which must be finite since each use of rule (b) increases the total of the lengths of the three non-# formulae in the 7-tuple. If the number of uses of rule (b) is one, the previous 7-tuple was deduced by rule (a) and hence was of the form $(\#, \#, \#, A_{-1}, A_0, B, \#)$, so that the application of rule (b) gives $(\#, \#, \#, A_0, A_1, B, \#)$ which is of the proper form. Now assume that all cases with the use of rule (b) n times gives the tuple $(\#, \#, \#, A_{n-1}, A_n, B, \#)$. Then the definition of rule (b) gives that the resulting 7-tuple is $(\#, \#, \#, A_n, A_{n+1}, B, \#)$ as desired. Q.E.D.

There remains to be shown that given a formula T, we can effectively tell if it is a consequence of the system **G3** or not. Consider the following procedure. Form the 7-tuple (#, #, #, #, #, #, T). If T is not of the form $B \supset (A \supset C)$ for some A, C, B, it is not a T_n for any n and so not a theorem. If T does have this appropriate form, then if it is a theorem, it must follow from (#, #, #, A, B, C, #) by rule (c). Now either this latter 7-tuple is of the form (#, #, #, $S \supset P$, $S \supset (P \supset Q)$, $S \supset Q$, #) for some formulae S, P, and Q, or it is not and we can effectively tell this. If the tuple is of the desired form, T is a consequence of the system **G3**, starting with the axiom (S, P, Q, #, #, #, #), applying rule (a) and then rule (c). If T is not of the desired form above, either (#, #, #, A, B, C, #) is of the form (#, #, #, B'', $B'' \supset A''$, C, #) for some B'' and A'' or it is not of this form. If it is not of this form, T is not a theorem, since then (#, #, #, A, B, C, #) could arise by none of the rules. If it is of this form, we argue on the sum of the lengths of A, B, and C. The only rule by which (#, #, #, A, B, C, #) might be deduced is rule (b), since we have eliminated all other cases. Hence, if T is to be a theorem, it must be the case that (#, #, #, A, B, C, #) was deduced by rule (b) from (#, #, #, A'', B'', C, #). But the sum of the lengths of A'', B'', and C is less than that for A, B, and C by the definition of rule (b), so by induction we can tell whether T is a theorem. Hence:

Theorem. *Given a well-formed formula T, there is an effective method by which one can tell whether T is a theorem of A3 or not, and by which, if T is a theorem, we can construct a proof of* (#, #, #, #, #, #, T) in **G3**. Q.E.D.

REFERENCES

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