Notre Dame Journal of Formal Logic
Volume XII, Number 2, April 1971

TRUTH-VALUE SEMANTICS FOR A LOGIC OF EXISTENCE

HUGUES LEBLANC

1. Introduction. Recall the opening moves in the interpretation of a first-order language $\mathfrak{L}$: (i) items, thought of as forming a domain D, are made the values of the (bound) individual variables of $\mathfrak{L}$, (ii) a member of D is assigned to each individual constant of $\mathfrak{L}$, (iii) a (possibly empty) set of members of D is assigned to each one-place predicate constant of $\mathfrak{L}$, (iv) a (possibly empty) set of pairs of members of D is assigned to each two-place predicate constant of $\mathfrak{L}$, and so on. Löwenheim's theorem of 1915 tells us that, as regards logical truth (i.e., truth under any interpretation whatever), logical falsehood (falsehood under any interpretation whatever), and the like, all but $\aleph_0$ members of any infinite domain D may be discounted.[1] A 1959 theorem of Beth's (implicit in results of Henkin, Hasenjaeger, and others) goes one better, and tells us that, as regards logical truth, logical falsehood, and the like, all but such members of any domain D as have been assigned to the individual constants of $\mathfrak{L}$ may be discounted, *provided* $\mathfrak{L}$ *has* $\aleph_0$ *individual constants*.[2] The latter result supplies the rationale for the "substitution" interpretation of the quantifiers, according to which a universal quantification $(\forall X)A$ of $\mathfrak{L}$ is true if every replacement of X everywhere in A by an individual constant of $\mathfrak{L}$ is true, an existential one $(\exists X)A$ true if some replacement of X everywhere in A by an individual constant of $\mathfrak{L}$ is true.[3]

Like considerations apply to the first-order quantificational calculus QC. Suppose that, as in many recent presentations of QC, two different runs of letters ($\aleph_0$ letters per run) serve as individual variables: one run—for which the appellation "individual variables" is often saved—occurring only bound in the well-formed formulas (wffs) of QC, and one run—called *individual parameters*—occurring only free in them. Löwenheim's theorem tells us that, as regards validity, contravalidity, and the like, any domain whose members are made the values of the individual variables of QC may be presumed of size $\aleph_0$; Beth's that all but such members as have been assigned to the individual parameters of QC may be discounted.

Beth's theorem issues into a fresh characterization of the valid wffs of QC—as a matter of fact, into a truth-value semantics for QC that runs largely like the ordinary semantics for the sentential calculus SC. Given

Received May 15, 1969

an assignment α of truth-values to the atomic wffs of QC, calculate the truth-value under α of a negation, conjunction, disjunction, and so on of QC as you would that of a negation, conjunction, disjunction, and so on of SC under an assignment of truth-values to the letters 'p', 'q', 'r', and so on; and certify a universal quantification $(\forall X)A$ (an existential quantification $(\exists X)A$) of QC true under α if every (some) replacement of X everywhere in A by an individual parameter of QC is true under α, otherwise certify the quantification false under α. This done, declare a wff of QC valid (contravalid) if it comes out true (false) under every assignment of truth-values to the atomic wffs of QC. It is readily shown that a wff of QC is valid (contravalid) in this truth-value sense of the word if and only if valid (contravalid) in the model-theoretic one of old.[4]

My main concern here will be to outfit Lambert's free logic FQ, rechristened for the occasion $QC!$, with a truth-value semantics.[5] Earlier free logics were, in effect, first-order languages with identity in which the requirement that individual constants designate something—plus in some cases the requirement that (bound) individual variables have values—was lifted. Lambert's $QC!$, by a slight but interesting contrast, is a first-order quantificational calculus (without identity) whose individual variables and parameters are free to have no values. Like so many of us, Lambert treats individual parameters (i.e., free individual variables) as placeholders for individual constants or singular terms, and hence has an excellent reason for allowing them to go without a value: not all singular terms designate something. He adduces the same reasons that Russell and others have for allowing individual variables to go without values. Beth's theorem suggests yet another one. Think indeed of Lambert's individual variables as ranging over just the values of his individual parameters. The former will have values so long as one or more of the latter has a value; otherwise, they won't.

As they stand, the truth conditions of paragraph three for $(\forall X)A$ and $(\exists X)A$ fail for $QC!$. Think of the individual parameter 'a' in the wff '$f(a)$' as standing for a singular term that does not designate, hence in effect think of 'a' as having no value, and think of 'f' as standing for 'does not exist'. Under this interpretation of 'a' and 'f', the existential quantification '$(\exists x)f(x)$' is clearly false even though replacement of 'x' everywhere in '$f(x)$' by 'a' is true. Our truth conditions are easily put to rights, however, as the predicate 'E!' from *Principia Mathematica*, *14, figures among the (primitive) signs of $QC!$. The wff '$E!a$', which is assigned a truth-value $\alpha(E!a)$ in any assignment of truth-values to the atomic wffs of $QC!$, is intended of course to be read: 'a exists', and for that reason I labelled Lambert's free logic *a logic of existence*. But 'a exists' is tantamount here to "'a' has a value,' or—to be more exact—'$\alpha(E!a) = T$' is tantamount to "'a' has a value under α,' the very qualification that is called for in the foregoing truth conditions for $(\forall X)A$ and $(\exists X)A$. Take indeed an existential quantification $(\exists X)A$ (a universal quantification $(\forall X)A$) of $QC!$ to be true under a truth-value assignment α if, *for some (every) individual parameter P of $QC!$ such that* $\alpha(E!P) = T$, replacement of X everywhere in A by P is true under α, and the above difficulty is met.

Note incidentally that, with the individual variables of $QC!$ taken to range over just the values of the individual parameters of $QC!$, at least one of the latter will be sure to have a value when the former have values. $QC!$ can be outfitted with a model-theoretical semantics under which no individual parameter of $QC!$ need ever have a value. The semantics in question may be more faithful to Lambert's intent, and it does make for a neater picture. I must save it, though, for another occasion.

After detailing in section 2 the key syntactical features of $QC!$ and some of the semantics that I intend here for Lambert's calculus, I establish in section 3 that every wff of $QC!$ that is provable in $QC!$ is valid in my sense, and *vice-versa*. Meyer and Lambert have already shown in [17] that $QC!$ is complete, but their characterization of the valid wffs of $QC!$ is of a model-theoretic (and rather complicated) sort, and the reasoning by which they arrive at their result is (to me, at any rate) somewhat roundabout. In section 4 I attend to the problem of implication in truth-value semantics, and show that $QC!$ is both strongly sound and strongly complete. Lastly, after axiomatizing in section 5 the valid wffs of $QC!$ that contain no 'E!', I study three further concepts of validity (one tantamount to provability in the sense of [14], another to provability in the sense of [13]).

2. The syntax and semantics of $QC!$. $QC!$ is to have as its (primitive) signs $\aleph_0$ sentence variables (among them 'p'); for each $m \geq 1$, $\aleph_0$ m-place predicate variables (among them the one-place 'f'); $\aleph_0$ individual variables (among them 'x' and 'y'); $\aleph_0$ individual parameters; the one-place predicate constant 'E!'; the two connectives '$\sim$' and '$\supset$'; the one quantifier letter '$\forall$'; the comma ','; and the two parentheses '(' and ')'. I shall understand by a *formula* of $QC!$ any finite sequence of primitive signs of $QC!$, and shall presume that the formulas of $QC!$ (hence, in particular, the individual variables of $QC!$, the individual parameters of $QC!$, and the formulas of $QC!$ to be soon acknowledged as well-formed) have been arranged in a definite order, to be known as the *alphabetical order* of the formulas of $QC!$. I shall say that a sign of $QC!$ is *foreign to a formula* of $QC!$ if it does not occur in the formula, and is *foreign to a set of formulas* of $QC!$ if it is foreign to every member of the set. I shall refer to the predicate variables of $QC!$ by means of the letter 'F'; to its individual variables by means of the letters 'X', 'Y', and 'Z'; to its individual parameters by means of the letters 'P', 'Q', and 'R'; to its individual signs (i.e., individual variables and individual parameters) by means of the letter 'I'; and to its formulas by means of the letters 'A', 'B', and 'C'. And I shall refer by means of '$(A)(I_1'/I_1)$' to the result of replacing I_1 everywhere in A by I_1'; by means of '$(A)(I_1',I_2'/I_1,I_2)$' to the result of replacing I_1 everywhere in A by I_1', and I_2 by I_2'; and so on.

By a *well-formed formula* (wff) of $QC!$ I shall understand any sentence variable of $QC!$, plus any formula of $QC!$ of one of the following five sorts: (i) $F(P_1, P_2, \ldots, P_m)$, where F is for some $m \geq 1$ an m-place predicate variable of $QC!$ and $P_1, P_2, \ldots, P_m$ are individual parameters of $QC!$, (ii) $E!P$, where P is an individual parameter of $QC!$, (iii) $\sim A$, where A is a wff of $QC!$, (iv) $(A \supset B)$, where A and B are wffs of $QC!$, and (v) $(\forall X)A(X/P)$, where A is a wff of $QC!$, X is an individual variable of $QC!$ that is foreign to A, and P is an individual parameter of $QC!$.[6] By an *atomic* wff of $QC!$ I shall understand any wff of $QC!$ that contains no occurrence of any one of '$\sim$', '$\supset$', and '$\forall$'; by an E!-*less* wff of $QC!$ any wff of $QC!$ that contains no occurrence of 'E!'; and by an *infinitely extendible* set of wffs of $QC!$ any set of wffs of $QC!$ to which $\aleph_0$ individual parameters of $QC!$ are foreign.[7] From now on I shall use the letters 'A', 'B', and 'C' to refer exclusively to wffs of $QC!$ and to results of replacing an individual parameter of $QC!$ in a wff of $QC!$ by an individual variable of $QC!$; and I shall use the letter 'S' to refer to sets of wffs of $QC!$. To abridge matters, I shall refer to '$\sim(p \supset p)$' by means of 'f', and to wffs of $QC!$ of the sorts $(\sim A \supset B)$, $\sim(A \supset \sim B)$, $\sim((A \supset B) \supset \sim(B \supset A))$, and $\sim(\forall X)\sim A$ by means of '$(A \vee B)$', '$(A \,\&\, B)$', '$(A \equiv B)$', and '$(\exists X)A$', respectively; and, when no ambiguity arises, I shall write '$A \supset B$', '$A \vee B$', '$A \equiv B$', '$A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n$', '$A(I_1'/I_1)$', '$A(I_1', I_2'/I_1, I_2)$', and so on, for '$(A \supset B)$', '$(A \vee B)$', '$(A \equiv B)$', '$(\ldots(A_1 \,\&\, A_2) \,\&\, \ldots) \,\&\, A_n$', '$(A)(I_1'/I_1)$', and '$(A)(I_1', I_2'/I_1, I_2)$', respectively.

A wff of $QC!$ will count as an *axiom* of $QC!$ if: (i) it is of one of the seven sorts

A1. $A \supset (B \supset A)$,
A2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$,
A3. $(\sim A \supset \sim B) \supset (B \supset A)$,
A4. $A \supset (\forall X)A$,
A5. $(\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$,
A6. $(\forall X)A \supset (\mathrm{E}!P \supset A(P/X))$,
A7. $(\forall X)\mathrm{E}!X$,

or (ii) it is of the sort $(\forall X)A(X/P)$, where A is an axiom of $QC!$.[8] B will be said to *follow from A and $A \supset B$ by Modus Ponens*. By a *derivation* in $QC!$ of A from S (for short, when S is the empty set ϕ, a *proof* of A in $QC!$) I shall understand any finite column of wffs of $QC!$ that closes with A and every one of whose entries belongs to S, is an axiom of $QC!$, or follows from two previous entries in the column by Modus Ponens. Lastly, I shall say that: (i) A is *derivable* from S in $QC!$ (for short, $S \vdash A$) if there is a derivation in $QC!$ of A from S, (ii) A is *provable* in $QC!$ (for short, $\vdash A$) if $\phi \vdash A$, (iii) S is *inconsistent* in $QC!$ if $S \vdash f$, and (iv) S is *consistent* in $QC!$ if S is not inconsistent in $QC!$.

By a *truth-value assignment* for $QC!$ I shall understand any function from the set of the atomic wffs of $QC!$ to $\{\mathrm{T}, \mathrm{F}\}$, where T is the truth-value "true" and F the truth-value "false". I shall say that A is *true* under a truth-value assignment α for $QC!$ if: (i) in the case that A is atomic, $\alpha(A) = \mathrm{T}$, (ii) in the case that A is (a negation) $\sim B$, B is not true under α, (iii) in the case that A is (a conditional) $B \supset C$, B is not true under α or C is, and (iv) in the case that A is (a quantification) $(\forall X)B$, $B(P/X)$ is true under α for every individual parameter P of $QC!$ such that $\alpha(\mathrm{E}!P) = \mathrm{T}$. I shall say that S is verifiable if there is a truth-value assignment for $QC!$ under which every member of S is true. And I shall say that A is *valid* if $\{\sim A\}$ is not verifiable, hence if A is true under every truth-value assignment for $QC!$

Note that if any existential quantification $(\exists X)A$ of $QC!$ is true under any truth-value assignment α for $QC!$, then—in view of clauses (ii) and (iv) above—$\alpha(\mathrm{E}!P)$ is sure to be T for at least one individual parameter P of $QC!$. Hence, in effect, if the individual variables of $QC!$ have values under any truth-value assignment for $QC!$, at least one individual parameter of $QC!$ is sure to have a value under α.

Attention will be paid in section 5 to truth-value assignments for $QC!$ of two special sorts: (i) those, to be called *null* assignments, under which $\mathrm{E}!P$ is false for every individual parameter P of $QC!$, and (ii) those, to be called *standard* assignments, under which $\mathrm{E}!P$ is true for every individual parameter P of $QC!$.

3. Soundness and completeness theorems for $QC!$. That a wff of $QC!$ is not provable in $QC!$ unless valid, nor valid unless provable in $QC!$, is established after four lemmas.

Lemma 1. *If A is an axiom of $QC!$, then A is valid.*
Proof: Let α be an arbitrary truth-value assignment for $QC!$. (1) Let A be an axiom of $QC!$ of one of the seven sorts A1-A7. It is easily ascertained that A is true under α. For suppose in particular that A is of the sort $B \supset (\forall X)B$. Since X is sure to be foreign to B, then $B(P/X)$ is the same as B. Hence, if B is true under α, so is $B(P/X)$ for every individual parameter P of $QC!$ such that $\alpha(\mathrm{E}!P) = \mathrm{T}$, and hence so is $(\forall X)B$. (2) Let A be as in (1), and P and Q be arbitrary individual parameters of $QC!$. It is easily ascertained that $A(Q/P)$ is of one of the seven sorts A1-A7, and hence in view of (1) is true under α. For suppose in particular that A is of the sort $(\sim B \supset \sim C) \supset (C \supset B)$, and hence that $A(Q/P)$ is of the sort $((\sim B \supset \sim C) \supset (C \supset B))(Q/P)$. Since $((\sim B \supset \sim C) \supset (C \supset B))(Q/P)$ is the same as $(\sim B \supset \sim C)(Q/P) \supset (C \supset B)(Q/P)$, $(\sim B \supset \sim C)(Q/P)$ the same as $(\sim B)(Q/P) \supset (\sim C)(Q/P)$, $(\sim B)(Q/P)$ the same as $\sim B(Q/P)$, and so on, then $A(Q/P)$ is the same as $(\sim B(Q/P) \supset \sim C(Q/P)) \supset (C(Q/P) \supset B(Q/P))$, and hence $A(Q/P)$ is of the sort A3. Or suppose that A is of the sort $(\forall X)B \supset (\mathrm{E}!R \supset B(R/X))$, and that R is the same as P. Since $((\forall X)B \supset (\mathrm{E}!P \supset B(P/X)))(Q/P)$ is the same as $((\forall X)B)(Q/P) \supset (\mathrm{E}!P \supset B(P/X))(Q/P)$, $((\forall X)B)(Q/P)$ the same as $(\forall X)B(Q/P)$, and so on, then $A(Q/P)$ is the same as $(\forall X)B(Q/P) \supset (\mathrm{E}!Q \supset (B(P/X))(Q/P))$. But $(B(P/X))(Q/P)$ is the same as $(B(Q/P))(Q/X)$. Hence $A(Q/P)$ is of the sort A6. (3) Let A be an axiom of $QC!$ of the sort $(\forall X)B(X/P)$, where B is of one of the seven sorts A1-A7. In view of (2) $B(Q/P)$ is true under α for every individual parameter Q of $QC!$. But $(B(X/P))(Q/X)$ is the same as $B(Q/P)$. Hence $(B(X/P))(Q/X)$ is true under α for every individual parameter Q of $QC!$ such that $\alpha(\mathrm{E}!Q) = \mathrm{T}$. Hence A is true under α. (4) Let A be an axiom of $QC!$ of the sort $(\forall X_1)(\forall X_2)\ldots(\forall X_n)B(X_1, X_2, \ldots, X_n/P_1, P_2, \ldots, P_n)$, where $n > 1$ and B is of one of the seven sorts A1-A7. Then by the same reasoning as in (3), but using (3) where (3) uses (2), A is true under α. (5) Let A be an axiom of $QC!$. Since A is sure to be as in (1), (3), or (4), A is sure to be true under α.

Lemma 2. *If S is inconsistent in $QC!$, then S is not verifiable.*
Proof: Suppose S is inconsistent in $QC!$, and the column made up of $A_1, A_2, \ldots, A_p$ constitutes a derivation in $QC!$ of f from S. It is easily

established by mathematical induction on i that $S \cup \{\sim A_i\}$ is not verifiable for any i from 1 to p. Suppose A_i is an axiom of $QC!$. Since in view of Lemma 1 A_i is true under every truth-value assignment for $QC!$, then $S \cup \{\sim A_i\}$ is not verifiable. Or suppose A_i follows from A_g and $A_g \supset A_i (= A_h)$ by Modus Ponens, and neither one of $S \cup \{\sim A_g\}$ and $S \cup \{\sim A_h\}$ is verifiable. Then $S \cup \{\sim A_i\}$ is not verifiable either. Hence $S \cup \{\sim f\}$ is not verifiable. But $\sim f$ is true under every truth-value assignment for $QC!$. Hence S is not verifiable.

Lemma 3. (a) *If A belongs to S or is an axiom of $QC!$, then $S \vdash A$.*
(b) *If $S \vdash A$, then there is a finite subset S' of S such that $S \vdash A$.*
(c) *If $S \vdash A$, then $S \cup S' \vdash A$.*
(d) $\vdash A \supset A$.
(e) *If $S \vdash A$ and $S \vdash A \supset B$, then $S \vdash B$.*
(f) *If $S \cup \{A\} \vdash B$, then $S \vdash A \supset B$.*
(g) *If $S \cup \{\sim A\}$ is inconsistent in $QC!$, then $S \vdash A$.*
(h) *If $S \vdash A$ and $S \vdash \sim A$ for any A, then S is inconsistent in $QC!$.*
(i) *If $S \vdash A$, then $S \cup \{\sim A\}$ is inconsistent in $QC!$.*
(j) *If $S \vdash (\forall X)A$ and $S \vdash (\forall X)(A \supset B)$, then $S \vdash (\forall X)B$.*
(k) *If $S \vdash A$, then $S \vdash (\forall X)A(X/P)$, so long as P is foreign to S.*
Proof: (a)-(c), (e), and (j) are immediate, and proof of (d) familiar from the literature.[9]

The proof of (f), suppose that the column made up of $C_1, C_2, \ldots, C_p$ constitutes a derivation in $QC!$ of $B (= C_p)$ from $S \cup \{A\}$. It is readily shown by mathematical induction on i that $S \vdash A \supset C_i$ for each i from 1 to p, and hence that $S \vdash A \supset C_p (= A \supset B)$. For suppose C_i belongs to $S \cup \{A\}$, but is other than A, or is an axiom of $QC!$. Then in view of (a) $S \vdash C_i$. But $C_i \supset (A \supset C_i)$ is an axiom of $QC!$. Hence in view of (a) and (e) $S \vdash A \supset C_i$. Or suppose C_i is A. Then in view of (d) and (c) $S \vdash A \supset C_i$. Or suppose C_i follows from C_g and $C_g \supset C_i (= C_h)$ by Modus Ponens, and suppose $S \vdash A \supset C_g$ and $S \vdash A \supset C_h$. Since $(A \supset C_h) \supset ((A \supset C_g) \supset (A \supset C_i))$ is an axiom of $QC!$, then in view of (a) and (e) $S \vdash (A \supset C_g) \supset (A \supset C_i)$, and hence in view again of (e) $S \vdash A \supset C_i$.

For proof of (g), suppose $S \cup \{\sim A\} \vdash f$, and hence in view of (f) $S \vdash \sim A \supset f$. Since $(\sim A \supset f) \supset ((p \supset p) \supset A)$ is an axiom of $QC!$, then in view of (a) and (e) $S \vdash (p \supset p) \supset A$. But in view of (d) and (c) $S \vdash p \supset p$. Hence in view of (e) $S \vdash A$.

For proof of (h), suppose $S \vdash \sim A$. Since $\sim A \supset (\sim f \supset \sim A)$ is an axiom of $QC!$, then in view of (a) and (e) $S \vdash \sim f \supset \sim A$. But $(\sim f \supset \sim A) \supset (A \supset f)$ is an axiom of $QC!$. Hence in view again of (a) and (e) $S \vdash A \supset f$. Hence, if $S \vdash A$, then in view of (e) S is inconsistent in $QC!$.

For proof of (i), suppose $S \vdash A$. Then in view of (c) $S \cup \{\sim A\} \vdash A$. But in view of (a) $S \cup \{\sim A\} \vdash \sim A$. Hence in view of (h) $S \cup \{\sim A\}$ is inconsistent in $QC!$.

For proof of (k), suppose the column made up of $B_1, B_2, \ldots, B_p$ constitutes a derivation in $QC!$ of A $(= B_p)$ from S, and let Y be X if X is foreign to $\{B_1, B_2, \ldots, B_{p-1}\}$ (with $(\forall X)A(X/P)$ presumed to be a wff of $QC!$, X is sure to be foreign to B_p), otherwise be the alphabetically earliest individual variable of $QC!$ that is foreign to $\{B_1, B_2, \ldots, B_p\}$. It can be shown by mathematical induction on i that $S \vdash (\forall Y)B_i(Y/P)$ for each i from 1 to p. Suppose B_i belongs to S. Since P is presumed to be foreign to S, then $B_i(Y/P)$ is the same as B_i. Hence in view of (a) $S \vdash B_i(Y/P)$. But $B_i(Y/P \supset (\forall Y)B_i(Y/P)(= B_i \supset (\forall Y)B_i)$ is an axiom of $QC!$. Hence in view of (a) and (e) $S \vdash (\forall Y)B_i(Y/P)$. Or suppose B_i is an axiom of $QC!$. Then so is $(\forall Y)B_i(Y/P)$, and hence in view of (a) $S \vdash (\forall Y)B_i(Y/P)$. Or suppose B_i follows from B_g and $B_g \supset B_i$ $(= B_h)$ by Modus Ponens, and suppose $S \vdash (\forall Y)B_g(Y/P)$ and $S \vdash (\forall Y)B_h(Y/P)$. Since $(\forall Y)B_h(Y/P)$ is the same as $(\forall Y)(B_g(Y/P) \supset B_i(Y/P))$, then in view of (j) $S \vdash (\forall Y)B_i(Y/P)$. Hence $S \vdash (\forall Y)A(Y/P)$, and hence $S \vdash (\forall X)A(X/P)$ if Y is X. Otherwise, since Y is foreign to A, $(A(Y/P))(P/Y)$ is the same as A, hence $(\forall Y)A(Y/P) \supset (E!P \supset A)$ is an axiom of $QC!$, and hence so is $(\forall X)((\forall Y)A(Y/P) \supset (E!X \supset A(X/P)))$. Hence in view of (a) $S \vdash (\forall X)((\forall Y)A(Y/P) \supset (E!X \supset A(X/P)))$. But $(\forall Y)A(Y/P) \supset (\forall X)(\forall Y)A(Y/P)$ is an axiom of $QC!$. Hence in view of (a) and (e) $S \vdash (\forall X)(\forall Y)A(Y/P)$. Hence in view of (j) $S \vdash (\forall X)(E!X \supset A(X/P))$. But $(\forall X)E!X$ is an axiom of $QC!$. Hence in view of (a) and (e) $S \vdash (\forall X)E!X$. Hence in view of (j) $S \vdash (\forall X)A(X/P)$.

Lemma 4. *Let S be an infinitely extendible set of wffs of QC!. If S is consistent in QC!, then S is verifiable.*
Proof: Let S be consistent in $QC!$.[10]

Part One: Take S_0 to be S; A_i being the alphabetically i-th wff at $QC!$, define S_i as follows for each i from 1 on:

(i) if $S_{i-1} \cup \{A_i\}$ is inconsistent in $QC!$, let S_i be $S_{i-1} \cup \{\sim A_i\}$,[11]

(ii) if $S_{i-1} \cup \{A_i\}$ is consistent in $QC!$ and A_i is not of the sort $\sim(\forall X)A$, let S_i be $S_{i-1} \cup \{A_i\}$, and

(iii) if $S_{i-1} \cup \{A_i\}$ is consistent in $QC!$ and A_i is a negated quantification $\sim(\forall X)A$, let S_i be $S_{i-1} \cup \{\sim(\forall X)A, E!P, \sim A(P/X)\}$, where P is the alphabetically earliest individual parameter of $QC!$ that is foreign to $S_{i-1} \cup \{\sim(\forall X)A\}$.

Let S_∞ be the union of S_0, S_1, S_2, ... It is easily ascertained that:

(1) For each $i \geq 0$, S_i is consistent in $QC!$,

(2) S_∞ is consistent in $QC!$,

(3) For every wff A of $QC!$, if A does not belong to S_∞, then $S_\infty \vdash \sim A$, and

(4) For every negated quantification $\sim(\forall X)A$ of $QC!$, if $S_\infty \vdash \sim(\forall X)A$, then there is an individual parameter P of $QC!$ such that $S_\infty \vdash E!P$ and $S_\infty \vdash \sim A(P/X)$.

For proof of (1), suppose S_i is as in (i), hence $S_{i-1} \cup \{A_i\}$ is inconsistent in $QC!$, and hence in view of Lemma 3(f) $S_{i-1} \vdash A \supset f$. If S_i is inconsistent in $QC!$, then in view of Lemma 3(g) $S_{i-1} \vdash A$, and hence in view of Lemma 3(e) S_{i-1} is inconsistent in $QC!$. Hence S_i is consistent in $QC!$ if S_{i-1} is. Next, suppose S_i is as in (ii). Then S_i is consistent in $QC!$, and hence is consistent in $QC!$ if S_{i-1} is. Lastly, suppose S_i is as in (iii) and is inconsistent in $QC!$. Then in view of Lemmas 3(g) and 3(f) $S_{i-1} \cup \{\sim(\forall X)A\} \vdash E!P \supset A(P/X)$. But P is presumed to be foreign to $S_{i-1} \cup \{\sim(\forall X)A\}$.

Hence in view of Lemma 3(k) $S_{i-1} \cup \{\sim(\forall X)A\} \vdash (\forall X)(\mathrm{E}!X \supset (A(P/X))(X/P))$. But, with P presumed to be foreign to $\sim(\forall X)A$ and hence to A, A is the same as $(A(P/X))(X/P)$. Hence $S_{i-1} \cup \{\sim(\forall X)A\} \vdash (\forall X)(\mathrm{E}!X \supset A)$. But $(\forall X)\mathrm{E}!X$ is an axiom of $QC!$. Hence in view of Lemma 3(a) $S_{i-1} \cup \{\sim(\forall X)A\} \vdash (\forall X)\mathrm{E}!X$. Hence in view of Lemma 3(j) $S_{i-1} \cup \{\sim(\forall X)A\} \vdash (\forall X)A$, and hence in view of Lemma 3(i) $S_{i-1} \cup \{\sim(\forall X)A\}$ $(= S_{i-1} \cup \{A_i\})$ is inconsistent in $QC!$, as against the hypothesis in (iii). Hence S_i is consistent in $QC!$, and hence is consistent in $QC!$ if S_{i-1} is. But S_0 is presumed to be consistent in $QC!$. Hence (1) by mathematical induction on i.

For proof of (2), suppose S_∞ is inconsistent in $QC!$. Then in view of Lemma 3(b) some finite subset of S_∞ is also inconsistent in $QC!$. But every finite subset of S_∞ is a subset of at least one of $S_0, S_1, S_2, \ldots$ Hence in view of Lemma 3(c) at least one of $S_0, S_1, S_2, \ldots$ is inconsistent in $QC!$, as against (1). Hence (2).

For proof of (3), suppose A does not belong to S_∞ and is the alphabetically i-th wff of $QC!$. Then S_i is $S_{i-1} \cup \{\sim A\}$, and hence in view of Lemma 3(a) $S_\infty \vdash \sim A$.

For proof of (4), suppose $S_\infty \vdash \sim(\forall X)A$ and $\sim(\forall X)A$ is the alphabetically i-th wff of $QC!$. Then $S_{i-1} \cup \{\sim(\forall X)A\}$ is consistent in $QC!$, for otherwise in view of Lemmas 3(f) and 3(c) $S_\infty \vdash (\forall X)A$, and hence in view of Lemma 3(h) S_∞ is inconsistent in $QC!$, as against (2). But, if $S_{i-1} \cup \{\sim(\forall X)A\}$ is consistent in $QC!$, then S_i is $S_{i-1} \cup \{\sim(\forall X)A, \mathrm{E}!P, \sim A(P/X)\}$, where P is as in (iii). Hence in view of Lemma 3(a) there is an individual parameter P of $QC!$ such that $S_\infty \vdash \mathrm{E}!P$ and $S_\infty \vdash \sim A(P/X)$.

Part Two: Let S' be the set S_∞ of *Part One*. It is easily ascertained with the aid of (2)-(4) in *Part One* that:

(5) For every negation $\sim A$ of $QC!$, $S' \vdash \sim A$ if and only if it is not the case that $S' \vdash A$,

(6) For every conditional $A \supset B$ of $QC!$, $S' \vdash A \supset B$ if and only if it is not the case that $S' \vdash A$ or it is the case that $S' \vdash B$.

(7) For every quantification $(\forall X)A$ of $QC!$, $S' \vdash (\forall X)A$ if and only if $S' \vdash A(P/X)$ for every individual parameter P of $QC!$ such that $S' \vdash \mathrm{E}!P$.

For proof of (5), suppose $S' \vdash \sim A$ and $S' \vdash A$. Then in view of Lemma 3(h) S' is inconsistent in $QC!$, as against (2). Suppose, on the other hand, it is not the case that $S' \vdash A$. Then in view of Lemma 3(a) A does not belong to S', and hence in view of (3) $S' \vdash \sim A$.

For proof of (6), suppose that $S' \vdash A \supset B$ and $S' \vdash A$. Then in view of Lemma 3(e) $S' \vdash B$. Next, suppose it is not the case that $S' \vdash A$. Then in view of (5) $S' \vdash \sim A$. But $\sim A \supset (\sim B \supset \sim A)$ is an axiom of $QC!$. Hence in view of Lemmas 3(a) and 3(e) $S' \vdash \sim B \supset \sim A$. But $(\sim B \supset \sim A) \supset (A \supset B)$ is an axiom of $QC!$. Hence in view of the same two lemmas $S' \vdash A \supset B$. Lastly, suppose that $S' \vdash B$. Since $B \supset (A \supset B)$ is an axiom of $QC!$, then in view of Lemmas 3(a) and 3(e) $S' \vdash A \supset B$.

For proof of (7), suppose $S' \vdash (\forall X)A$. Since $(\forall X)A \supset (\mathrm{E}!P \supset A(P/X))$ is an axiom of $QC!$, then in view of Lemmas 3(a) and 3(e) $S' \vdash \mathrm{E}!P \supset A(P/X)$, and hence in view again of Lemma 3(e) $S' \vdash A(P/X)$ for every individual parameter P of $QC!$ such that $S' \vdash E!P$. Suppose, on the other hand, it is not the case that $S' \vdash (\forall X)A$. Then in view of (5) $S' \vdash \sim (\forall X)A$, hence in view of (4) there is an individual parameter P of $QC!$ such that $S' \vdash E!P$ and $S' \vdash \sim A(P/X)$, and hence in view of (5) there is an individual parameter P of $QC!$ such that $S' \vdash E!P$ and it is not the case that $S' \vdash A(P/X)$.

Part Three: Let α be the truth-value assignment for $QC!$ such that, for every *atomic wff* A of $QC!$, $\alpha(A) = T$ if and only if $S' \vdash A$. Given (5)-(7) in *Part Two*, it is easily shown by mathematical induction on the number of occurrences of '$\sim$', '$\supset$', and '$\forall$' in an arbitrary *wff* A of $QC!$, that A is true under α if and only if $S' \vdash A$. For suppose in particular that A is a quantification $(\forall X)B$. By the hypothesis of the induction $B(P/X)$ is true under α if and only if $S' \vdash B(P/X)$, and this for every individual parameter P of $QC!$. Hence $B(P/X)$ is true under α for every individual parameter P of $QC!$ such that $\alpha(E!P) = T$ if and only if $S' \vdash B(P/X)$ for every individual parameter P of $QC!$ such that $\alpha(E!P) = T$. But, $E!P$ being atomic, $\alpha(E!P) = T$ if and only if $S' \vdash E!P$. Hence in view of (7) $(\forall X)B$ is true under α if and only if $S' \vdash (\forall X)B$. Consider then an arbitrary *member* A *of* S. Since A belongs to S', then in view of Lemma 3(a) $S' \vdash A$. Hence A is true under α. Hence every member of S is true under α.

Our soundness and completeness theorems for $QC!$ are now at hand.

Theorem 1. *If* $\vdash A$, *then* A *is valid.*
Proof: Suppose $\vdash A$. Then in view of Lemma 3(i) $\{\sim A\}$ is inconsistent in $QC!$, hence in view of Lemma 2 $\{\sim A\}$ is not verifiable, and hence A is valid.

Theorem 2. *If* A *is valid, then* $\vdash A$.
Proof: Suppose A is valid, and hence $\{\sim A\}$ is not verifiable. Since $\{\sim A\}$ is infinitely extendible, then in view of Lemma 4 $\{\sim A\}$ is inconsistent in $QC!$, and hence in view of Lemma 3(g) $\vdash A$.

4. That matter of implication. In 1919 Löwenheim's theorem was generalized by Skolem, who showed that, as regards the satisfiability of infinite sets of wffs (a set S of wffs of QC being satisfiable when S has a model), the implication of wffs by infinite sets of wffs (a wff A of QC being implied by a set S of wffs of QC when every model of S is one of $\{A\}$), and the like, any domain whose members are made the values of the individual variables of QC may be of size $\aleph_0$.[12] Unfortunately, Beth's theorem does not likewise generalize, as Dunn and Belnap, Thomason, and others have noticed.[13] The set $\{f(a_1), f(a_2), f(a_3), \ldots, \sim(\forall x)f(x)\}$, where '$a_1$', '$a_2$', '$a_3$', ... are presumed to be all the individual parameters of QC, is satisfiable in the model-theoretic sense, and hence the set $\{f(a_1), f(a_2), f(a_3), \ldots\}$ does not imply the wff '$(\forall x)f(x)$', since '$f(a_1)$', '$f(a_2)$', '$f(a_3)$', ..., '$\sim(\forall x)f(x)$' all come out true when 'a_1', 'a_2', 'a_3', ... are all assigned the same member of any domain D of size 2 and 'f' is assigned either one of the two subsets of D other than ϕ and D. Yet there is no assignment of truth-values to the atomic wffs of QC under which all the members of $\{f(a_1), f(a_2), f(a_3), \ldots, \sim(\forall x)f(x)\}$ are true. The same difficulty arises with the set $\{f(a_1), f(a_2), f(a_3), \ldots, E!a_1, E!a_2, E!a_3, \ldots, \sim(\forall x)f(x)\}$, one member of which is bound to be false under any assignment of truth-values to the atomic wffs of $QC!$.

With nameability by $\aleph_0$ individual parameters falling short of denumerability at this juncture, we cannot pronounce a wff A of $QC!$ implied by a set S of wffs of $QC!$ if A is true under any assignment of truth-values to the atomic wffs of $QC!$ under which the members of S are all true. One way of meeting the difficulty is to pronounce A implied by a finite set of wffs of $QC!$, say $\{B_1, B_2, \ldots, B_n\}$, if $B_1 \supset (B_2 \supset (\ldots \supset (B_n \supset A)\ldots))$ is valid, and pronounce it implied by an infinite set of wffs of $QC!$ if it is implied by some finite subset of the set. It is rather ad hoc. Another, favored by Belnap and Dunn in the case of QC, is to pronounce A implied by a set S of wffs of $QC!$ if, for every parametric extension of $QC!$ (i.e., every plying of $QC!$ with fresh individual parameters) and every assignment of truth-values to the atomic wffs of the extension, A is true under the assignment if every member of S is. I resort to a third, which—as I mentioned in [12]—I owe in part to Hintikka.[14]

Where S is a set of wffs of $QC!$, Σ_S is the set of all the individual parameters of $QC!$ occurring in members of S, and M is a one-to-one mapping of Σ_S into the set of all the individual parameters of $QC!$, (1) I shall take the *M-image* $M(A)$ of a member of A to be A itself when no individual parameter of $QC!$ occurs in A, otherwise to be $A(M(P_1), M(P_2), \ldots, M(P_m)/P_1, P_2, \ldots, P_m)$, where the $P_1, P_2, \ldots, P_m$ are in alphabetical order all the individual parameters of $QC!$ that occur in A, and (2) I shall take the *M-image* $M(S)$ of S to be S itself when S is empty, otherwise to be the set of the M-images of the various members of S. Where S and S' are sets of wffs of $QC!$, and Σ_S is as above, I shall say that S' is *isomorphic* to S if S' is $M(S)$ for some one-to-one mapping of Σ_S into the set of all the individual parameters of $QC!$. And, where S is a set of wffs of $QC!$ and A is a wff of $QC!$, I shall say that S *implies* A (or, to use Tarski's terminology, A is a semantic consequence of S) if no set of wffs of $QC!$ that is isomorphic to $S \cup \{\sim A\}$ is verifiable.

Proof that $S \vdash A$ if and only if S implies A in my sense of the word (and, hence, that $QC!$ is both strongly sound and strongly complete) calls for a few extra lemmas.[15]

Lemma 5. *If S is inconsistent in $QC!$, then so is every set of wffs of $QC!$ that is isomorphic to S.*

Proof: Let Σ_S consist of all the individual parameters of $QC!$ occurring in members of S; let M be an arbitrary one-to-one mapping of Σ_S into the set of all the individual parameters of $QC!$; and let the column made up of $A_1, A_2, \ldots, A_p$ constitute a derivation in $QC!$ of $f(=A_p)$ from S. It is easily verified that the column made up of $M(A_1), M(A_2), \ldots, M(A_p)$ constitutes a derivation in $QC!$ of $f(=M(A_p))$ from $M(S)$. For suppose A_i belongs to S. Then $M(A_i)$ belongs to $M(S)$. Or suppose A_i is an axiom of $QC!$. Then so is $M(A_i)$. Or suppose A_i follows from A_g and $A_g \supset A_i$ $(= A_h)$ by Modus Ponens. Since $M(A_h)$ is the same as $M(A_g) \supset M(A_i)$, then $M(A_i)$ follows from $M(A_g)$ and $M(A_h)$ by Modus Ponens. Hence $M(S)$ is inconsistent in $QC!$ if S is.

Lemma 6. *If S is inconsistent in $QC!$, then no set of wffs of $QC!$ that is isomorphic to S is verifiable.*

Proof by Lemma 2 and Lemma 5.

Lemma 7. *Let S be a set of wffs of QC!; let Σ_S be the set of all the individual parameters of QC! occurring in members of S; and let M be the one-to-one mapping of Σ_S into the set of all the individual parameters of QC! such that, where P is the alphabetically i-th individual parameter of QC!, M(P) is the alphabetically 2i-th individual parameter of QC!. If S is consistent in QC!, then so is M(S).*
Proof: Let the column made up of $A_1, A_2, \ldots, A_p$ constitute a derivation in *QC*! of $f(= A_p)$ from $M(S)$. It is easily verified that the column made up of $M'(A_1), M'(A_2), \ldots, M'(A_p)$, where M' is the inverse of M, constitutes a derivation in *QC*! of $f(= M'(A_p))$ from $M'(M(S))$. But $M'(M(S))$ is S. Hence $M(S)$ is consistent in *QC*! if S is.

Lemma 8. *If S is consistent in QC!, then at least one set of wffs of QC! that is isomorphic to S is verifiable.*
Proof: Suppose S is infinitely extendible. Since S is isomorphic to itself, then Lemma 8 by Lemma 4. Suppose on the other hand, S is not infinitely extendible, and let M be as in Lemma 7. $M(S)$ is infinitely extendible, and in view of Lemma 7 is consistent in *QC*! if S is. Hence Lemma 8 by Lemma 4.

Theorem 3. *$S \vdash A$ if and only if S implies A.*[16]
Proof like that of Theorems 1-2, but using Lemmas 6 and 8 in place of Lemmas 2 and 6.

Hence Corollary 1, in view of which validity is tantamount-as expected-to implication by the null set:

Corollary 1. *A is valid if and only if ϕ implies A.*
Proof by Theorems 1-3.

5. *The dispensability of* 'E!'. Since 'E!' does not figure in *QC*, the valid E!-less wffs of *QC*! are of special interest. I have recently discovered that the wffs in question are axiomatizable,[17] hence—in effect—that a first-order quantificational calculus whose individual variables and individual parameters need have no values (under the present understanding of things, whose individual variables need have no values, and only one of whose individual parameters need have a value when its individual variables have values), can be had without recourse to Lambert's 'E!' or the identity predicate of the free logics of [7], [13], and [15].

Count an E!-less wff of *QC*! as an *axiom** of *QC*! if it is of one of the five sorts A1-A5 in section 2, or of the sort

A6.* $(\forall Y)((\forall X)A \supset A(Y/X))$,

or of the sort $(\forall X)A(X/P)$, where A is an axiom* of *QC*!.[18] Count a finite column of E!-less wffs of *QC*! as a *proof** in *QC*! of an E!-less wff A of *QC*! if the column closes with A and every entry in the column is an axiom* of *QC*! or follows from previous entries in the column by Modus Ponens. And take an E!-less wff A of *QC*! to be *provable** in *QC*! if there is a proof* of A in *QC*!.

That A is provable* in *QC*! if valid, can be shown as follows. Suppose the column made up of the wffs $B_1, B_2, \ldots, B_p$ of $QC!$ qualifies as a proof of A in $QC!$; and suppose $P_1, P_2, \ldots, P_m$ $(m \geq 0)$ are in alphabetical order all the individual parameters of $QC!$, and $F_1, F_2, \ldots, F_n$ $(n \geq 0)$ are in alphabetical order all the predicate variable of $QC!$ that occur in anyone of $B_1, B_2, \ldots, B_p$. Next, in the case that $n > 0$, take $C_j(I,I')$ to be for each j from 1 to n and for any two individual signs I and I':

(i) when F_j is 1-place, the biconditional $F_j(I) \equiv F_j(I')$,
(ii) when F_j is 2-place, the conjunction of biconditionals
$F_j(I,I) \equiv F_j(I',I')$ & $F_j(I,I') \equiv F_j(I',I')$ & $F_j(I',I) \equiv F_j(I',I')$ &
$F_j(I,P_1) \equiv F_j(I',P_1)$ & $F_j(I,P_2) \equiv F_j(I',P_2)$ & ... & $F_j(I,P_m) \equiv F_j(I',P_m)$ &
$F_j(P_1,I) \equiv F_j(P_1,I')$ & $F_j(P_2,I) \equiv F_j(P_2,I')$ & ... & $F_j(P_m,I) \equiv F_j(P_m,I')$ &
$(\forall X)(F_j(I,X) \equiv F_j(I',X))$ & $(\forall X)(F_j(X,I) \equiv F_j(X,I'))$,
where X is the alphabetically earliest individual variable of $QC!$ that is foreign to $B_1, B_2, \ldots, B_p$, and distinct from I and I',[19] and so on.

Lastly, for each i from 1 to p, let B'_i be the result of replacing: (a) $E!P$ everywhere in B_i by $f(P) \supset f(P)$ if $n = 0$, otherwise by $(\exists Y)(C_1(Y,P)$ & $C_2(Y,P)$ & ... & $C_n(Y,P))$, where Y is the alphabetically earliest individual variable of $QC!$ that is foreign to $B_1, B_2, \ldots, B_p$, and (b) $E!X$ everywhere in B_i by $f(X) \supset f(X)$ if $n = 0$, otherwise by $(\exists Y)(C_1(Y,X)$ & $C_2(Y,X)$ & ...& $C_n(Y,X))$.[20] Then the column made up of $B'_1, B'_2, \ldots, B'_p$ qualifies as a proof* in A of $QC!$ or can be mechanically turned into one. But, if A is valid, then A is provable* in $QC!$. On the other hand, since every E!-less wff of the sort A6* is valid, it follows from Lemma 2 that A is valid if provable* in $QC!$. Hence an E!-less wff of $QC!$ is provable* in $QC!$ if and only if valid.

Recalling the various kinds of truth-value assignments for $QC!$ described at the close of section 2, pronounce a wff of $QC!$ *valid*$_1$ if true under every null and every standard truth-value assignment for $QC!$, *valid*$_2$ if true under every non-null truth-value assignment for $QC!$, and *valid*$_3$ if true under every standard truth-value assignment for $QC!$. With regards to validity$_1$ every individual parameter of $QC!$ has to have a value (and, hence, the individual variables of $QC!$ have to have values) if anyone does, the option considered in [14]. With regards to validity$_2$ the individual variables of $QC!$ have to have values (hence, under the present understanding of things, at least one individual parameter of $QC!$ has to have a value), the stand taken in [13]. And with regards to validity$_3$ every individual parameter of $QC!$ has to have a value (and, hence, the individual variables of $QC!$ have to have values), the norm outside free logic.

Addition of

A8$_1$. $(\exists x)E!x \supset E!P$

to the axiom schemata A1-A7 of section 2 will permit proof in $QC!$ of all the wffs of $QC!$ that are valid$_1$; addition of the antecedent

A8$_2$. $(\exists x)E!x$

of A8$_1$ permit proof in $QC!$ of all those that are valid$_2$; and addition of the consequent

A8$_3$. E!P

of A8$_1$ permit proof in QC! of all those that are valid$_3$.[21] Consider indeed the set S' and the truth-value assignment α in the proof of Lemma 4. If wffs of QC! of the sort A8$_1$ count as extra axioms of QC!, then in view of Lemma 3(a) $S' \vdash (\exists x)\mathrm{E}!x \supset \mathrm{E}!P$ for every individual parameter P of QC!, hence $(\exists x)\mathrm{E}!x \supset \mathrm{E}!P$ is true under α for every individual parameter P of QC!, and hence $\alpha(\mathrm{E}!P) = F$ for every individual parameter P of QC! or $\alpha(\mathrm{E}!P) = \mathrm{T}$ for every individual parameter P of QC!. If A8$_2$ counts as an extra axiom of QC!, then in view of Lemma 3(a) $S' \vdash (\exists x)\mathrm{E}!x$, hence '$(\exists x)\mathrm{E}!x$' is true under α, and hence $\alpha(\mathrm{E}!P) = \mathrm{T}$ for at least one individual parameter P of QC!. And if wffs of QC! of the sort A8$_3$ count as extra axioms of QC!, then in view of Lemma 3(a) $S' \vdash \mathrm{E}!P$ for every individual parameter P of QC!, and hence $\alpha(\mathrm{E}!P) = \mathrm{T}$ for every individual parameter P of QC!.

The E!-less wffs of QC! that are valid$_1$ are axiomatizable, as was shown in [14]. So are the valid$_2$ ones, as I recently discovered. And of course so are the valid$_3$ ones, which coincide with the wffs of QC! that are valid in the sense of section 1, paragraph three. Indeed, with the addition of

A9$^*_2$. $(\forall X)A \supset A$,

the axiom schemata A1-A5, A6* permit proof* of every valid$_2$ E!-less wff of QC!; with

A6$^*_1$. $(\forall X)A \supset ((\exists x)(f(x) \vee {\sim}f(x)) \supset A(P/X))$

substituting for A6*, they permit proof* of every valid$_1$ E!-less wff of QC!; and with

A6$^*_3$. $(\forall X)A \supset A(P/X)$

substituting for A6*, they permit proof* of every valid$_3$ E!-less wff of QC!. Note in connection with A9$^*_2$ that since '$(\forall y)((\forall x){\sim}(f(x) \vee {\sim}f(x)) \supset {\sim}(f(y) \supset {\sim}(f(y)))$' is an axiom* of QC!, then '$(\forall y)((f(y) \vee {\sim}f(y)) \supset (\exists x)(f(x) \vee {\sim}f(x)))$' is sure to be provable* in QC!, hence so is '$(\forall y)(f(y) \vee {\sim}f(y)) \supset (\forall y)(\exists x)(f(x) \vee {\sim}f(x))$', and hence so is '$(\forall y)(\exists x)(f(x) \vee {\sim}f(x))$'. Hence, if '$(\forall y)(\exists x)(f(x) \vee {\sim}f(x)) \supset (\exists x)(f(x) \vee {\sim}f(x))$' also counts as an axiom* of QC!, then '$(\exists x)(f(x) \vee {\sim}f(x))$'—a common rendering of "Something exists"—is provable* in QC!.[22]

NOTES

1. See [16]. Skolem's generalization of Löwenheim's theorem is discussed in section 4.

2. See [1], Section 89. Beth's theorem (not to be confused with his more celebrated theorem on definability) was anticipated in [4] and [5].

3. The interpretation, which goes back to Russell, has recently been championed by Ruth Barcan Marcus and others.

4. For further details, see [1], [10], [12], and [20]. Hintikka's model-set semantics in [6] and later papers is but another brand of what I call here *truth-value semantics*.

5. See [9], [11], and [17]. The version of $QC!$ that I employ here comes from [17], and is credited by Lambert to Meyer. The one in [9], which uses '∃' rather than '∀' as primitive quantifier letter, has very attractive axioms, but proves less handy to work with.

6. By requiring in (v) that X be foreign to A, I forego wffs of the sort $(\forall X)A$ with a component of the sort $(\forall X)B$, but avoid difficulties that would otherwise beset the substitution of parameters for variables. Essentially the same point is made in [19], p. 15, footnote 1.

7. I owe the phrase "infinitely extendible" to my friend R. K. Meyer.

8. So far as I know, the trick of counting $(\forall X)A\,(X/P)$ as an axiom if A is an axiom, and thereby dispensing with Generalization as a rule of inference, stems from [3]. It clears up all the difficulties that in some formulations of QC have beset, in others have blocked, proof of Lemma 3(c) in Section 3 (see [18] on this matter). Note as regards A4 that with $A \supset (\forall X)A$ presumed to be a wff of $QC!$, X is sure to be foreign to A.

9. When Generalization serves as a rule of inference, proof of (c)—as noted in footnote 8—can be tricky, and proof of (f) has an extra case (which can be tricky too).

10. The following proof borrows from [5], [12], [15], and [22]. The simplification brought in [22], p. 96, to the argument of [5] is credited by Smullyan to Henkin.

11. Or, equivalently, let S_i be S_{i-1}. The course adopted in the text makes for a shorter proof of (3) below, the one reported here for a shorter proof of (1).

12. See [21].

13. See [2] and [10].

14. For further details on this whole matter, see [10] and [12].

15. In what follows I borrow in part from [12].

16. One half of the standard Compactness Theorem clearly holds for $QC!$, namely: If a set S of wffs is verifiable, so is every finite subset of S. The other half fails, since every finite subset of $\{f(a_1), f(a_2), f(a_3), \ldots, E!a_1, E!a_2, E!a_3, \ldots, \sim(\forall x)f(x)\}$ is verifiable, but the set itself is not. The following weakening of that half holds, however: If every finite subset of S is verifiable, then so is some set of wffs of $QC!$ that is isomorphic to S. Suppose indeed that no set of wffs of $QC!$ that is isomorphic to S is verifiable. Then in view of Lemma 8 S is inconsistent in $QC!$, hence in view of Lemma 3(b) some finite subset of S is inconsistent in $QC!$, and hence in view of Lemma 2 some finite subset of S is not verifiable. The following also holds in view of Theorem 3 and Lemmas 3(b) and 3(c): S implies a wff A of $QC!$ if and only if some finite subset of S implies A.

17. The discovery came as a surprise to me. My more perspicacious friends Meyer and van Fraassen were sure all along, though, that the wffs in question could be axiomatized.

18. The axiom schema $(\forall Y)(\exists Y)(A \equiv A(Y/X))$ can do duty for A6*. Note indeed that A1-A5 permit proof of the following analogue of A6: $(\forall X)A \supset ((\exists X)(A \equiv A(P/X)) \supset A(P/X))$, and hence of $(\forall Y)(\exists X)(A \equiv A(Y/X)) \supset (\forall Y)((\forall X)A \supset A(Y/X))$. A5, A6*, and axiom schemata to the same effect as A1-A3 already appear in [8].

19. The need for the two conjuncts $(\forall X)(F_j(I,X) \equiv F_j(I',X))$ and $(\forall X)(F_j(X,I) \equiv F_j(X,I'))$ was brought to my attention by Meyer.

20. The transformation of B_i into $B_i^!$ was arrived at by first thinking of, say, $E!P$ as short for $(\exists Y)(Y = P)$, and then hunting for a suitable paraphrase of $Y = P$.

21. With $A8_3$ at hand (and $(\forall X)A(X/P)$ counting as an axiom when A does), A7 is of course redundant.

22. My thanks go to Lambert, Meyer, van Fraassen, and John T. Kearns who read an earlier version of my paper. The results proved here were announced at a Symposium on *The Logic of Existence* held at Indiana University in the spring of 1969.

REFERENCES

[1] Beth, E. W., *The Foundations of Mathematics*, North Holland Publishing Co., Amsterdam (1959).

[2] Dunn, M., and N. D. Belnap, Jr., "The substitution interpretation of the quantifiers," *Noûs*, vol. 2 (1968), pp. 177-185.

[3] Fitch, F. B., "Intuitionistic modal logic with quantifiers," *Portugaliae Mathematica*, vol. 7 (1948), pp. 113-118.

[4] Hasenjaeger, G., "Eine Bemerkung zu Henkin's Beweis für die Vollständigkeit des Prädikatenkalküls der ersten Stufe," *The Journal of Symbolic Logic*, vol. 18 (1953), pp. 42-48.

[5] Henkin, L., "The completeness of the first-order functional calculus," *The Journal of Symbolic Logic*, vol. 14 (1949), pp. 159-166.

[6] Hintikka, K. J. J., *Two Papers on Symbolic Logic*, in *Acta Philosophica Fennica*, fasc. 8, Societas Philosophica, Helsinki (1955).

[7] Hintikka, K. J. J., "Existential presuppositions and existential commitments," *The Journal of Philosophy*, vol. 56 (1959), pp. 125-137.

[8] Lambert, K., "Existential import revisited," *Notre Dame Journal of Formal Logic*, vol. 4 (1963), pp. 288-292.

[9] Lambert, K., "Free logic and the concept of existence," *Notre Dame Journal of Formal Logic*, vol. 8 (1967), pp. 133-144.

[10] Leblanc, H., "A simplified account of validity and implication for quantificational logic," *The Journal of Symbolic Logic*, vol. 33 (1968), pp. 231-235.

[11] Leblanc, H., "On Meyer and Lambert's quantificational calculus FQ," *The Journal of Symbolic Logic*, vol. 33 (1968), pp. 275-280.

[12] Leblanc, H., "A simplified strong completeness proof for QC=," *Akten des XIV. Internationalen Kongresses für Philosophie, Wien: 2.-9. September 1968*, Verlag Herder, Wien (1969).

[13] Leblanc, H., and T. Hailperin, "Nondesignating singular terms," *The Philosophical Review*, vol. 68 (1959), pp. 129-136.

[14] Leblanc, H., and R. K. Meyer, "Open formulas and the empty domain," forthcoming in *Archiv für Mathematische Logik und Grundlagenforschung.*

[15] Leblanc, H., and R. H. Thomason, "Completeness theorems for some presupposition-free logics," *Fundamenta Mathematicae*, vol. 62 (1968), pp. 125-164.

[16] Löwenheim, L., "Über Möglichkeiten im Relativkalkül," *Mathematische Annalen*, vol. 76 (1915), pp. 447-470.

[17] Meyer, R. K., and K. Lambert, "Universally free logic and standard quantification theory," *The Journal of Symbolic Logic*, vol. 33 (1968), pp. 8-26.

[18] Montague, R., and L. Henkin, "On the definition of 'formal deduction'," *The Journal of Symbolic Logic*, vol. 21 (1956), pp. 129-136.

[19] Prawitz, D., *Natural Deduction, A Proof-Theoretical Study*, Almqvist & Wiksell, Stockholm (1965).

[20] Schütte, K., *Lecture Notes in Mathematical Logic*, The Pennsylvania State University.

[21] Skolem, T., "Logisch- kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen," *Skrifter utgit av Videnskappsselskapet i Kristiana, I, Mathematisk-naturvidenskalbelig klasse 1919*, no. 3 (1920), pp. 1-36.

[22] Smullyan, A., *First-Order Logic*, in *Ergebnisse der Mathematik und Ihrer Grenzgebiete*, vol. 43, Springer-Verlag New York Inc. (1968).

Temple University
Philadelphia, Pennsylvania