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CONVENTIONALIST AND CONTINGENCY-ORIENTED MODAL LOGICS

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No modal logic so far presented adequately represents radical conventionalism. Yet conventionalism about modalities is a very pervasive doctrine. In this paper we make a start on filling this serious gap in the literature.

Radical conventionalism is distinguished by the thesis:

(**R**). *All assertions of modalities are contingent.*

Consequently according to the radical conventionalist, $\nabla\Box p$, $\nabla\Diamond p$ and $\nabla\nabla p$ are true, that is statements of necessity, possibility and contingency are themselves contingent. Strawson has recommended adoption of "the convention . . . that intensional statements are contingent".[1] For a large class of languages, including the systems considered below, Strawson's thesis that intensional statements are contingent is tantamount to (**R**).

Radical conventionalism is inconsistent with *all* Lewis modal logics, as we shall show. It does not follow, however, that radical conventionalism is, as has often been assumed, inconsistent. One of our main aims is to exhibit a class of consistent modal logics in which thesis (**R**) is satisfied. This is a major step towards showing that radical conventionalism is a consistent doctrine.

We distinguish radical conventionalism from two other main positions regarding modality which have also been called *conventionalisms*. For radical conventionalism has frequently been confused with these other doctrines, to the detriment of each position since each pair of positions entails mutually inconsistent principles. The positions inconsistent with radical conventionalism are these:

(1) *Primitive conventionalism* according to which there are really no necessary or—what is equivalent—no noncontingent propositions. Those propositions which are taken to be necessary are really a special kind of contingent propositions, i.e., generalisations about linguistic usage of conventions. J. S. Mill is an early advocate of this view.[2] Developed versions of primitive conventionalism, designed to counter obvious objec-

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tions such as that there *are* noncontingent propositions, commonly reappear as reductionist programmes according to which all propositions normally classed as necessary reduce under analysis or translation to contingent propositions.

Taken literally, primitive conventionalisms assert $(p)\sim\Box p$ or $(p)\nabla p$, and so entail the postulate of universal possibility $(p)\Diamond p$. These are of course inconsistent with all those modal systems which assert $\Box q$, for some q. They are however consistent with a number of modal logics developed by Lemmon (in [10]), namely with those Lemmon systems contained in system E (of [11, p. 214]). E coincides with *the Falsum system*, the system which *appears to encapsulate the sentential modal logic of primitive conventionalism.* The falsum system, with postulate set consisting of some independent axiom set for sentential logic, uniform substitution, material detachment, and the axiom

$A0 \quad \sim\Box p \qquad (\text{or } A0' \, \nabla p)$

is consistent and complete over the usual two-valued matrices for sentential logic together with the matrix:

$$\begin{array}{c|cc} p & T & F \\ \hline \Box p & F & F \end{array}.$$

The postulates of this Falsum system are independent and the system does not reduce modality, since $p \supset \Box p$ is inconsistent with it.

(2) *Linguistic theories of logical necessity,* according to which logically necessary propositions are propositions true in virtue of, or as a consequence of linguistic data of some sort, such as data about the senses of expressions. Carnap's approach [2, p. 10] is typical of these theories:

> A sentence $\mathfrak{S}_i$ is **L-true** in a semantical system S if and only if $\mathfrak{S}_i$ is true in S in such a way that its truth can be established on the basis of the semantical rules of the S alone, without any reference to (extra-linguistic) facts.

The modal logics associated with linguistic theories are inconsistent with $\sim\Box p$ and $\nabla\Box p$, but are compatible with S5 modal structures.

Radical conventionalism is inconsistent with primitive conventionalism. For according to the radical conventionalist doctrine, which we now outline, there are necessary sentences, such as $\sim(p \,\&\, \sim p)$, and impossible sentences do not reduce to contingent sentences. But *that* they are necessary or impossible is itself contingent; and this contingency is ascribed to (linguistic) conventions of some kind.

We avoid going into all the details that have to be filled out satisfactorily to make radical conventionalism a cogent doctrine, namely an account of exactly how the necessity that necessary statements have may be ascribed to conventions of whatever kind, and convincing arguments for the positions. For these details, even if they could be satisfactorily elaborated, are not needed in investigating the sentential modal logics associated with radical conventionalisms.

What is important, however, for the choice of a system-notation is that radical conventionalism operates at the linguistic level. For it is sentences and their parts, not propositions, that are affected by linguistic conventions. Radical conventionalism involves nominalism at least in this respect: that if modalities were assigned to standard propositions, rather than sentences say, the case for asserting that $\Box p$ is contingent, rather than necessary when true, would be undermined. Consider, to illustrate, $p \supset p$ where it is supposed both $\Box(p \supset p)$ and $\nabla\Box(p \supset p)$. Given the usual sentence/proposition distinctions any variation in the necessity of $p \supset p$, designed to show $\nabla\Box(p \supset p)$, would be attributed to the view that the sentence '$p \supset p$' expresses a different proposition, the modal value of which is invariant. In this way it would be guaranteed that $\Box(p \supset p)$ is necessary, i.e., the S4 thesis $\Box\Box(p \supset p)$ would be ensured. Thus a conventionalist may still adopt the S4 postulates provided he is prepared to say that different propositions are expressed when conventions are changed. Radical conventionalism would be transformed into a linguistic theory. Part of the contrast then, between radical conventionalism and the linguistic theory is that these theories adopt different identity criteria for propositions, that the theories are coupled with different theories of propositions.

To vindicate (**R**) along conventionalist lines, the theory of propositions or sentences to which modal values are assigned, which carry modalities, has to be appropriately specified so that certain modal values may vary under change of conventions. Moreover the conventionalist has always to connect with his propositions associated sentences, for as has already been remarked it is these that are affected by linguistic conventions. (Here the familiar difficulty over the modal values of unexpressed propositions arises). The simplest solution, and one which follows at once given the Occamist or nominalist assumptions conventionalists commonly do make, is to eliminate propositions altogether, and to take sentences as the carriers of modality. Conventionalism and nominalism, though logically distinct, are sympathetic positions, and they fit together easily in empiricist and Occamist schemes. Indeed conventionalists usually *assume* that sentences carry modalities, and argue on the basis of this to (**R**).

For these reasons we assume that sentence-variables 'p', 'q', 'r' etc., not propositional variables, carry modalities. There is room then for the conventionalist to argue that whilst a sentence may be necessary—or contingent—because of suitable linguistic conventions, its being necessary—or contingent—is always a matter that is contingent on contingent linguistic conventions.[4] Thus when '∇', '$\Box$' and '$\Diamond$' are considered as sentence-forming operators on sentences, the sentences '$\nabla\Box p$', '$\nabla\nabla p$' and '$\nabla\Diamond p$' are *prima facie* true. We find that these theses are consistent with substantial fragments of systems up to and including S3, and that S3 augmented by $\nabla\nabla p$, though not by $\nabla\Box p$, is consistent.

Since $\nabla\nabla p$ entails $\Diamond\Diamond p$, radical conventionalist systems entail the principle $\Diamond\Diamond p$. Modal systems containing postulates asserting the universal possibility of possibility were first suggested by Lewis (in [12, p. 497]). The motivation which led Lewis to consider his *C13*, $\Diamond\Diamond p$, was this: that whilst it would be allowed that there are necessary propositions it could well be claimed, in opposition to the S4 thesis that these propositions are necessarily necessary, that there are no necessarily necessary propositions, that is that $\sim\Box\Box p$, and so $\Diamond\Diamond\sim p$, is true for every proposition p. But on what sort of grounds could this be claimed? Unfortunately Lewis does not give any reason why someone might want to assert $(p) \sim\Box\Box p$.

We offer two broad sets of reasons why a person may want to claim that no propositions are necessarily necessary: (a) that already mentioned, that the person subscribes to a conventionalistic theory of logical necessity, and (b) that the person adheres to a psychologistic theory of (logical) necessity. Typical psychologistic theories are provided by British empiricists such as Hume (according to Pap's exposition in [18]) and Bentham (see [17]), and by Kant (see Pap [18]). But both these sorts of theories lead, even at the sentential level, to systems distinct from Lewis's system S6, and beyond systems subsequently discussed in the literature.

Just as it is worth considering the stand taken by a person who rejects the characteristic S4 thesis outright, so it is interesting to investigate the effect of the outright rejection of S5. Now we know that $\sim\nabla\nabla p$ added even to S1 is sufficient to generate S5; consequently this axiom typifies S5 better than does the usual S5 postulate $\Diamond p \supset \Box\Diamond p$ since this postulate when added to S1 does not lead to S5 at all. (For details see Montgomery and Routley [15] and [16]). To mimic the Lewis case for investigating the addition of $\Diamond\Diamond p$ one would say: It might well be claimed, in opposition to the S5 thesis, that all statements are contingently contingent. Universal rejection of the characteristic S5 thesis leads too to a new class of consistent modal logics, which include universal contingency of contingency postulates. In particular, addition of $\nabla\nabla p$ to S2 and S3 leads to contingency-oriented systems which properly include Lewis systems S6 and S7 respectively.

Since Lewis's time systems S6, S7, S8 and S7.5 ([1], [7], [9], [23]) have been briefly discussed in the literature. Completeness results for the first three of these systems have been claimed by Kripke, but the semantical models sketched cast no real light on the problems of interpreting the systems. Some more illuminating suggestions as to how to interpret systems like S6 and its extensions are worth following out, at least for philosophical and historical reasons. These interpretations fall, as before, into two classes: (a) linguistic and conventionalistic interpretations, and (b) psychologistic interpretations of modality.

An interesting proposal as to how to interpret S6 along linguistic lines has been made by J. L. Mackie (in [13])[5]. Mackie has suggested interpreting '$\Box p$' as 'qu$(\sim p)$ is forbidden by linguistic rules', or as 'qu(p) is demanded by linguistic rules'. On this interpretation $\sim\Box\Box p$, for according to Mackie, it is false that the sentence 'sentence qu$(\sim p)$ is forbidden by linguistic rules' is demanded by linguistic rules. At the same time, a system for this linguistic interpretation should not contain the S8 thesis $\Box\Diamond\Diamond p$, for it is a simple fact, not a requirement of language, that it is not necessary that language requires what it does. Thus if we confine attention to Lewis systems then we are restricted to S6 or S7. Mackie presents these points for deciding in favour of S6: "In attempting to provide a logical system for this interpretation we are proposing that the requirements of language are at least consistent. We may also assume that they are consequential in the sense that if both $p \supset q$ and p are linguistically required, so is q, so that we have the thesis $\Box(p \supset q) . \Box p \supset \Box q$ and its equivalent $\Box(p \supset q) \supset . \Box p \supset \Box q$. But this does not entail the distinctive thesis of S3 and S7, $\Box[\Box(p \supset q) \supset . \Box(\Box p \supset \Box q)]$, and I can see no reason why this should hold for this interpretation.'' Mackie concludes that *of* the Lewis systems S6 is the most appropriate. However the interpretation does not verify all the postulates, and it verifies various principles not included in S6. As to the first point, the interpretation, though it verifies $\Box p \supset p$, does not validate the S6 law $\Box(\Box p \supset p)$. According to Mackie ''we assume that language is at least so far adequate to the world that nothing forbidden by linguistic rules would be true;'' and such a requirement on linguistic rules does suffice for the vindication of $\Box p \supset p$. But it is not always forbidden by the linguistic rules that linguistic rules forbid the exclusion of truth, a fact well-known to totalitarian regimes which can, and do, fix linguistic rules to preclude (linguistic) statement of certain truths. Thus $\Box(\Box p \supset p)$ is definitely rejected under this linguistic interpretation, unless some attempt is made to introduce special metalinguistic rules.

To arrive at a system which satisfies Mackie's linguistic interpretation we must retreat at least to $S6^e$, a system obtained from S6 by weakening $\Box p \dashv p$ to $\Box p \supset p$. Since Mackie's consequentiality requirement is likewise not necessitated by linguistic rules, without yet another suitable metalinguistic rule, we are forced to retreat further—perhaps back to E6, a system got from Lemmon's E2 by adding the axiom $\Diamond\Diamond p$. A linguistic interpretation can be given to E6. To be formally specific let us symbolise 'qu(p) is forbidden by linguistic rules' as 'Fp'. We require that 'F' satisfy the following conditions—formulated using sentential logic as the underlying logic—

(*1*) $Fp \supset \sim p$	Mackie's requirement that what is forbidden by the linguistic rules (of L) is false.
(*2*) $F(p \& \sim q) \ \& \ Fq \supset . \ Fp$.	Mackie's consequentiality requirement.
(*3*) $A \supset B \rightarrow FB \supset FA$	A related consistency rule.
(*4*) $\sim F \sim Fp$.	A conventionalist requirement.

These requirements on 'F' are consistent and reasonable. If we require further that 'F' satisfies *just* these conditions at the sentential level, then it is easy to show that E6 is consistent and complete under Mackie's linguistic interpretation. This completeness result does not provide a decision procedure for E6; but since E6 has the finite model property (as we shall show elsewhere) it is decidable.

That the linguistic account so construed is compatible with primitive conventionalism is a shortcoming. For the account no longer asserts any sentence as necessary. To surmount this problem the following obvious requirement on 'F' should be added:

(*5*) The negations of theses of sentential logic are forbidden by the linguistic rules (of L).

However in the presence of rule (*3*) this leads to inconsistency. We explain the point with respect to E6. If we simply add to E6 the thesis $\Box(p \supset p)$, without curtailing the rules of E6 to apply to theses of E6 *only*, inconsistency results as follows:

(a) The rule of necessitation is derivable thus:

$$\begin{aligned} A &\rightarrow p \supset p \supset . A \\ &\rightarrow \Box(p \supset p) \supset . \Box A \\ &\rightarrow \Box A \end{aligned}$$

(b) Hence $\Box\Box(p \supset p)$, contradicting $\Diamond\Diamond \sim(p \supset p)$.

Thus in the presence of (*5*), rule (*3*) should be abandoned or replaced. If (*3*) is simply dropped, then an interpretation for the system (S0.5 + $\Diamond\Diamond p$) is provided. But a readily acceptable replacement for (*3*) is:

(*3′*) $F(A \& \sim B) \rightarrow F(FB \& \sim FA)$

that is, if A without B is linguistically forbidden then we can infer that B's being forbidden without A's being forbidden is also forbidden. The resulting linguistic interpretation provides a complete interpretation for yet another uninvestigated modal logic, namely one obtained from Lemmon's P2 (of [10]) by restricting necessitation just to tautologies and adding the axiom $\Diamond\Diamond p$.

By demanding for the interpretation a language L_s which imposes special metalinguistic rules—to the effect, first, that it is linguistically forbidden in L_s that what is forbidden be true and, second, that if it can be shown that A linguistically necessitates B in L_s then B's being forbidden linguistically necessitates in L_s A's being forbidden—an interpretation for S6 results. For, let 'Fp' read 'sentence qu(p) is forbidden by the linguistic rules of language L_s'. Then in place of conditions (*1*) and (*3*) on 'F', under this interpretation with its metalinguistic rules, we demand the stronger conditions, (*3′*) and:

(*1′*) $F(Fp \& p)$

The oddness of the metalinguistic rules of L_s should not be underemphasised; especially odd is the rule which forbids that anything forbidden is true. For this rule effectively prevents the desired and intended conventionalistic interpretations. If, in defiance of this, we require that 'F' satisfies just the conditions (*1′*), (*2*), (*3′*), (*4*) and (*5*) then S6 is consistent and complete with respect to this interpretation.

It is quite unsatisfactory to require that 'F' satisfies just this last set of conditions or just (*1-5*). For on the understood reading of 'F', 'F' satisfies further conditions; for instance not just does (*4*) hold under the understood reading, so does $\sim FFp$. If the linguistic interpretation correctly admits $\Diamond\Diamond p$ there would seem to be no grounds for excluding $\nabla\nabla p$. But addition of $\nabla\nabla p$ would properly extend E6 and S6 to new systems QE2 and Q2. Nor are these new systems complete with respect to pre-analytic linguistic interpretations. For the conventionalist thesis $\nabla\Box p$, and its equivalent $\nabla \Diamond p$, are not theorems of QE2 or of Q2. Indeed $\nabla \Box p$ is not even consistent with Q2 or S6; $\nabla \Box p$ cannot be consistently added even to S1. Yet it is plain that Mackie should adopt $\nabla \Box p$; for according to him "it is clear that with this interpretation 'p is necessary' is itself not a necessary statement, but a statement of a contingent linguistic fact".

One trouble with a metalinguistic rule ensuring $\Box(\Box p \supset p)$ is that it is inconsistent, on linguistically incontestible assumptions, with the linguistic thesis that possibility statements are contingent linguistic statements, that is with $\nabla \Diamond p$—as follows:

1.	$\Box(p \supset \Diamond p)$	contraposing $\Box(\Box p \supset p)$
2.	$\Box p \supset \Box\Diamond p$	distributing 1.
3.	$\Box(p \supset p)$	
4.	$\Box\Diamond(p \supset p)$	from 2, 3.
5.	$\sim\Diamond(p \supset p)$	from 4.

Alternatively the rule form $\Box(A \supset B), \Box A \rightarrow \Box B$, is sufficient.

$\Box p \prec p$ and $p \prec \Diamond p$ are indefensible under linguistic interpretations of modality which concede that there are (or may be) necessary truths: although a sentence may in fact be necessary, *this* is contingent according to linguistic interpretations; so the consequent of 2 is always false. Thus if there is any true sentence of the form $\Box A$, $\Box p \supset \Box\Diamond p$ fails. Consequently, since $\Box(p \supset q) \supset . \Box p \supset \Box q$ appears valid under linguistic and conventionalistic interpretations, $\Box p \prec p$ and $p \prec \Diamond p$ are linguistically indefensible.

That these laws come out as indefensible on this interpretation is a telling objection to such linguistic interpretations of modality. For on these accounts necessity is not sufficient to necessarily, or logically, guarantee truth; necessary truth is not a kind of truth; what is necessary might not even be true! Worse, on conventionalist theses to which these linguistic interpretations lead, it is just an accident that the actual world isn't an impossible one! For given Wittgenstein's dictum that the world is everything that is the case, then the conjunction of all true statements, W (the world), is as a matter of fact possible since $W \supset \Diamond W$ and W. But since $(p)\ \Box p$, $\Box\Diamond W$; hence $\Diamond\sim\Diamond W$, i.e. it is possible that the world is impossible. The same formal anomolies occur under psychologistic interpretations.

Some of the systems we present are open to an alternative psychologistic interpretation. Under such interpretations modal assignments are interpreted in terms of some psychological notion such as actual conceivability, indubitability or imaginability, e.g. '$\Diamond p$' is interpreted as 'p is imaginable'. *Under these interpretations the variables of conventionalistic* systems should be reinterpreted as ranging over propositions or contexts.

On the face of it imaginability, indubitability and such like vary from person to person, according to the person's psychological features. It would seem then that '$\Diamond$' psychologically interpreted should be relativized to '$\Diamond_x$' where the subscript variable ranges over persons. Such an account of modality is in fact implicit in Bentham (see Ogden [17, p. 50]). But such a relative interpretation offers no account of the standard person-invariant modalities; some assumptions are needed in order to reach a psychologistic theory of the standard modalities. To overcome the problem certain well-known but unconvincing shuffles are made, for example an assumption of similarity of humans in respect of relevant psychological abilities is introduced, or presupposed as when 'conceivability by the human mind' or 'conceivability by men' is adopted, or a rationality assumption is made. (This last assumption *may* lift the interpretation out of the psychological class, depending on how it is elaborated).

A typical outcome is Hume's imaginability criterion of possibility: according to Hume (on Pap's well documented reading, [18, pp. 75-85]) p is possible if and only if p is humanly imaginable. The modal reduction theses $\Diamond\Diamond p$, $\nabla \Diamond p$ and $\nabla \Box p$ appear to hold straight off under Hume's account. For what is imaginable, according to Hume, is certainly contingent on which senses human beings have. So it is contingent that $\Diamond p$; hence, accounting for *this* contingency in Hume's way, $\nabla \Diamond p$. It follows, by transformations Hume could hardly refuse, that $\Diamond\Diamond p$ and $\nabla \Box p$. It might be objected that this is all very well where what is at stake is, for example, the imaginability of a colour or a smell, but that there is nothing contingent about the imaginability of a contradiction such as $p \,\&\, {\sim} p$. The point is well taken if some test of contradictoriness (entailing impossibility) stands over and above the test of unimaginability. It is not unambiguously clear where Hume stands on this issue, but for the psychologistic interpretation here described it is assumed that imaginability is the sole criterion.

Since certain distribution principles such as $\Box(p \supset q) \supset. \Box p \supset \Box q$ and some strengthenings of this principle appear valid under the human imaginability criterion, various of the logics we develop are susceptible of a Humean interpretation. Under a psychologistic interpretation such as Hume's the principles $\Box(\Box p \supset p)$ and $\Box(p \supset \Diamond p)$ are not valid, and even $\Box p \supset p$ and $p \supset \Diamond p$ are precarious. It seems, however, that *human imaginability* can be doctored up so that $\Box p \supset p$ and $p \supset \Diamond p$ do hold though not necessarily, by denying that human imaginability is so weak that there are actual states of affairs, such as sounds of a very high pitch (Pap's example [18, pp. 79-80]), which humans cannot imagine.

By suitabily strait-jacketing 'p is humanly imaginable' symbolised 'Hp' we can prove certain of our systems consistent and complete under the relevant psychologistic interpretations, much as we did for related conventionalistic interpretations. For example if we require that 'H' satisfy just the following conditions, formulated with sentential logic as sublogic—

(1) ${\sim} H(p \,\&\, {\sim} q) \supset. Hp \supset Hq$
(2) $p \supset Hp$
(3) $CHp \,\&\, CCp$

where $CA =_{df} HA \,\&\, H \sim A$, then system QR0.5 is consistent and complete with respect to this psychologistic interpretation.

In spite of the semantical motivation for many of the systems developed the remainder of the paper is descriptive rather than semantical.

However features of the foundational systems and their extensions, and consistency and independence results for these systems, are often of philosophical as well as formal interest, for reasons we have elaborated; and the way is prepared for subsequent semantical investigations.

Foundation Systems. Church's bracketing and notational conventions ([3, pp. 74-81]) are in general adopted, with certain additions. The '&' is used for conjunction. Numerals preceded by '*F*' are references to items designated by the same numerals in Feys *Modal Logics* [5]. References to the rule of uniform substitution *US* are omitted. '*MD*' denotes the rule of detachment for material implication and '*SD*' the rule for strict implication. 'SL' refers to standard results obtainable from sentential logic and '*SSE*' the rule of substitutivity of strict equivalents. '$(X + Y)$' denotes the system formed by augmenting the system X by the postulates Y; '$(X - Y)$' the system obtained by deleting Y from the postulates of X. Systems with rules additional to *US* and *MD* are often distinguished by appending subscripts to the system label. These correspond to the rule number.

The basic apparatus for the systems is now presented.

Primitive connectives: $\Box, \sim, \supset$.

Definitions:

$$A \,\&\, B =_{df} \sim(A \supset \sim B)$$
$$A \vee B =_{df} \sim A \supset B$$
$$A \equiv B =_{df} (A \supset B) \,\&\, (B \supset A)$$
$$\Diamond A =_{df} \sim\Box\sim A$$
$$A \mathrel{-\!\!3} B =_{df} \Box(A \supset B)$$
$$A = B =_{df} (A \mathrel{-\!\!3} B) \,\&\, (B \mathrel{-\!\!3} A)$$
$$\nabla A =_{df} \Diamond A \,\&\, \Diamond\sim A$$
$$\triangle A =_{df} \Box A \vee \Box\sim A$$

Axioms:

A1	$p \supset . q \supset p$	*As1*	$p \mathrel{-\!\!3} . q \supset p$
A2	$p \supset (q \supset r) \supset . p \supset q \supset . p \supset r$	*As2*	$p \supset (q \supset r) \mathrel{-\!\!3} . p \supset q \supset . p \supset r$
A3	$\sim p \supset \sim q \supset . q \supset p$	*As3*	$\sim p \supset \sim q \mathrel{-\!\!3} . q \supset p$
A4	$p \mathrel{-\!\!3} q \supset . \Box p \supset \Box q$	*As4*	$p \mathrel{-\!\!3} q \mathrel{-\!\!3} . \Box p \supset \Box q$
A4'	$p \mathrel{-\!\!3} q \supset . \Box p \mathrel{-\!\!3} \Box q$	*As4'*	$p \mathrel{-\!\!3} q \mathrel{-\!\!3} . \Box p \mathrel{-\!\!3} \Box q$
		As5	$(p \mathrel{-\!\!3} q) \,\&\, (q \mathrel{-\!\!3} r) \mathrel{-\!\!3} . p \mathrel{-\!\!3} r$
		As6	$\Box p \,\&\, \Box q \mathrel{-\!\!3} \Box(p \,\&\, q)$
A7	$\Box p \supset p$	*As7*	$\Box p \mathrel{-\!\!3} p$
A8	$\Box\sim p \supset \sim\Box p$	*As8*	$\Box\sim p \mathrel{-\!\!3} \sim\Box p$
A9	$\Diamond\Diamond p$	*As9*	$\Box\Diamond\Diamond p$
A10	$\Box p \mathrel{-\!\!3} \Box q \supset . \Box p \supset \Box q$		
A11	$\nabla\nabla p$		
A12	$\nabla\Box p$		
A13	$\Diamond p$		

Rules:

R1	*US*	$A \rightarrow \mathsf{S}^{b}_{B} A \mid$
R2	*MD*	$A,\ A \supset B \rightarrow B$
R3		$\Box A \rightarrow A$

R4	$A = B \rightarrow \Box A = \Box B$
R5	$A \prec B \rightarrow \Box A \prec \Box B$
R6	$A \supset B \rightarrow \Box A \supset \Box B$
R7	*If A is a theorem of* SL *then* $\Box A$
R8	*SSE*

The primitive connectives and definitions and rules *R1* and *R2* are common to all systems. The five systems following are formed by adding these sets of postulates:

$$\begin{array}{lll}
S0.5^{o}_{3} & = \{As1, As2, As3, A4 & ;\ R3\} \\
S1^{o}_{3,4} & = \{As1, As2, As3, As4, As5 & ;\ R3, R4\} \\
S2^{o}_{3,5} & = \{As1, As2, As3, As4 & ;\ R3, R5\} \\
S2.5^{o}_{3} & = \{As1, As2, As3, A4', As6 & ;\ R3\} \\
S3^{o}_{3} & = \{As1, As2, As3, As4', As6 & ;\ R3\}
\end{array}$$

Corresponding to each of these systems are further systems $S0.5^{e}$, $S1^{e}_{4}$, $S2^{e}_{5}$, and $S3^{e}$, formed by replacing the rule *R3* by the axiom *A7*, i.e., schematically

$$SX^{e} = (SX^{o} - R3 + A7).$$

Thus $S0.5^{e}$, $S2.5^{e}$ and $S3^{e}$ are rule simplified systems. Further systems $S0.5^{d}_{3}$, $S1^{d}_{3,4}$, $S2.5^{d}_{3,5}$ and $S3^{d}_{3}$ are obtained from zero systems by adding *A8*; thus

$$SX^{d} = (SX^{o} + A8)$$

The notation was chosen so as to parallel, as far as standard notation permits, Lemmon's choice (in [10 and 11]) of notation for his epistemic systems (E systems) and deontic systems (D systems). Thus just as Lemmon's D systems are obtained from his C systems by adding *A8* so are super-d systems obtained from zero systems by addition of *A8*, and as E systems are obtained from C systems by adding *A7* so super-e systems result from zero systems by addition of *A7*.

Lewis systems are obtained by adding to SX^{e} systems axiom *As7*. Thus

$$SX = (SX^{e} + As7)\ (X = 1 - 3).$$

Systems based on Lemmon's C-, D-, and E-systems ([11, p. 47], [10]) are also introduced. For a distinctive set of conventionalistic systems can be based on these Lemmon systems. The relevance of the Lemmon systems will emerge still further when the semantics of our non-normal modal logics are investigated. (As to how see Lemmon [11, II]). Lemmon's 2-systems have the following postulate sets:

$$\begin{array}{ll}
C2_6 = \{A1, A2, A3, A4 & ;\ R6\} \\
D2_6 = \{A1, A2, A3, A4, A8 & ;\ R6\} \\
E2_6 = \{A1, A2, A3, A4, A7 & ;\ R6\}
\end{array}$$

There is a deliberate jump from D2 to D3, the weakest of Lemmon's 3-systems (in [10]), to guarantee reduction theses in D3. To obtain D3 Lemmon introduces the axiom schema:

(D) *If A is fully modalised, then* $\vdash \Box A \supset A$

where A is fully modalised if and only if all occurrences of variables in A are within the scope of a modal operator.

We shall, however, also introduce more conservative 3-systems, reached just as stronger 3-systems like E3 and D3 are reached, from corresponding 2-systems. Accordingly we define the systems:

$$\begin{aligned} C3^1_6 &= \{A1, A2, A3, A4, A4' \quad ; R6\} \\ D3^1_6 &= \{A1, A2, A3, A4, A4', A8 ; R6\} \\ E3_6 &= \{A1, A2, A3, A4', A7 \quad ; R6\} \end{aligned}$$

Lemmon's systems D3 and a related system C3 are defined instead as follows:

$$\begin{aligned} C3_6 &= \{A1, A2, A3, A4', (D) \quad ; R6\} \\ D3_6 &= \{A1, A2, A3, A4', A8, (D) ; R6\} \end{aligned}$$

It appears that there are various intermediate systems to be investigated as well, e.g., the systems $C3^2_6$ and $D3^2_6$ obtained from $C3^1_6$ and $D3^1_6$ by omitting *A4* from the postulate sets and adding *A10*.

Extension of Foundation Systems. A class of modal systems asserting the universal possibility of possibilities is obtained by adding *A9* to systems already introduced. These systems are a first step in the direction of contingency-oriented and conventionalist systems. 6-systems ($S6^{\circ}$, $S6^{d}$, $S6^{e}$, S6, C6, D6, E6, etc.) are obtained by adding *A9* to the postulate sets of 2-systems ($S2^{\circ}$, $S2^{d}$, $S2^{e}$, S2, C2, D2, E2, etc.); 6.5-systems ($S6.5^{\circ}$ etc.) by adding *A9* to the postulates of corresponding 2.5-systems; 7-systems ($S7^{\circ}$, etc.) by adding *A9* to postulates of corresponding 3-systems ($S3^{\circ}$, etc.).

Addition of the anticonventionalist axiom *As9* asserting the necessity of the universal possibility of possibilities leads to a stronger class of modal systems. For example there are consistent systems obtained by adding *As9* to 2-systems, 2.5-systems and to 3-systems.

Systems like the 6- and 8-systems may be obtained by adding *A9* or *As9* to systems weaker than $S2^{\circ}$. But whereas in $S6^{\circ}$ $\Diamond\Diamond p$ guarantees $\Diamond(\Diamond p \,\&\, \Diamond q)$—by:

1.	$\Diamond\Diamond(p \,\&\, q)$	*US.*
2.	$\Diamond(p \,\&\, q) \prec \Diamond p \,\&\, \Diamond q$	*F41.3.*
3.	$\Diamond\Diamond(p \,\&\, q) \supset \Diamond(\Diamond p \,\&\, \Diamond q)$	2, *F33.321, MD.*
4.	$\Diamond(\Diamond p \,\&\, \Diamond q)$	3, 1, *MD.*

—in systems which lack the consistency postulate (effectively *F40.1*) universal possibility of *joint* possibility does not follow from universal possibility of possibility. Consequently in weaker systems *A9* (and also *As9*) may be viewed as only the first of a sequence of progressively stronger assumptions of possibility of joint possibility. Once sentential variables are suitably ordered and indexed, e.g., $p_1, p_2, \ldots, p_n, \ldots$, the sequence of assumptions is given by:

A9ⁿ $$\Diamond \bigwedge_{i=1}^{i=n} (\Diamond p_i)$$

for $n = 1, 2, 3, \ldots\ldots$ Then $A9^1 = A9$. The sequence of inclusive systems based on S0.5 by adding $A9^1, A9^2, \ldots, A9^n, \ldots; A9^n$ was pointed out to us by M. K. Rennie. There appear to be similar extensions of systems of S0.5 order such as S0.5, and of systems of S1 order.

Contingency-oriented systems result by adjoint *A11*, $\nabla\nabla p$, instead of *A9*. *A11* asserts the universal contingency of contingency. On its own it is not sufficient to provide the distinctive features of conventionalistic systems, since linguistic conventionalism asserts the universal contingency of *all* assignments of logical modalities. Conventionalist systems insist on both *A11* and *A12*, $\Box p$. *A12* yields $\Diamond p$ in all our foundations systems and we find that *A11* is derivable in systems ($S2^o + A12$) and stronger, though not in ($S0.5 + A12$). To label the wealth of resulting systems we adopt the following notation:

'Q . . . ' represents the system (. . . . + *A11*) ,
'R . . . ' represents the system (. . . . + *A12*) ,
'QR . . . ' represents the system (. . . . + *A11* + *A12*).

In cases where the system label begins with 'S' the 'S' is dropped; e.g. ($S2^o + A12$) is called $R2^o$

Some Results on Foundational Systems.

Theorem 1. *Any system which includes one of the following sets of postulates*: {*A1, A2, A3; R1, R2*}, {*As1, As2, As3; R1, R2, R3*} *or* {*As1, As2, As3, A7; R1, R2*} *deductively includes* SL.

Proof: The first set provides Church's postulates ([3, p. 119]) for P_2, and the axioms of this set are immediately forthcoming from either of the other sets by application of *R3* or *A7*, *R1* and *R2*. Hence all the systems introduced deductively include SL.

Theorem 2. *R7 is a derived rule of any system having one of the following sets of postulates*: {*As1, As2, As3; one of A4, A4', As4, As4', R4, R5, R6; one of A7, R3; and R1, R2*}.

Proof: In any such system the rule $\vdash A \dashv B \rightarrow \vdash \Box A \supset \Box B$ is derivable. The result follows by consideration of the axioms *As1, As2, As3*, this derived rule, *R1*, *R2*, induction on the length of a proof, and theorem 1.

Theorem 3. $S0.5^e$ *is deductively equivalent to the system* S0.5 (of Lemmon [10, p. 180]).

Proof: (1) $S0.5^e$ deductively includes S0.5; for $S0.5^e$ deductively includes SL by theorem 1, and Lemmon's postulates A$''$, (1$'$), (2) are *R7, A4* and *A7*, and *R7* is derivable in $S0.5^e$ by theorem 2.

(2) S0.5 deductively includes $S0.5^e$; for *As1, As2, As3* are derivable in S0.5 by SL and *R1*; *A4* and *A7* are (1) and (2); and *R1* and *R2* are part of the SL formulation of S0.5.

Theorem 4. $S1^o{}_{3,4}$ *is deductively equivalent to* $S1^o$[(Feys 5, pp. 43-45]).

Proof: (1) $S1^\circ{}_{3,4}$ deductively includes $S1^\circ$. The postulates for $S1^\circ$ are derivable in $S1^\circ{}_{3,4}$. *SD* (*F30.23*) is a derived rule by *R3* and *R2*. $S1^\circ{}_{3,4}$ contains SL by theorem 1. *F30.22* is derivable by SL. *F30.11*, *F30.12*, *F30.13*, *F30.14* follow by SL and theorem 2. *F30.15* is *As5*. *F30.24* (*SSE*), follows by induction using *R4* and the rules

$$A = B \rightarrow (C \supset A) = (C \supset B)$$
$$A = B \rightarrow (A \supset C) = (B \supset C)$$
$$A = B \rightarrow \sim A = \sim B$$

which may be derived (as in [10, p. 178-179]). By SL, theorem 2 and *SSE*, the definitions of $S1^\circ$ are all provable as strict equivalences in $S1^\circ{}_{3,4}$.

(2) $S1^\circ$ deductively includes $S1^\circ{}_{3,4}$. The postulates of $S1^\circ{}_{3,4}$ are derivable in $S1^\circ$: *As1*, *As2*, *As3* follow from *F34.1*. *As4* is *F33.311*, *As5* is *F30.15*, *R1* is *F30.21*, *R2* is *F32.211*, *R3* is *F34.2*, *R4* is *F31.19*. The definitions follow as in (1) above.

Note that although *R3* is derivable is $S1^\circ$ it is independent in ($S1^\circ{}_{3,4}$ - *R3*) since it is needed to make *R2* operative.

Theorem 5. $S2^\circ{}_{3,5}$ *is deductively equivalent to* $S2^\circ$.

Proof: (1) $S2^\circ{}_{3,5}$ deductively includes $S2^\circ$. It suffices to show that $S2^\circ{}_{3,5}$ contains *As5*, *R4*, *F40.1*. The result then follows from theorem 4.

ad *R4*: Immediate from *R5* and SL.
ad *As5*: *R7* is a derived rule of $S2^\circ{}_{3,5}$ by theorem 2.

1.	$p \supset q \prec .q \supset r \supset . p \supset r$	SL, *R7*.
2.	$p \supset (q \supset r) \prec . p \,\&\, q \supset r$	SL, *R7*.
3.	$p \prec (q \supset r) \prec . p \,\&\, q \prec r$	2, *R5*.
4.	$p \prec q \prec . q \supset r \prec . p \supset r$	1, *R5*.
5.	$q \supset r \prec (p \supset r) \prec . q \prec r \supset . p \prec r$	*As4*.
6.	$p \prec q \supset . q \supset r \prec . p \supset r$	4, *R3*.
7.	$q \supset r \prec (p \supset r) \supset . q \prec r \supset . p \prec r$	5, *R3*.
8.	$p \prec q \supset . q \prec r \supset . p \prec r$	6, 7, SL.
9.	$p \prec q \prec . q \prec r \supset . p \prec r$	8, 4, 5, *R2*.
10.	$(p \prec q) \,\&\, (q \prec r) \prec . p \prec r$	3, 9, *SD*.

ad *F40.1*:

1.	$\sim p \prec \sim (p \,\&\, q)$	Theorem 2, *R7*
2.	$\Box \sim p \prec \Box \sim (p \,\&\, q)$	1, *R5*.
3.	$p \supset q \prec . \sim q \supset \sim p$	SL, *R7*.
4.	$p \prec q \prec . \sim q \prec \sim p$	3, *R5*.
5.	$\sim \Box \sim (p \,\&\, q) \prec \sim \Box \sim p$	4, 2, *SD*.
6.	$\Diamond(p \,\&\, q) \prec \Diamond p$	5, *Df* $\Diamond$.

(2) $S2^\circ$ deductively includes $S2^\circ{}_{3,5}$. It suffices to show that *R5* is derivable in $S2^\circ$, and this is demonstrated as *F46.1*.

Theorem 6. $S3^\circ{}_3$ *is deductively equivalent to* $S3^\circ$ (Sobociński [22, p. 53]).

Proof: (1) $S3^{\circ}_{3}$ deductively includes $S3^{\circ}$: $S3^{\circ}_{3}$ deductively includes $S1^{\circ}$ by a modification of the argument in the proof (1) of theorem 4. Proofs are required for *F30.15* and *F30.24* (*SSE*).

ad *F30.15*: *R7* is derivable by theorem 2.

1.	$(p \supset q) \;\&\; (q \supset r) \prec . \, p \supset r$	*R7*
2.	$(p \prec q) \;\&\; (q \prec r) \prec \Box\{(p \supset q) \;\&\; (q \supset r)\}$	*As6*
3.	$p \supset q \prec . \, q \supset r \supset . \, p \supset r$	*R7*
4.	$p \prec q \prec . \, q \supset r \prec . \, p \supset r$	*As4'*, 3, *SD*
5.	$q \supset r \prec (p \supset r) \prec . \, q \prec r \prec . \, p \prec r$	*As4'*
6.	$q \supset r \prec (p \supset r) \supset . \, q \prec r \prec . \, p \prec r$	5, *R3*
7.	$p \prec q \supset . \, q \supset r \prec . \, p \supset r$	4, *R3*
8.	$p \prec q \supset . \, q \prec r \prec . \, p \prec r$	6, 7, SL.
9.	$\Box\{(p \supset q) \;\&\; (q \supset r)\} \prec . \, p \prec r$	*As4'*, 1, *SD*
10.	$(p \prec q) \;\&\; (q \prec r) \prec . \, p \prec r$	8, 2, 9, *MD*, *SD*

ad *F30.24*: The derivation is as in theorem 4 except that *R4* must be *As4'* and SL.

To complete this half of the proof it remains to show that *F50.01* is provable in $S3^{\circ}_{3}$.
ad *F50.01*:

1.	$\sim q \prec \sim p \prec . \, \Box \sim q \prec \Box \sim p$	*As4*
2.	$p \prec q \prec . \sim q \prec \sim p$	SL, *R7*, *As4*, *SD*
3.	$\Box \sim q \prec \Box \sim p \prec . \sim \Box \sim p \prec \sim \Box \sim q$	2
4.	$p \prec q \prec . \, \Diamond p \prec \Diamond p$	1, 2, 3, *F30.15* SL, *SD*, *Df* $\Diamond$.

(2) $S3^{\circ}$ deductively includes $S3^{\circ}_{3}$. The postulates of $S3^{\circ}_{3}$ with the exception of *As4'* and *As6* have been derived in $S1^{\circ}$ in the course of the proof of theorem 4 above, *As6* is *F44.1*, a theorem of $S2^{\circ}$, and *As4'* is derivable in $S3^{\circ}$ as in the proof of *F51.11* in S3.

Theorem 7. *The systems* $S0.5^{e}_{3}$, $S1^{e}_{4}$, $S2^{e}_{5}$, $S2.5^{e}$, $S3^{e}$ *are proper extensions of systems* $S0.5^{\circ}_{3}$, $S1^{\circ}_{3,4}$, $S2^{\circ}_{3,5}$, $S2.5^{\circ}_{3}$, $S3^{\circ}_{3}$ *respectively*.

Proof: The following set of matrices satisfies the postulates of the second set of systems, but does not satisfy *A7*.

$\supset$	1	2	3	4	$\sim$	$\Box$
*1	1	2	3	4	4	1
2	1	1	3	3	3	3
3	1	2	1	2	2	3
4	1	1	1	1	1	3

These are Lewis and Langford Group IV but with only the value 1 designated. *A7* fails for $p = 2$.

Theorem 8. *The systems* $S1_{4}$, $S2_{5}$, S2.5, S3 *are proper extensions of systems* $S1^{e}_{4}$, $S2^{e}_{5}$, $S2.5^{e}$, $S3^{e}$ *respectively*.

Proof: The following set of matrices satisfies the postulates of the second set of systems, but does not satisfy *F36.0*, $p \dashv \Diamond p$, a theorem of S1.

	$\supset$	1	2	3	4	5	6	7	8	$\sim$	$\Box$
*	1	1	2	3	4	5	6	7	8	8	2
*	2	1	1	3	3	5	5	7	7	7	6
*	3	1	2	1	2	5	6	5	6	6	6
*	4	1	1	1	1	5	5	5	5	5	6
	5	1	2	3	4	1	2	3	4	4	6
	6	1	1	3	3	1	1	3	3	3	6
	7	1	2	1	2	1	2	1	2	2	6
	8	1	1	1	1	1	1	1	1	1	6

The set is adapted from *F50.1* (due to Parry). *F36.0* fails for $p = 1$.

Theorem 9. *The postulates of* $S3^{\circ}{}_3$ *are independent.*

Proof: (1) *As1* is independent by the matrices:

	$\supset$	1	2	3	4	$\sim$	$\Box$
*	1	1	4	4	4	4	1
*	2	1	2	3	4	3	2
	3	1	2	2	4	2	3
	4	1	1	1	1	1	4

These satisfy the remaining postulates but fail for *As1* ($p = 2$, $q = 1$).

(2) *As2* is independent, for every set of postulates for S3 contains at least one axiom with three variables (Diamond and McKinsey [4, p. 962]). ($S3^e + \Box p \dashv p$) is deductively equivalent to S3. Hence *As2* is independent in ($S3^e + \Box p \dashv p$) and hence in $S3^e$.

(3) *As3* is independent by the matrices

	$\supset$	1	2	$\sim$	$\Box$
*	1	1	2	1	1
	2	1	1	2	2

which satisfy the remaining postulates and fail for *As3* ($p = 2$, $q = 1$)

(4) *As4'* is independent by Parry's matrices *F50.1* which fail for *As4'* ($p = 1$, $q = 2$).

(5) *As6* is independent by the following matrices (due to Ivo Thomas):

	$\supset$	1	2	3	4	$\sim$	$\Box$
*	1	1	2	3	4	4	1
*	2	1	1	3	3	3	3
	3	1	2	1	2	2	3
	4	1	1	1	1	1	4

These satisfy the remaining postulates but fail for *As6* ($p = 2$, $q = 3$).

(6) *R1* is independent since without it the longest theorem cannot be longer than the longest axiom.

(7) *R2* is independent since without it the shortest theorem cannot be shorter than the shortest axiom (and $p \supset p$ is a theorem).

(8) *R3* is independent since with it *R2* is inoperative.

Theorem 10. (i) *The postulates of* $S3^e$ *are independent, and* (ii) *the postulates of* S3 *are independent.*

Proof: (i) As in theorem 9 except that the independence of *A7* follows for the same reason that *R3* is independent in $S3^o{}_3$.

(ii) Except for *As7* the result follows from (i). *As7* is independent by the Lewis Group IV matrices *F36.1.*

Theorem 11. $S3^e$ *properly includes* S3* (Sobociński [22, p. 53]).

Proof: It is easily shown that $S3^e$ contains the postulates of S3*, and *As5*, a theorem of $S3^e$ by theorem 6, is not provable in S3* (Thomas [24, p. 199]).

Theorem 12. *If A is a thesis of* SL *then* $\Box A$ *is a thesis of* $S1^o$, $S2^o$, $S2.5^o$ *and* $S3^o$ *and all systems which include them.*

Proof: By induction over proofs in SL, as outlined in *F34.1.*

Theorem 13. *Each of the following inclusions is proper*:

$$\begin{array}{lllll} S2^o_{3,5} & \subset & S2.5^o_3 & \subset & S3^o_3 \\ S2^d_{3,5} & \subset & S2.5^d_3 & \subset & S3^d_3 \\ S2^e_5 & \subset & S2.5^e & \subset & S3^e \\ S2_5 & \subset & S2.5 & \subset & S3 \end{array}$$

Proof: Improper inclusions of $S2.5^i$ in $S3^i$ are immediate. To show that $S2^i$ is improperly included in $S2.5^i$ it is necessary to prove that *As4* is a theorem of $S2.5^o_3$.

(i) $\vdash_{S2.5^o_3} (p \prec q) \,\&\, (q \prec r) \prec .\, p \prec r$

1.	$p \prec q \prec .\, q \supset r \prec .\, p \supset r$	*A4′*, SL, Theorem 2.
2.	$q \supset r \prec (p \supset r) \supset .\, q \prec r \prec .\, p \prec r$	*A4′*
3.	$p \prec q \supset .\, q \supset r \prec .\, p \supset r$	1, *R3*.
4.	$p \prec q \supset .\, q \prec r \prec .\, p \prec r$	3, 2, SL.
5.	$(p \prec q) \,\&\, (q \prec r) \prec \Box\{(p \supset q) \,\&\, (q \supset r)\}$	*As6*.
6.	$\Box\{(p \supset q) \,\&\, (q \supset r)\} \prec .\, p \prec r$	*A4′*, SL, Theorem 2.
7.	$(p \prec q) \,\&\, (q \prec r) \prec .\, p \prec r$	4, 5, 6, *MD, SD*.

(ii) $\vdash_{S2.5^o_3} (p \prec q) \,\&\, \Box p \prec \Box q$

1.	$(p \supset q) \,\&\, p \prec q$	Theorem 12, SL.
2.	$\Box((p \supset q) \,\&\, p) \prec \Box q$	*A4′, MD*
3.	$\Box(p \supset q) \,\&\, \Box p \prec \Box((p \supset q) \,\&\, p)$	*As6*
4.	$(p \prec q) \,\&\, \Box p \prec \Box q$	2, 3, (i), SL.

(iii) $\vdash_{S2.5^o_3} p \,\&\, q \prec r \prec .\, p \prec (q \supset r)$ Theorem 12, *A4′*

Finally $\vdash_{S2.5^o_3} p \prec q \prec .\, \Box p \supset \Box q$ (iii) and (ii) by *SD*.

The inclusions are however proper. For *A4′* is not provable in $S2_5$ (the most comprehensive of the $S2^i$ systems) by the Parry-Huntington matrix *F50.1*, with values 1 and 2 designated; and *As4'* in S2.5 by the same matrix *F50.1* with values 1, 2, 3, 4, designated.

Corollary: *SSE is a derived rule of* $S2.5^i$.

Theorem 14. *The postulates of* $S2.5^o_3$ *are independent.*

Proof: As for theorem 9. *A4'* is independent by *F50.1* with $p = 1$ and $q = 2$.

Theorem 15. (i) *The postulates of* $S2.5^e$ *are independent, and* (ii) *the postulates of* S2.5 *are independent.*

Proof: As for theorem 10. *A4'* is independent by *F50.1.*

Results on Contingency-oriented and Conventionalist Extensions.

Theorem 16. *The systems* $Q0.5^i$, $Q2^i$, $Q2.5^i$ *and* $Q3^i$ *are consistent.*

Proof: The following matrices satisfy S0.5, S2, S2.5, S3 and the axiom *A11.*

	&	1	2	3	4	5	6	7	8	~	□	▽
*	1	1	2	3	4	5	6	7	8	8	2	7
*	2	2	2	4	4	6	6	8	8	7	6	3
*	3	3	4	3	4	7	8	7	8	6	8	3
*	4	4	4	4	4	8	8	8	8	5	8	3
	5	5	6	7	8	5	6	7	8	4	6	3
	6	6	6	8	8	6	6	8	8	3	6	3
	7	7	8	7	8	7	8	7	8	2	8	3
	8	8	8	8	8	8	8	8	8	1	8	7

Theorem 17. *Where, e.g.* $Q2^i$ *is the* Q2 *system corresponding to the* S6 *system; e.g.* $Q2^e$ *corresponds to* $S6^e$,

$$\begin{array}{lcl} S6^i & \subset & Q2^i \\ S6.5^i & \subset & Q2.5^i \\ S7^i & \subset & Q3^i \end{array}$$

Proof: (1) $\Diamond\Diamond p$ is provable in $Q2^i$, $Q2.5^i$ and $Q3^i$.

1.	$\nabla\nabla p$	*A11.*
2.	$\Diamond(\Diamond p \,\&\, \Diamond{\sim} p) \,\&\, \Diamond{\sim}(\Diamond p \,\&\, \Diamond{\sim} p)$	1, *Df* ▽.
3.	$\Diamond(\Diamond p \,\&\, \Diamond{\sim} p)$	2, SL.
4.	$\Diamond\Diamond p$	*F35.32*, 3, *F35.11.*

$\Diamond\Diamond p$ is similarly derivable in $Q0.5^o$ (using *A4* etc. for Line 4).

(2) *A11* is independent of S6, S6.5 and S7, since S6 and S7 satisfy the matrices *F56.1* (Lewis and Langford, Group I), but *A11* does not.

Theorem 18. (S2 + *As9* + *A11*), (S2.5 + *As9* + *A11*) *and* (S3 + *As9* + *A11*) *are inconsistent.*

Proof: It suffices to prove inconsistence for (S2 + *As9* + *A11*).

1.	$\nabla\nabla p$	*A11.*
2.	$\Diamond\nabla p \;\&\; \sim\Box\nabla p$	1, *Df* ∇ , *F33.211, SSE.*
3.	$\sim\Box(\Diamond p \;\&\; \Diamond\sim p)$	2, *Df* ∇ , SL.
4.	$\sim(\Box\Diamond p \;\&\; \Box\Diamond\sim p)$	3, *F44.1, SSE.*
5.	$\Box\Diamond p \supset \sim\Box\Diamond\sim p$	4, SL.
6.	$\Box\Diamond p \supset \Diamond\Box p$	5, *Df* $\Diamond$, *F33.21, SSE.*
7.	$\Box\Diamond\Diamond p \supset \Diamond\Box\Diamond p$	6.
8.	$\Box p \prec \Box\Diamond\Box p$	*F32.2.*
9.	$\Diamond\Box\Diamond p \prec \Diamond p$	*F31.34,* 8, *SD, F33.211. F33.22, F33.23, SSE.*
10.	$\Box\Diamond\Diamond p \supset \Diamond p$	*F37.12,* 9, *SD,* 7, SL.
11.	$\Diamond p$	10, $\Box\Diamond\Diamond p$, *MD.*

Inconsistency follows immediately as in *F91.0.*

Theorem 19. (Q2 + $\Diamond p \supset \Box\Diamond p$) *is inconsistent; hence too* (S3.5 + *A11*) *is inconsistent.*

Proof:

1.	$\Diamond p \supset \Box\Diamond p$	*A4.*
2.	$\Diamond\Diamond p \supset \Box\Diamond\Diamond p$	1.
3.	$\Box\Diamond\Diamond p$	2, *F91.10, MD.*

Inconsistency follows as in theorem 18.

The following are theorems of Q2; the proofs present no difficulties:

$$\Box p \supset \nabla\Box p \qquad \Box\sim p \supset \nabla(\Box\sim p)$$
$$p \;\&\; \nabla p \supset \nabla(p \;\&\; \nabla p) \qquad \sim p \;\&\; \nabla p \supset \nabla(\sim p \;\&\; \nabla p)$$

It is clear from the first two theorems that Q2 would be a radical conventionalist system if it were also provable that $\nabla p \supset \nabla\Box p$. But such a result is not provable: in fact it is not even consistently adjoinable to Q2. We show then that S6, Q2 and extensions of these systems are not properly conventionalistic.

Theorem 20. (S1 + *A12*) *is inconsistent.*

Proof:

1.	$\nabla p \equiv \nabla\sim p$	*Df* ∇, SL.
2.	$\nabla\Box p \supset \nabla\sim\Box p$	1, SL.
3.	$\nabla\Diamond p$	2, *A12, MD, Df* $\Diamond$.
4.	$p \prec \Diamond p$	*F36.0.*
5.	$\Box p \supset \Box\Diamond p$	*F33.311,* 4, *SD.*
6.	$\sim\Box\Diamond p \supset \sim\Box p$	5, SL.
7.	$\nabla\Diamond p \supset \Diamond\Diamond p \;\&\; \sim\Box\Diamond p$	SL, *Df* ∇, *F33.211, SSE.*
8.	$\sim\Box p$	7, 3, 6, SL.
9.	$\Diamond p$	8, *Df* $\Diamond$.

Inconsistency follows as in theorem 3.

However by weakening the S1 axiom $p \prec \Diamond p$ to $p \supset \Diamond p$ this sort of derivation can be avoided.

Theorem 21. QR3^{e} *and its subsystems are consistent.*

Proof: The matrices used in theorem 8 can be used to prove this result.

Theorem 22. QR0.5 *is a proper extension of* S0.5, *of* Q0.5 *and of* R0.5.

Proof: *A11* and *A12* are each independent of S0.5 by the Lewis Group III matrices (*F56.3*). *A12* is independent of Q0.5 by the Lewis Group V matrices (*F30.5*). *A11* is independent of R0.5 by the Parry matrices (*F50.1*) for '$\sim$' and '&', with 1, 2, 3, 4, designated and the following matrix for '$\Box$':

p	1	2	3	4	5	6	7	8
$\Box p$	2	5	5	5	5	5	5	5

Theorem 23. *The pairs of systems* QR2^{o}, R2^{o}; QR2^{e}, R2^{e}; QR2.5^{o}, R2.5^{o}; QR2.5^{e}, R2.5^{e}; QR3^{o}, R3^{o}; QR3^{e}, R3^{e} *are deductively equivalent.*

Proof: It is sufficient to prove *A11* is a theorem of R2^{o}.

1.	$\nabla\Box p$	*A12.*
2.	$\Diamond\Box p \,\&\, \Diamond{\sim}\Box p$	1, *Df* ∇.
3.	$\Diamond\Diamond(p \,\&\, {\sim}p)$	2, SL, *Df* $\Diamond$.
4.	$\Diamond\Box(p \,\&\, {\sim}p)$	2, SL.
5.	$p \,\&\, {\sim}p \prec q$	SL, *F34.1, F43.2, SD.*
6.	$\Diamond(p \,\&\, {\sim}p) \prec \Diamond q$	*F46.2*, 5.
7.	$\Diamond(p \,\&\, {\sim}p) \prec \Diamond{\sim}q$	5, *F46.2.*
8.	$\Diamond(p \,\&\, {\sim}p) \prec \nabla q$	6, 7, SL, *F42.21, SD, Df* ∇.
9.	$\Diamond\Diamond(p \,\&\, {\sim}p) \prec \Diamond\nabla q$	8, *F46.2.*
10.	$\Diamond\nabla q$	9, 3, *SD.*
11.	$\Box(p \,\&\, {\sim}p) \prec \Box q$	5, *F46.1.*
12.	$\Box q \prec \Box q \vee \Box{\sim}q$	SL, *F34.1.*
13.	$\Box q \prec {\sim}\nabla q$	12, SL, *F33.2, SSE, Df* ∇.
14.	$\Box(p \,\&\, {\sim}p) \prec {\sim}\nabla q$	11, 13, *F31.021.*
15.	$\Diamond\Box(p \,\&\, {\sim}p) \prec \Diamond{\sim}\nabla q$	14, *F46.2.*
16.	$\Diamond{\sim}\nabla q$	15, 4, *SD.*
17.	$\nabla\nabla q$	10, 16, SL, *Df* ∇.

Theorem 24. *The systems* Q2^{o}, Q2^{e}, Q2.5^{o}, Q2.5^{e}, Q3^{o} *and* Q3^{e} *are proper subsystems of the corresponding* QR *systems.*

Proof: The inclusions are immediate, and since the matrices of the proof of theorem 16 satisfy the Q systems and reject *A12*, it follows the inclusions are proper.

Corollary. *The systems* Q2^{e}, Q2.5^{e} *and* Q3^{e} *are proper subsystems of* R2^{e}, R2.5^{e} *and* R3^{e} *respectively.*
The proof is by theorems 23 and 24.

The system $S2^e$ is Feys system $S2^o$ augmented by *A7*, and we give the system ($S2^o$ - *F30.23* + *A7* + *R2*) the label $S2^e_*$.

Theorem 25. *The system* $S2^e_*$ *is deductively equivalent to the systems* $S2^e$ *and* $S2^e_5$.

Proof: The deduction equivalence of $S2^e$ and $S2^e_5$ is immediate from theorem 5. It is therefore sufficient to show that $S2^e_*$ is deductively equivalent to $S2^e$. This result follows since *F30.23* is derivable in $S2^e_*$ by *A7* and *R2*, *R2* is a derived rule of $S2^e$ (*F32.211*) and *A7* is a postulate of $S2^e$.

Theorem 26. $\vdash_{R2^e} A$ *if and only if* $\vdash_{S2^e} \Diamond\Diamond(p \,\&\, \sim p) \,\&\, \Diamond\Box(p \,\&\, \sim p) \supset. A$ (*where 'p' is some variable not occurring in A*).

Proof: The proof considers the formulations $R2^e_*$ and $S2^e_*$ of $R2^e$ and $S2^e$.

(1) If $\vdash_{S2^e} \Diamond\Diamond(p \,\&\, \sim p) \,\&\, \Diamond\Box(p \,\&\, \sim p) \supset. A$ then $\vdash_{R2^e} A$.

The proof follows the first four lines of the proof of theorem 23 thence by Adjunction and *MD*.

(2) If $\vdash_{R2^e} A$ then $\vdash_{S2^e} \Diamond\Diamond(p \,\&\, \sim p) \,\&\, \Diamond\Box(p \,\&\, \sim p) \supset. A$.

The proof is by induction over the length of the proof of A in $R2^e$. For the initial clause, if A is an axiom of $S2^e$ then the result follows by $p \supset. q \supset p$, and if A is *A12* then:

1.	$p \,\&\, \sim p \prec q$	SL, *F34.1, F43.2, SD.*
2.	$\Box(p \,\&\, \sim p) \prec \Box q$	1, *F46.1.*
3.	$\Diamond\Box(p \,\&\, \sim p) \prec \Diamond\Box q$	2, *F46.2.*
4.	$\Diamond(p \,\&\, \sim p) \prec \Diamond\sim q$	1, *F46.2.*
5.	$\Diamond\Diamond(p \,\&\, \sim p) \prec \Diamond\sim\Box q$	4, *F33.2, SSE, F46.2.*
6.	$\Diamond\Diamond(p \,\&\, \sim p) \,\&\, \Diamond\Box(p \,\&\, \sim p) \prec \Diamond\Box q \,\&\, \Diamond\sim\Box q$	5, 3, SL, *F42.31, SD.*
7.	$\Diamond\Diamond(p \,\&\, \sim p) \,\&\, \Diamond\Box(p \,\&\, \sim p) \supset \nabla\Box q$	6, *Df* ∇, *F34.2.*

For the inductive clause, the result is immediate for *US* (*F30.21*) and *SSE* (*F30.24*), and almost immediate by SL for Adjunction (*F30.22*) and *MD* (*R2*).

Theorem 27. $\vdash_{R2.5^e} A$ *if and only if* $\vdash_{S2.5^e} \Diamond\Diamond(p \,\&\, \sim p) \,\&\, \Diamond\Box(p \,\&\, \sim p) \supset. A$ *and* $\vdash_{R3^e} A$ *if and only if* $\vdash_{S3^e} \Diamond\Diamond(p \,\&\, \sim p) \,\&\, \Diamond\Box(p \,\&\, \sim p) \supset. A$.

Proof: Since these are rule-simplified systems containing $R2^e$ and $S2^e$ respectively, the proof is as in the proof of theorem 26 with a simplified induction step.

Theorem 28. *The system* $R2^e$ *is decidable.*

Proof: $S2^e$ is decidable (for proof see Routley [19]). Hence by theorem 26, $R2^e$ is decidable.

NOTES

1. P. F. Strawson, "Necessary propositions and entailment-statements," *Mind*, vol. 57 (1948), pp. 184-200, p. 185. That Strawson's thesis is tantamount to (**R**)

in the systems we consider is evident from footnote 2, p. 184. J. Bennett in "Iterated modalities," *Philosophical Quarterly*, vol. 5 (1955), pp. 45-56, p. 49, mistakenly attributes the S8 thesis $\Box\sim\Box\Box p$ to Strawson; he appears to think this thesis is a consequence of Strawson's convention. It is not: indeed the S8 thesis is inconsistent with radical conventionalism.

2. See, for example, *A System of Logic*, Book I, Chapter VI.

3. Kneale's claim (in [8], p. 644) that conventionalism involves nominalism is nowhere substantiated in his book.

4. An argument which will show that the radical conventionalist is in serious difficulties elaborating his position satisfactorily is given in [21]. What can be shown is that there is no feasible way of elaborating the conventionalist case as sketched in the text here.

5. Mackie's linguistic *interpretation* is not to be confused with the linguistic *theory* of logical necessity explained earlier. We have replaced Mackie's single quote marks by the quotation function notation of [6].

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