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ATOMISTIC MEREOLOGY I

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In Leśniewski's mereology the existence of the mereological atoms cannot be proved without an addition of some new axioms to the original axiom system of mereology. The strongest form of the possible axioms concerning the existence of mereological atoms in the field of mereology is an assumption which implies that every object A is either a mereological atom or a mereological class constructed from the atoms which are the mereological elements of A. I shall call an extension of mereology obtained by adding the above assumption, as a new axiom, to the axiom system of mereology, atomistic mereology. On the other hand it is possible to construct an entirely different extension of mereology by adding an assumption that no atom exists in the field of mereology. Such a system which is called the atomless mereology will not be discussed in this paper. Up to now these two extreme extensions of mereology which, obviously, are mutually incompatible were investigated very little. Only, in a still unpublished part of his doctoral thesis, *cf.* [5], chapter II, sections 1 and 2, pp. 72-100, Clay has established several metatheorems about general properties of these two ramifications of mereology. In [5], p. 83, Clay has remarked that since there is no mereological zero element, i.e. an element which would correspond to Boolean algebraic zero element, in mereology, the definition of an atom in the latter system is more simple than it is in Boolean algebra. And, using mereological functor "**pr**" he defined a notion of a mereological atom, as follows:

CD1 $[A]: A\varepsilon A \,.\, [B] \,.\, \sim(B\varepsilon\, \mathbf{pr}\,(A)) \,.\equiv.\, A\,\varepsilon\, \mathbf{atm}$

But, although in [5] he has proved several metamereological theorems concerning atomistic mereology, Clay did not axiomatize this system. Recently, in connection with his investigations which are not yet published concerning a certain geometrical system, V. F. Rickey observed that the axiomatized atomistic mereology would be very useful for this research. Consequently, he defined an atom using the mereological functor "**el**" in the field of mereology, as follows:

RD1 $[A]\therefore A\varepsilon A : [B] : B\varepsilon\, \mathbf{el}(A) \,.\supset.\, B = A :\equiv.\, A\varepsilon\, \mathbf{atm}$

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And, he based his system of atomistic mereology by adding a new axiom:

$RA1$ $[A]::A\varepsilon A.\supset.\because.[\exists a].\because.A\varepsilon\mathbf{Kl}(a):[BC]:B\varepsilon a.C\varepsilon\mathbf{el}(B).\supset.C=B$

in which "**Kl**" is the symbol of mereological class, to the ordinary mereology. Moreover, Rickey defined a new functor in the field of mereology, namely "atom of an object B", in symbols "**at** (B)", as follows:

$RD2$ $[AB]:A\varepsilon\mathbf{atm}.A\varepsilon\mathbf{el}(B).\equiv.A\varepsilon\mathbf{at}(B)$

and he has shown that this notion can be used as the single primitive functor of his system of atomistic mereology.

Since, as it is well-known, e.g., *cf.* [18], pp. 333-334, note 1, that in a certain sense mereology is closely related to the system of complete Boolean algebra, the natural questions arise: 1) whether the definitions *CD1* and *RD1* and the mereological formulas corresponding to the definitions of a Boolean atom introduced by Schröder, *cf.* [12], pp. 318-349, and analyzed by Tarski in the field of complete Boolean algebra, *cf.* [18], p. 334, are mutually equivalent in mereology; and 2) whether Rickey's axiom *RA1* and mereological formulas analogous to the Boolean axioms concerning the existence of atoms proposed by Schröder and proved to be inferentially equivalent in the field of complete Boolean algebra by Tarski, *cf.* [18], pp. 335-336, also are equivalent in the field of mereology. In this paper I shall show that in mereology *CD1*, *RD1* and all mereological formulas corresponding to Schröder's definitions are mutually equivalent, and that in the field of mereology *RA1* and all formulas analogous to the atomistic axioms discussed in [18] also are inferentially equivalent.

Moreover, there will be given an axiom system of atomistic mereology in which Rickey's functor "**at**" will be used as a single primitive mereological notion. Namely, I shall show that the following axiom system:

$S1$ $[AB]:A\varepsilon\mathbf{at}(B).\supset.B\varepsilon B$

$S2$ $[ABC]:A\varepsilon\mathbf{at}(B).C\varepsilon\mathbf{at}(A).\supset.C=A$

$S3$ $[AB]\therefore A\varepsilon A.B\varepsilon B:[C]:C\varepsilon\mathbf{at}(A).\equiv.C\varepsilon\mathbf{at}(B):\supset.A=B$

$S4$ $[Aa]::A\varepsilon a.\supset.\because.[\exists B].\because.[\exists E].E\varepsilon\mathbf{at}(B):[C]:C\varepsilon\mathbf{at}(B).\equiv.[\exists D].C\varepsilon\mathbf{at}(D).D\varepsilon a$

is inferentially equivalent to an axiom-system of atomistic mereology which contains only two axioms, namely the single axiom of general mereology A which will be presented below at the beginning of section **2** and an additional atomistic axiom:

V $[A]::A\varepsilon A.\supset.\because.[\exists B].\because.B\varepsilon\mathbf{el}(A):[C]:C\varepsilon\mathbf{el}(B).\supset.C=B$

which will be discussed in **3.2**. It should be noticed that in the latter axiom-system instead of V, Rickey's axiom *RA1* can be used.

An elementary acquaintance with mereology and Leśniewski's system of logic, i.e. protothetic and ontology, on which mereology is based is presupposed. In order to understand the proofs given below, an acquaintance with Leśniewski's ontology (about which, eventually, *cf.* [9], last part, [10],

[16], [13], [11], [1], [2], [3], and [4]) is especially important. The notation used in this paper is the well-known Peano-Russell symbolism slightly adjusted to the requirements of Leśniewski's system; *cf.* [14].

1 The symbol "ε" which occurs in the formulas given above means "is" in the sense of Leśniewski's ontology. This primitive ontological notion differs in some respect from "ϵ" which can be found in the other systems of logic and in set theory. For this reason in ontology besides the usual logical theorems there are also such theses which have no corresponding theorems in the fields of the other logical systems. Since some of these special ontological theorems will be used in the proofs which will appear below, they are presented in this section without the explanations why they are valid in the field of ontology. The logical notions "an object (an individual) A is identical with an object B", "A is an object" and "a or b" (a logical addition) are defined by "ε" in ontology, as follows:

Df1 $[A\,B]:A\,\varepsilon\,B\,.\,B\,\varepsilon\,A\,.\equiv.\,A=B$

Df2 $[A]:A\,\varepsilon\,A\,.\equiv.\,A\,\varepsilon\,\mathsf{V}$

Df3 $[A\,a\,b]\therefore A\,\varepsilon\,A:A\,\varepsilon\,a\,.\mathsf{v}.\,A\,\varepsilon\,b:\equiv.\,A\,\varepsilon\,a\cup b$

respectively. And, the following theorems are provable in this system:

T1 $[A\,a]:A\,\varepsilon\,a\,.\supset.\,A\,\varepsilon\,A$

T2 $[A\,B\,a]:A\,\varepsilon\,B\,.\,B\,\varepsilon\,a\,.\supset.\,A=B$

T3 $[A\,B]\therefore A\,\varepsilon\,B:[C\,D]:C\,\varepsilon\,B\,.\,D\,\varepsilon\,B\,.\supset.\,C=D:\supset.\,A=B$

T4 $[A\,B\,a]:A\,\varepsilon\,B\,.\,B\,\varepsilon\,a\,.\supset.\,A\,\varepsilon\,a$

These formulas mean: *T1*—if A is something, then A is A (A is an object); *T2* (called the characteristic law of ontology)—if A is B and B is something, then A is identical with B; *T3*—if A is B and B is unique, then A is identical with B; *T4* shows that functor "ε" is transitive. It is not prescribed by a rule in Leśniewski's system, but it is only a custom that if we know that a variable stands for an object, then a capital letter is used. On the other hand the small letters represent the variables standing for the general names (sets) in the formulas. Besides *T1* - *T4*, the following, theorems concerning extensionality which are provable in ontology,

E1 $[A\,B\,\varphi]:A=B\,.\,\varphi\{A\}\,.\supset.\,\varphi\{B\}$

E2 $[a\,b]\therefore[A]:A\,\varepsilon\,a\,.\equiv.\,A\,\varepsilon\,b:\equiv:[\varphi]:\varphi\{a\}\,.\equiv.\,\varphi\{b\}$

will be used in the proofs, for the most cases tacitly. Concerning the laws of extensionality in Leśniewski's system, *cf.* e.g. [3], and for a formalized derivation of *T1* - *T4*, *E1* and *E2* see [17].

2 In mereology the notion of a mereological element, in symbol "**el**", can be used as a single primitive functor of this theory. In this case the following thesis:

A $[A\,B]\cdot\!\cdot:A\,\varepsilon\,\mathbf{el}(B)\,.\equiv::B\,\varepsilon\,B::[C\,a]::[D]\therefore D\,\varepsilon\,C\,.\equiv:[E]:E\,\varepsilon\,a\,.\supset.$
$E\,\varepsilon\,\mathbf{el}(D):[E]:E\,\varepsilon\,\mathbf{el}(D)\,.\supset.[\exists F\,G].\,F\,\varepsilon\,a\,.\,G\,\varepsilon\,\mathbf{el}(F).\,G\,\varepsilon\,\mathbf{el}(E)\therefore B\,\varepsilon\,\mathbf{el}(B).$
$B\,\varepsilon\,a\,.\supset.A\,\varepsilon\,\mathbf{el}(C)$

can be adopted as a single axiom of this system. Axiom A is due to Lejewski, *cf.* [7], pp. 138-139, and [8], and it is a slight modification of my single axiom of mereology which I obtained in 1945 and published in [14], section VIII, p. 257, *cf.* also [16], p. 38. In this section the following mereological theorems, which are the consequences of axiom A and which will be used in our further considerations, are presented without proof.

2.1 It is rather easy to prove that on the basis of ontology axiom A is inferentially equivalent to the following set of formulas:

$A1$ $[A\,B]:A\varepsilon\,\mathbf{el}(B)\,.\supset.\,B\varepsilon B$
$A2$ $[A\,B\,C]:A\,\varepsilon\,\mathbf{el}(B)\,.\,B\varepsilon\,\mathbf{el}(C)\,.\supset.A\,\varepsilon\,\mathbf{el}(C)$
$D1$ $[A\,a]\,.\because\,A\varepsilon A:[B]:B\,\varepsilon\,a\,.\supset.B\varepsilon\,\mathbf{el}(A):[B]:B\,\varepsilon\,\mathbf{el}(A)\,.\supset.[\exists E\,F].E\varepsilon\,a\,.$ $F\varepsilon\,\mathbf{el}(E)\,.\,F\varepsilon\,\mathbf{el}(B):\equiv.A\,\varepsilon\,\mathbf{Kl}(a)$
$A3$ $[A\,B\,a]:A\,\varepsilon\,\mathbf{Kl}(a)\,.\,B\,\varepsilon\,\mathbf{Kl}(a)\,.\supset.A=B$
$A4$ $[A\,a]:A\,\varepsilon\,a\,.\supset.[\exists B].\,B\varepsilon\,\mathbf{Kl}(a)$

Since we have $\{A\}\rightleftarrows\{A1, A2, D1, A3, A4\}$ and the second axiom-system is more convenient for our purposes, in further argumentations we shall refer to it rather than to the axiom A. The formula $D1$ given above is a definition of a mereological class by "**el**". And, it should be noticed that $D1$ can be easily eliminated in an equivalent way from the second axiom-system by replacing the axioms $A3$ and $A4$ by some theorems in which the defined functor "**Kl**" does not occur.

2.2 Furthermore, the following theses are the consequences of A (or $\{A1, A2, D1, A3, A4\}$):

$A5$ $[A]:A\,\varepsilon\,A\,.\supset.A\,\varepsilon\,\mathbf{el}(A)$

Concerning the provability of $A5$ from $\{A1, A2, D1, A3, A4\}$, *cf.* [6].

$A6$ $[A\,B]\,.\because\,A\varepsilon A:[D]:D\varepsilon\,\mathbf{el}(A)\,.\supset.[\exists F].\,F\varepsilon\,\mathbf{el}(D)\,.\,F\varepsilon\,\mathbf{el}(B):\supset.A\,\varepsilon\,\mathbf{el}(B)$

$A6$ is a very strong mereological theorem. Its proof given by Leśniewski in [9] is rather difficult. An entirely different, but also difficult, proof which is not yet published has been obtained by R. E. Clay.

$A7$ $[A\,B]:A\,\varepsilon\,\mathbf{el}(B)\,.\,B\varepsilon\,\mathbf{el}(A)\,.\supset.A=B$
$A8$ $[A\,a]:A\,\varepsilon\,\mathbf{Kl}(a)\,.\supset.A\,\varepsilon\,\mathbf{el}(\mathbf{Kl}(a))$
$A9$ $[A\,a]:A\,\varepsilon\,a\,.\supset.A\,\varepsilon\,\mathbf{el}(\mathbf{Kl}(a))$
$D2$ $[A\,B]:A\,\varepsilon\,\mathbf{el}(B)\,.\sim(A=B)\,.\equiv.A\,\varepsilon\,\mathbf{pr}(B)$

$D2$ is a definition of the mereological notion of a part: an object A is a part of an object B iff A is an element of B and A is not identical with B.

$A10$ $[A].\sim(A\,\varepsilon\,\mathbf{pr}(A))$
$A11$ $[A\,B]\,.\because\,A\,\varepsilon\,\mathbf{el}(B)\,.\equiv:A\,\varepsilon\,\mathbf{pr}(B)\,.\vee.A=B$
$A12$ $[A\,B]:B\varepsilon\,\mathbf{pr}(A)\,.\supset.A\,\varepsilon\,\mathbf{Kl}(\mathbf{pr}(A))$
$D3$ $[A\,B]\,.\because\,A\,\varepsilon A\,.\,B\varepsilon\,B:[C]:\,C\varepsilon\,\mathbf{el}(A)\,.\supset.\sim(C\,\varepsilon\,\mathbf{el}(B)):\equiv.\,A\varepsilon\,\mathbf{ex}(B)$

$D3$ is a definition of the mereological notion to be outside: an object A is outside of an object B iff A and B are the objects and no element of A is an element of B.

$A13$ $[A\,B\,C]:A\,\varepsilon\,\mathbf{el}(B).\,B\,\varepsilon\,\mathbf{ex}(C).\supset.A\,\varepsilon\,\mathbf{ex}(C)$
$A14$ $[A\,B]:A\,\varepsilon\,\mathbf{el}(B).\supset.\sim(A\,\varepsilon\,\mathbf{ex}(B)).\sim(B\,\varepsilon\,\mathbf{ex}(A))$
$A15$ $[A\,B]:A\,\varepsilon A.\,B\,\varepsilon\,B.\sim(A\,\varepsilon\,\mathbf{el}(B)).\supset.[\exists C].\,C\,\varepsilon\,\mathbf{el}(A).\,C\,\varepsilon\,\mathbf{ex}(B)$
$D4$ $[A\,B\,C]:A\,\varepsilon\,\mathbf{Kl}(B\cup C).\,B\,\varepsilon\,\mathbf{ex}(C).\equiv.A\,\varepsilon\,B+C$

D4 is a definition of mereological addition: an object A is a sum of two objects B and C iff A is a class constructed from the logical addition of two objects B and C and B is outside C. Concerning this mereological functor, *cf.* [15], section 2.

$A16$ $[A\,B]:A\,\varepsilon\,\mathbf{el}(B).\sim(A=B).\supset.[\exists C].\,B=A+C$
$A17$ $[A\,B\,C]:A\,\varepsilon\,\mathbf{el}(B+C).\,A\,\varepsilon\,\mathbf{ex}(C).\supset.\,A\varepsilon\,\mathbf{el}(B)$
$D5$ $[A\,B\,C]:A\,\varepsilon A.\,B\varepsilon A+C.\equiv.A\,\varepsilon\,B\dot{-}C$

D5 is a definition of mereological subtraction: an object A is obtained from an object B from which an object C is removed iff B is an object obtained by the mereological addition of C to A.

$D6$ $[A]:A\,\varepsilon\,\mathbf{Kl}(\mathsf{V}).\equiv.\,A\varepsilon\,\mathbf{Un}$

D6 is a definition of Universe in the field of mereology: A is the Universe iff A is the mereological class of all existing objects.

$A18$ $[A]:\,A\varepsilon A.\supset.\,A\varepsilon\,\mathbf{el}(\mathbf{Un})$
$A19$ $[A]:\,A\varepsilon A.\supset.\,\mathbf{Un}=(\mathbf{Un}\dot{-}A)+A$

Previously given theses *A4* and *A3* imply that a mereological class of the given objects a exists, if at least one object a exists, and that if such class exists, then it is unique. Hence, by *D6*, the Universe exists and is unique under condition that at least one arbitrary object exists. *A18* shows that every existing object is an element of **Un**. Therefore, in a certain way **Un** corresponds to Boolean-algebraic unit element. On the other hand, in mereology the following theorem is easily provable:

$A20$ $[A\,B]:\,A\varepsilon A.\,B\varepsilon B.\sim(A=B).\supset.\sim([\exists C]\therefore C\varepsilon C:[D]:\,D\varepsilon D.\supset.\,C\varepsilon\,\mathbf{el}(D))$

i.e., if there are two different objects, then there does not exist an object which would be an element of every existing object. Therefore, in the field of mereology, which is not degenerate by an additional assumption that there is one and only one object, there is no constant which would correspond to the zero element of non-degenerate Boolean algebra.

3 As was mentioned previously, in [18] Tarski has proved in the field of complete Boolean algebra that the conditions of various definitions of Boolean atom given in [12] are equivalent (this fact already was known to Schröder), and that several formulas which Schröder proposed as possible atomistic axioms also are inferentially equivalent. It will be proved here that in mereology the same holds for the mereological formulas which correspond to the above mentioned Boolean ones. Since in mereology there is no constant corresponding to Boolean algebraic zero element, the proofs presented below will differ considerably from those which Tarski gave in his paper. The Boolean formulas discussed in this section will be written in the same symbolism which is used in [18]. In particular, the capital letter B always will indicate the carrier set of the given, say $\mathfrak{B}$, system of complete Boolean algebra.

3.1 Definitions of a Boolean atom. In [18], p. 334, Tarski accepted the following definition of a Boolean atom:

$\mathfrak{C}$ *$x \epsilon At$ (1) if and only if $x \epsilon B$ and $x \neq 0$, and (2) for every element $y \epsilon B$, the formulas $y < x$ and $y \neq 0$ imply $y = x$.*

And, he noticed, *cf.* his Theorem 3, that in the field of complete Boolean algebra the conditions of definition $\mathfrak{C}$ and of the following definitions of Schröder are equivalent:

$\mathfrak{C}1$ *$x \epsilon At$ if and only if $x \epsilon B$ and every element $y \epsilon B$ satisfies one and only one of the two formulas $x < y$ or $x < y'$.*

$\mathfrak{C}2$ *$x \epsilon At$ if and only if $x \epsilon B$, $x \neq 0$ and formula $x = y + z$, for all elements $y, z \epsilon B$, implies $x = y$ or $x = z$.*

$\mathfrak{C}3$ *$x \epsilon At$ if and only if $x \epsilon B$, $x \neq 0$ and the formula $x < y + z$, for all elements $y, z \epsilon B$, implies $x < y$ or $x < z$.*

$\mathfrak{C}4$ *$x \epsilon At$ if and only if for every set $X \subseteq B$, the formula $x = \sum_{y \epsilon X} y$ implies $x \epsilon X$.*

$\mathfrak{C}5$ *$x \epsilon At$ if and only if $x \epsilon B$ and for every set $X \subseteq B$, the formula $x < \sum_{y \epsilon X} y$ implies $x < y$.*

3.2 In [18], pp. 335-336, in order to obtain the atomistic system of Boolean algebra, Tarski added the following formula:

$\mathfrak{D}$ *If $x \epsilon B$ and $x \neq 0$, then there is an element $y \epsilon At$ such that $y < x$*

as a new axiom to a certain axiom system which he assumed previously as a postulate system of complete Boolean algebra. And, he has proved that on the base of that postulate system axiom $\mathfrak{D}$ is inferentially equivalent to each of the following formulas:

$\mathfrak{D}1$ $1 = \sum_{y \epsilon At} y$.

$\mathfrak{D}2$ *If $x \epsilon B$, then $x = \sum_{y \epsilon At \text{ and } y < x} y$.*

$\mathfrak{D}3$ *If $x, y \epsilon B$ and for every element $z \epsilon At$ the formula $z < x$ implies $z < y$, then $x < y$.*

$\mathfrak{D}4$ *If $x, y \epsilon B$ and if the formulas $z < x$ and $z < y$ are equivalent for every $z \epsilon At$, then $x = y$.*

3.3 Besides set-theoretical symbols ϵ and $\subseteq$, in the formulas which appear in **3.1** and **3.2** the symbols "0", "1", "$x < y$", "$x = y$", "$x + y$", "x'" and "$\sum_{y \epsilon X} y$" have, obviously, the following Boolean meanings "the zero element," "the unit element", "x is included in y", "x is equal to y", "x join y", "the complement of the element x" and "the (Boolean) sum of all elements of the set X" respectively. Now, in order to construct mereological formulas analogous to ones which are given in **3.1** and **3.2** we should substitute the Boolean expressions occurring in those formulas by the suitable mereological ones. For this end instead of "1", "$x < y$", "$x = y$", "$x + y$", "x'" and "$\sum_{x \in X} y$" we shall use "**Un**", "$x\ \varepsilon\ \mathbf{el}(y)$", "$x = y$", "$x + y$", "$\mathbf{Un} \dot{-} x$" and "$\mathbf{Kl}(X)$" respectively. Moreover, since in mereology there are no such notions as a carrier set or zero element, the condition "$x \in B$" will be substituted by the condition "x is an object", and the condition "$x \neq 0$" will be dropped altogether. It is clear that instead of "$x \in At$" formula "$x\ \varepsilon\ \mathbf{atm}$" will be used, and that the condition "$y \in At$ *and* $y < x$" which occurs in *D2* will be substituted by "$y\ \varepsilon\ \mathbf{at}(x)$". Then:

(A) Using the mereological expressions which are discussed above we can introduce the following definitions of mereological atoms correctly constructed according to the rules of procedure of Leśniewski's system:

*D*I $[A] .\because A \varepsilon A : [B] : B \varepsilon \mathbf{el}(A) . \supset . B = A : \equiv . A \varepsilon \mathbf{atm}$

*D*I which we accept as the standard definition of an atom in mereology is Rickey's definition *RD1*. In an obvious way it corresponds to definition $\mathfrak{C}$ from **3.1**. The factor "$A \varepsilon A$" occurring according to rules of procedure in the definiens of *D*I guarantees that A is an object.

*D*I.1 $[A] :: A \varepsilon A .\because [B] .\because B \varepsilon B . \supset : A \varepsilon \mathbf{el}(B) . \vee . A \varepsilon \mathbf{el}(\mathbf{Un} - B) : \equiv . A \varepsilon \mathbf{atm}_1$

*D*I.1 corresponds to definition $\mathfrak{C}1$ from **3.1**. In the latter definition the second factor of its definiens has the following form: "*every element* $y \in B$ *satisfies one and only one of the two formulas* $x < y$ or $x < y'$", i.e. in symbolic form:

(a) $[y] .\because y \in B . \supset : x < y . \vee . x < y' : x < y . \supset . \sim (x < y')$

It is clear that in the field of Boolean algebra formula (a) is inferentially equivalent to:

(b) $x \neq 0 .\because [y] .\because y \in B . \supset : x < y . \vee . x < y'$

Hence in *D*I.1 the restriction "one and only one" can be omitted.

*D*I.2 $[A] :: A \varepsilon A .\because [BC] .\because A \varepsilon B + C . \supset : A = B . \vee . A = C .\because \equiv . A \varepsilon \mathbf{atm}_2$
*D*I.3 $[A] :: A \varepsilon A .\because [BC] .\because A \varepsilon \mathbf{el}(B + C) . \supset : A \varepsilon \mathbf{el}(B) . \vee . A \varepsilon \mathbf{el}(C) .\because \equiv . A \varepsilon \mathbf{atm}_3$

*D*I.2 and *D*I.3 correspond to $\mathfrak{C}2$ and $\mathfrak{C}3$ respectively. It follows at once from *A1*, *D4*, and *D3*, *cf.* **2.1**, that B and C which occur in the definiens of *D*I.2 and *D*I.3 are objects.

*D*I.4 $[A] .\because A \varepsilon A : [a] : A \varepsilon \mathbf{Kl}(a) . \supset . A \varepsilon a . \equiv . A \varepsilon \mathbf{atm}_4$
*D*I.5 $[A] .\because A \varepsilon A : [a] : A \varepsilon \mathbf{el}(\mathbf{Kl}(a)) . \supset . [\exists B] . B \varepsilon a . A \varepsilon \mathbf{el}(B) : \equiv . A \varepsilon \mathbf{atm}_5$

Clearly, *D*I.4 and *D*I.5 correspond to $\mathfrak{C}4$ and $\mathfrak{C}5$ respectively. Finally, we add to this set of definitions Clay's *CD1*:

*D*I.6 $[A] : A \varepsilon A . [B] . \sim (B \varepsilon \mathbf{pr}(A)) . \equiv . A \varepsilon \mathbf{atm}_6$

and

(B) In the introductory remarks to this paper we already assumed that atomistic mereology is a system which is obtained by adding the following formula:

$V \quad [A]::A \varepsilon A . \supset .\therefore [\exists B] .\therefore B \varepsilon \mathbf{el}(A) : [C] : C \varepsilon \mathbf{el}(B) . \supset . B = A$

as a new axiom, to the single axiom A of mereology. Now, we add definition DI, *cf.* (A), and Rickey's *RD2*:

$D\text{II} \quad [A\,B] : A \varepsilon \mathbf{atm} . A \varepsilon \mathbf{el}(B) . \equiv . A \varepsilon \mathbf{at}(B)$

to this theory. Then, using the same translation of Boolean expressions into mereological ones which was established in **3.3** we obtain the formulas:

$V1 \quad [A] : A \varepsilon A . \supset . [\exists B] . B \varepsilon \mathbf{at}(A)$
$V2 \quad [A] : A \varepsilon A . \supset . \bar{\mathbf{Un}} = \mathbf{Kl}(\mathbf{atm})$
$V3 \quad [A] : A \varepsilon A . \supset . A \varepsilon \mathbf{Kl}(\mathbf{at}(A))$
$V4 \quad [A\,B] .\therefore A \varepsilon A : [C] : C \varepsilon \mathbf{at}(A) . \supset . C \varepsilon \mathbf{at}(B) : \supset . A \varepsilon \mathbf{el}(B)$
$V5 \quad [A\,B] .\therefore A \varepsilon A . B \varepsilon B : [C] : C \varepsilon \mathbf{at}(A) . \equiv . C \varepsilon \mathbf{at}(B) : \supset . A = B$

which, obviously, are the mereological analogues of $\mathfrak{D}$, $\mathfrak{D}1$, $\mathfrak{D}2$, $\mathfrak{D}3$ and $\mathfrak{D}4$ respectively. In connection with the structure of $V2$, it should be remarked that its antecedent, viz. "$A \varepsilon A$", is necessary in mereology, since in its field the existence of a class of the given objects can be assumed only if at least one such object exists, *cf.* $A4$. Furthermore, we add Rickey's axiom

$V6 \quad [A]::A \varepsilon A . \supset .\therefore [\exists a] .\therefore A \varepsilon \mathbf{Kl}(a) : [B\,C] : B \varepsilon a . C \varepsilon \mathbf{el}(B) . \supset . C = B$

to the formulas $V1$ - $V5$.

3.4 We shall prove that in the field of mereology the formulas DI and DI.1 - DI.6 given in **3.3**, point (A), define exactly the same concept, viz. mereological atom. Let us assume, as the single axiom of mereology, thesis A. Hence, we have at our disposal all definitions and theorems given in **2**. Then:

$B1 \quad [A\,B] :: [B] .\therefore B \varepsilon B . \supset : A \varepsilon \mathbf{el}(B) . \vee . A \varepsilon \mathbf{el}(\mathbf{Un} \dot{-} B) .\therefore B \varepsilon \mathbf{el}(A) . \sim (B = A) .$
$\quad \supset . B = A$

PR $[A\,B] :: \mathrm{Hp}(3) .\therefore \supset :$

4.	$A \varepsilon \mathbf{el}(B) . \vee . A \varepsilon \mathbf{el}(\mathbf{Un} \dot{-} B) :$	[$T1$; 1; 2]
5.	$A = B . \vee . A \varepsilon \mathbf{el}(\mathbf{Un} \dot{-} B) :$	[4; $A7$; 2]
6.	$A \varepsilon \mathbf{el}(\mathbf{Un} \dot{-} B) .$	[5; 3]
7.	$\mathbf{Un} \dot{-} B \varepsilon \mathbf{Un} \dot{-} B ,$	[$A1$; 6]
8.	$\mathbf{Un} \varepsilon (\mathbf{Un} \dot{-} B) + B .$	[$D5$; 7]
9.	$(\mathbf{Un} \dot{-} B) \varepsilon \mathbf{ex}(B) .$	[$D4$; 8]
10.	$A \varepsilon \mathbf{ex}(B) .$	[$A13$; 9; 6]
11.	$\sim (A \varepsilon \mathbf{ex}(B)) .$	[$A14$; 2]
	$B = A$	[10; 11]

$B2 \quad [A\,B] .\therefore A \varepsilon A : [B] : B \varepsilon \mathbf{el}(A) . \supset . B = A : B \varepsilon B . \sim (A \varepsilon \mathbf{el}(B)) . \supset .$
$\quad A \varepsilon \mathbf{el}(\mathbf{Un} \dot{-} B)$

PR $[A\,B] .\therefore \mathrm{Hp}(4) : \supset :$

$[\exists C] .$

5. $C \varepsilon \mathbf{el}(A)\,.$ $[A15;\ 1;\ 3;\ 4]$

6. $C \varepsilon \mathbf{ex}(B)\,.$

7. $C = A\,.$ $[2;\ 5]$

8. $A \varepsilon \mathbf{ex}(B)\,.$ $[6;\ 7]$

9. $\mathbf{Un} = (\mathbf{Un} \dot{-} B) + B.$ $[A19;\ 3]$

10. $A \varepsilon \mathbf{el}(\mathbf{Un})\,.$ $[A18;\ 1]$

11. $A \varepsilon \mathbf{el}((\mathbf{Un} \dot{-} B) + B)\,.$ $[9,\ 10]$

$A \varepsilon \mathbf{el}(\mathbf{Un} \dot{-} B)\,.$ $[A17;\ 11;\ 8]$

B3 $[A] :: A \varepsilon A : [B] : B \varepsilon \mathbf{el}(A) . \supset . B = A : \equiv .\therefore A \varepsilon A .\therefore [B] .\therefore B \varepsilon B . \supset : A \varepsilon \mathbf{el}(B)$ $. \vee . A \varepsilon \mathbf{el}(\mathbf{Un} \dot{-} B)$ $[B1;\ B2]$

B4 $[A\,B] :: [B\,C] .\therefore A \varepsilon B + C . \supset : A = B . \vee . A = C .\therefore B \varepsilon \mathbf{el}(A) . \sim (B = A) .$ $\supset . B = A$

PR $[A\,B] :: \mathrm{Hp}(3) .\therefore \supset .\therefore$

4. $\sim (B \varepsilon \mathbf{ex}(A)) .\therefore$ $[A14;\ 2]$

$[\exists C] .\therefore$

5. $A = B + C\,.$ $[A16;\ 2;\ 3]$

6. $A \varepsilon B + C.$ $[Df1;\ 5]$

7. $B \varepsilon \mathbf{ex}(C) :$ $[D4;\ 6]$

8. $A = B . \vee . A = C :$ $[2;\ 6]$

9. $A = B . \vee . B \varepsilon \mathbf{ex}(A) .\therefore$ $[8;\ 7]$

$B = A$ $[9;\ 4]$

B5 $[A\,B\,C] .\therefore [B] : B \varepsilon \mathbf{el}(A) . \supset . B = A : A \varepsilon B + C . \supset : A = B . \vee . A = C$

PR $[A\,B\,C] :: \mathrm{Hp}(2) : \supset .\therefore$

3. $A \varepsilon \mathbf{Kl}(B \cup C)\,.$ $[D4;\ 2]$

4. $B \varepsilon \mathbf{ex}(C)\,.$ $[D4;\ 2]$

5. $B \varepsilon \mathbf{el}(A)\,.$ $[T1;\ 4;\ D1;\ 3]$

6. $C \varepsilon \mathbf{el}(A)\,.$ $[D3;\ 4;\ T1;\ D1;\ 3]$

7. $A \varepsilon \mathbf{el}(A) .\therefore$ $[T1;\ 2;\ A5]$

$[\exists E\,F] .\therefore$

8. $E \varepsilon B \cup C.$

9. $F \varepsilon \mathbf{el}(E)\,.$ $[D1;\ 3;\ 7]$

10. $F \varepsilon \mathbf{el}(A)\,.$

11. $F = A.$ $[1;\ 10]$

12. $A \varepsilon \mathbf{el}(E) :$ $[9;\ 11]$

13. $E = B . \vee . E = C .\therefore$ $[8;\ T1;\ 4;\ 6]$

14. $A \varepsilon \mathbf{el}(B) . \vee . A \varepsilon \mathbf{el}(C) :$ $[12;\ 13]$

$A = B . \vee . A = C$ $[14;\ A7;\ 5;\ 6]$

B6 $[A] :: A \varepsilon A : [B] : B \varepsilon \mathbf{el}(A) . \supset . B = A : \equiv .\therefore A \varepsilon A .\therefore [B\,C] .\therefore A \varepsilon B + C.$ $\supset : A = B . \vee . A = C$ $[B4;\ B5]$

B7 $[A\,B] :: [B\,C] .\therefore A \varepsilon \mathbf{el}(B + C) . \supset : A \varepsilon \mathbf{el}(B) . \vee . A \varepsilon \mathbf{el}(C) .\therefore B \varepsilon \mathbf{el}(A) .$ $\sim (B = A) . \supset . B = A$

PR $[A\,B] :: \mathrm{Hp}(3) .\therefore \supset .\therefore$

4. $A \varepsilon \mathbf{el}(A) .\therefore$ $[A1;\ A5;\ 2]$

$[\exists C] .\therefore$

5. $A = B + C\,.$ $[A16;\ 2;\ 3]$

6. $B \varepsilon \mathbf{ex}(C)\,.$ $[Df1;\ 5;\ D4]$

7. $A\,\varepsilon\,\mathsf{el}(B+C):$ [5; 4]
8. $A\,\varepsilon\,\mathsf{el}(B).\mathsf{v}.A\,\varepsilon\,\mathsf{el}(C):$ [1; 7]
9. $B=A.\mathsf{v}.B\,\varepsilon\,\mathsf{el}(C):$ [8; *A7*; 2; *A2*; 2]
10. $B=A.\mathsf{v}.\sim(B\,\varepsilon\,\mathsf{ex}(C)):$ [9; *A14*]
$B=A$ [10; 6]

B8 $[A\,B\,C]\therefore[B]:B\,\varepsilon\,\mathsf{el}(A).\supset.B=A:A\,\varepsilon\,\mathsf{el}(B+C).\supset:A\,\varepsilon\,\mathsf{el}(B).\mathsf{v}.A\,\varepsilon\,\mathsf{el}(C)$

PR $[A\,B\,C]::\mathrm{Hp}(2)\therefore\supset\therefore$

3. $B+C\,\varepsilon\,B+C.$ [*A1*; 2]
4. $B\,\varepsilon\,\mathsf{ex}(C).$ [*D4*; 3]
5. $B+C=\mathsf{Kl}(B\cup C).$ [*D4*; *A3*; *T3*; 3]
6. $A\,\varepsilon\,\mathsf{el}(\mathsf{Kl}(B\cup C)).$ [*E1*; 2; 5]
7. $\mathsf{Kl}(B\cup C)\,\varepsilon\,\mathsf{Kl}(B\cup C)\therefore$ [*A1*; 6]
$[\exists E\,F]\therefore$
8. $E\,\varepsilon\,B\cup C.$
9. $F\,\varepsilon\,\mathsf{el}(E).$ [*D1*; 6; 7]
10. $F\,\varepsilon\,\mathsf{el}(A).$
11. $F=A.$ [1; 10]
12. $A\,\varepsilon\,\mathsf{el}(E):$ [9; 8]
13. $E=B.\mathsf{v}.E=C\therefore$ [8; *Df4*; *T2*; 4; *D3*]
$A\,\varepsilon\,\mathsf{el}(B).\mathsf{v}.A\,\varepsilon\,\mathsf{el}(C)$ [*E1*; 13; 12]

B9 $[A]::A\,\varepsilon\,A:[B]:B\,\varepsilon\,\mathsf{el}(A).\supset.B=A:\equiv\therefore A\,\varepsilon\,A\therefore[B\,C]\therefore A\,\varepsilon\,\mathsf{el}(B+C).$
$\supset:A\,\varepsilon\,\mathsf{el}(B).\mathsf{v}.A\,\varepsilon\,\mathsf{el}(C)$ [*B8*; *B7*]

B10 $[A\,B]\therefore[a]:B\,\varepsilon\,\mathsf{Kl}(a).\supset.B\,\varepsilon\,a:B\,\varepsilon\,\mathsf{el}(A).\sim(B=A).\supset.B=A$

PR $[A\,B]\therefore\mathrm{Hp}(3):\supset.$

4. $B\,\varepsilon\,\mathsf{pr}(A).$ [*D2*; 2; 3]
5. $A\,\varepsilon\,\mathsf{Kl}(\mathsf{pr}(A)).$ [*A12*; 4]
6. $A\,\varepsilon\,\mathsf{pr}(A).$ [1; 5]
$B=A$ [*A10*; 6]

B11 $[A\,a]\therefore[B]:B\,\varepsilon\,\mathsf{el}(A).\supset.B=A:A\,\varepsilon\,\mathsf{Kl}(a).\supset.A\,\varepsilon\,a$

PR $[A\,a]\therefore\mathrm{Hp}(2):\supset.$

$[\exists B].$
3. $B\,\varepsilon\,a.$ [*T1*; *A5*; 2; *D1*]
4. $B\,\varepsilon\,\mathsf{el}(A).$ [*D1*; 2; 3]
$A\,\varepsilon\,a$ [1; 4; 3]

B12 $[A]\therefore A\,\varepsilon\,A:[B]:B\,\varepsilon\,\mathsf{el}(A).\supset.B=A:\equiv:A\,\varepsilon\,A:[a]:A\,\varepsilon\,\mathsf{Kl}(a).\supset.A\,\varepsilon\,a$
[*B11*; *B10*]

B13 $[A\,C]\therefore[a]:A\,\varepsilon\,\mathsf{el}(\mathsf{Kl}(a)).\supset.[\exists B].B\,\varepsilon\,a.A\,\varepsilon\,\mathsf{el}(B):C\,\varepsilon\,\mathsf{el}(A).\sim(C=A).$
$\supset.C=A$

PR $[A\,C]\therefore\mathrm{Hp}(3):\supset.$

4. $C\,\varepsilon\,\mathsf{pr}(A).$ [*D2*; 2; 3]
5. $A\,\varepsilon\,\mathsf{el}(\mathsf{Kl}(\mathsf{pr}(A))).$ [*A12*; *A8*; 4]
$[\exists B].$
6. $B\,\varepsilon\,\mathsf{pr}(A).$
7. $A\,\varepsilon\,\mathsf{el}(B).$ [1; 5]
8. $A=B.$ [*D2*; *A7*; 6; 7]
$C=A$ [*D2*; 6; 8]

B14 $[A\,a] \therefore [B] : B \varepsilon \mathbf{el}(A) . \supset . B = A : A \varepsilon \mathbf{el}(\mathbf{Kl}(a)) . \supset . [\exists B] . B \varepsilon a . A \varepsilon \mathbf{el}(B)$

PR $[A\,a] \therefore \mathrm{Hp}(2) : \supset .$

3. $\mathbf{Kl}(a) \varepsilon \mathbf{Kl}(a) .$ [*A1*; 2]

$[\exists B F] .$

4. $B \varepsilon a .$
5. $F \varepsilon \mathbf{el}(B) .$ [*D1*; 3; 2]
6. $F \varepsilon \mathbf{el}(A) .$
7. $F = A .$ [1; 6]

$[\exists B] . B \varepsilon a . A \varepsilon \mathbf{el}(B)$ [4; 5; 7]

B15 $[A] \therefore A \varepsilon A : [B] : B \varepsilon \mathbf{el}(A) . \supset . B = A : \equiv : A \varepsilon A : [a] : A \varepsilon \mathbf{el}(\mathbf{Kl}(a)) .$
$\supset . [\exists B] . B \varepsilon a . A \varepsilon \mathbf{el}(B)$ [*B14*; *B13*]

B16 $[A] \therefore A \varepsilon A : [B] : B \varepsilon \mathbf{el}(A) . \supset . B = A : \equiv . A \varepsilon A . [B] . \sim (B \varepsilon \mathbf{pr}(A))$
[*A11*; *D2*]

Since *B3*, *B6*, *B9*, *B12*, *B15* and *B16* are theorems of mereology, as it was proved above, we obtained a proof that definitions *D*I and *D*I.1 - *D*I.6 define exactly the same concept, viz. of mereological atom.

3.5 It will be shown now that in the field of general mereology the atomistic axioms presented in **3.3**, point (B), viz. the formulas *V* and *V1* - *V6* are mutually equivalent. Again, *cf.* **3.4**, let us assume axiom *A* and its consequences which are given in **2**. And, we introduce to the system definitions *D*I and *D*II in order to define "**atm**" and "**at**" in the field of mereology. Then:

3.5.1 $\{V\} \rightleftarrows \{V1\}$. A proof, by *A1*, *D*I and *D*II, is obvious.

3.5.2 $\{V1\} \rightleftarrows \{V2\}$. (a) Assume *V1*. Then:

Z1 $[B\,C\,D] : C \varepsilon \mathbf{Kl}(\mathbf{atm}) . D \varepsilon \mathbf{el}(B) . \supset . [\exists F] . F \varepsilon \mathbf{el}(D) . F \varepsilon \mathbf{el}(C)$

PR $[B\,C\,D] : \mathrm{Hp}(2) . \supset .$

$[\exists F] .$

3. $F \varepsilon \mathbf{at}(D)$ [*T1*; *V1*; 2]
4. $F \varepsilon \mathbf{atm} .$ [*D*II; 3]
5. $F \varepsilon \mathbf{el}(D) .$ [*D*II; 3]
6. $F \varepsilon \mathbf{el}(C) .$ [*D1*; 1; 4]

$[\exists F] . F \varepsilon \mathbf{el}(D) . F \varepsilon \mathbf{el}(C)$ [5; 6]

V2 $[A] : A \varepsilon A . \supset . \mathbf{Un} = \mathbf{Kl}(\mathbf{atm})$

PR $[A] :\cdot: \mathrm{Hp}(1) . \supset ::$

2. $A \varepsilon \mathsf{V} ::$ [*Df2*; 1]

$[\exists B] ::$

3. $B \varepsilon \mathbf{Kl}(\mathsf{V}) .$ [*A4*; 2]
4. $B = \mathbf{Un}.$ [*A3*; *T3*; 3; *D6*]

$[\exists C] .$

5. $C \varepsilon \mathbf{atm} \therefore$ [*V1*; *D*II; 1]

$[\exists C] \therefore$

6. $C \varepsilon \mathbf{Kl}(\mathbf{atm}) :$ [*A4*; 5]
7. $[D] : D \varepsilon \mathbf{el}(B) . \supset . [\exists F] .$
$F \varepsilon \mathbf{el}(D) . F \varepsilon \mathbf{el}(C) :$ [*Z1*; 6]

8.	$B\,\varepsilon\,\mathsf{el}(C)\,.$	[$T1$; 3; $A6$; 7]
9.	$C\,\varepsilon\,\mathsf{el}(B)\,.$	[$T1$; 6; $D1$; 3]
10.	$C = B::$	[$A7$; 9; 10]
	$\mathsf{Un} = \mathsf{Kl}(\mathsf{atm})$	[$A3$; $T3$; 6; 10; 4]

Thus, $\{V1\} \rightarrow \{V2\}$. (b) Now, assume $V2$. Then:

$V1$	$[A]:A\,\varepsilon\,A\,.\supset.\,[\exists B].\,B\,\varepsilon\,\mathsf{at}(A)$	
PR	$[A]:\mathrm{Hp}(1)\,.\supset.$	
2.	$A\,\varepsilon\,\mathsf{el}(\mathsf{Un}).$	[$A18$, 1]
3.	$\mathsf{Un} = \mathsf{Kl}(\mathsf{atm})\,.$	[$V2$; 1]
4.	$A\,\varepsilon\,\mathsf{el}(\mathsf{Kl}(\mathsf{atm}))\,.$	[2; 3]
5.	$\mathsf{Kl}(\mathsf{atm})\,\varepsilon\,\mathsf{Kl}(\mathsf{atm})\,.$	[$A1$; 4]
	$[\exists E\,B].$	
6.	$E\,\varepsilon\,\mathsf{atm}\,.$	
7.	$B\,\varepsilon\,\mathsf{el}(E)\,.$	[$D1$; 5; 4]
8.	$B\,\varepsilon\,\mathsf{el}(A)\,.$	
9.	$B = E\,.$	[DI; 6; 7]
	$[\exists B].\,B\,\varepsilon\,\mathsf{at}(A)$	[DII; 6; 9; 8]

Thus, $\{V2\} \rightarrow \{V1\}$. Hence, by (a), $\{V1\} \rightleftarrows \{V2\}$.

3.5.3 $\{V1\} \rightleftarrows \{V3\}$. (a) Assume $V1$, Then:

$Z1.$	$[A\,B\,D]:B\,\varepsilon\,\mathsf{Kl}(\mathsf{at}(A))\,.\,D\,\varepsilon\,\mathsf{el}(B)\,.\supset.\,[\exists F].\,F\,\varepsilon\,\mathsf{el}(D)\,.\,F\,\varepsilon\,\mathsf{el}(A)$	
PR	$[A\,B\,D]:\mathrm{Hp}(2)\,.\supset.$	
	$[\exists E\,F].$	
3.	$E\,\varepsilon\,\mathsf{at}(A)\,.$	
4.	$F\,\varepsilon\,\mathsf{el}(E)\,.$	[$D1$; 1; 2]
5.	$F\,\varepsilon\,\mathsf{el}(D)\,.$	
6.	$F\,\varepsilon\,\mathsf{el}(A)\,.$	[DII; $A2$; 4; 3]
	$[\exists F].\,F\,\varepsilon\,\mathsf{el}(D)\,.\,F\,\varepsilon\,\mathsf{el}(A)$	[5; 6]

$Z2$	$[A\,B\,C]:\,B\,\varepsilon\,\mathsf{Kl}(\mathsf{at}(A))\,.\,D\,\varepsilon\,\mathsf{el}(A)\,.\supset.\,[\exists F].\,F\,\varepsilon\,\mathsf{el}(D)\,.\,F\,\varepsilon\,\mathsf{el}(B)$	
PR	$[A\,B\,C]\therefore\mathrm{Hp}(2)\,.\supset:$	
	$[\exists K]:$	
3.	$K\,\varepsilon\,\mathsf{at}(D)\,.$	[$T1$; $V1$; 2]
4.	$K\,\varepsilon\,\mathsf{at}(A)\,.$	[DII; $A2$; 2; 3]
5.	$K\,\varepsilon\,\mathsf{el}(B)\,.$	[$D1$; 1; 4]
	$[\exists E\,F].$	
6.	$E\,\varepsilon\,\mathsf{at}(A)\,.$	
7.	$F\,\varepsilon\,\mathsf{el}(E)\,.$	[$D1$; 1; 5]
8.	$F\,\varepsilon\,\mathsf{el}(K)\,.$	
9.	$F\,\varepsilon\,\mathsf{el}(D)\,.$	[DII; $A2$; 8; 3]
10.	$F = E\,.$	[DII; DI; 6; 7]
11.	$F\,\varepsilon\,\mathsf{el}(B):$	[DI; 1; 6; 10]
	$[\exists F].\,F\,\varepsilon\,\mathsf{el}(D)\,.\,F\,\varepsilon\,\mathsf{el}(B)$	[9; 11]

$V3$	$[A]:A\,\varepsilon\,A\,.\supset.\,A\,\varepsilon\,\mathsf{Kl}(\mathsf{at}(A))$	
PR	$[A]::\mathrm{Hp}(1)\,.\supset\therefore$	
	$[\exists C].$	

2. $C \varepsilon \mathbf{at}(A) .\therefore$ [$V1$; 1]

$[\exists B] .\because$

3. $B \varepsilon \mathbf{Kl}(\mathbf{at}(A)):$ [$A4$; 2]

4. $[D]: D \varepsilon \mathbf{el}(B) . \supset . [\exists F] . F \varepsilon \mathbf{el}(D) . F \varepsilon \mathbf{el}(A):$ [$Z1$; 3]

5. $[D]: D \varepsilon \mathbf{el}(A) . \supset . [\exists F] . F \varepsilon \mathbf{el}(D) . F \varepsilon \mathbf{el}(B):$ [$Z2$; 3]

6. $B \varepsilon \mathbf{el}(A) .$ [$T1$; $A6$; 3; 4]

7. $A \varepsilon \mathbf{el}(B) .$ [$A6$; 1; 5]

8. $A = B .\because$ [$A7$; 7; 6]

$A \varepsilon \mathbf{Kl}(\mathbf{at}(A))$ [3; 8]

Thus, $\{V1\} \rightarrow \{V3\}$. (b) Now, assume $V3$. Then:

$V1$ $[A]: A \varepsilon A . \supset . [\exists B] . B \varepsilon \mathbf{at}(A)$

PR $[A]: \mathrm{Hp}(1) . \supset .$

2. $A \varepsilon \mathbf{el}(A) .$ [$A5$; 1]

3. $A \varepsilon \mathbf{Kl}(\mathbf{at}(A)) .$ [$V1$; 1]

$[\exists B] . B \varepsilon \mathbf{at}(A)$ [$D1$; 3; 2]

Thus, $\{V3\} \rightarrow \{V1\}$. Hence, by (a), $\{V1\} \rightleftarrows \{V3\}$.

3.5.4 $\{V1\} \rightleftarrows \{V4\} \rightleftarrows \{V5\}$. (a) Assume $V1$. Then:

$Z1$ $[A B D] .\because [C]: C \varepsilon \mathbf{at}(A) . \supset . C \varepsilon \mathbf{at}(B): D \varepsilon \mathbf{el}(A) . \supset . [\exists F] . F \varepsilon \mathbf{el}(D) . F \varepsilon \mathbf{el}(B)$

PR $[A B D] .\because \mathrm{Hp}(1): \supset .$

$[\exists F] .$

3. $F \varepsilon \mathbf{at}(D) .$ [$T1$; $V1$; 2]

4. $F \varepsilon \mathbf{at}(A) .$ [DII; $A2$; 3; 2]

5. $F \varepsilon \mathbf{at}(B) .$ [1; 4]

$[\exists F] . F \varepsilon \mathbf{el}(D) . F \varepsilon \mathbf{el}(B)$ [DII; 3; 5]

$V4$ $[A B] .\because A \varepsilon A : [C]: C \varepsilon \mathbf{at}(A) . \supset . C \varepsilon \mathbf{at}(B): \supset . A \varepsilon \mathbf{el}(B)$ [$Z1$; $A6$]

Thus, $\{V1\} \rightarrow \{V4\}$. (b) Assume $V4$. Then:

$V5$ $[A B] .\because A \varepsilon A . B \varepsilon B : [C]: C \varepsilon \mathbf{at}(A) . \equiv . C \varepsilon \mathbf{at}(B): \supset . A = B$ [$V4$; $A7$]

Thus, $\{V4\} \rightarrow \{V5\}$. (c) Assume $V5$. Then:

$Z1$ $[A B C]: C \varepsilon \mathbf{el}(A) . B \varepsilon \mathbf{at}(C) . \supset . B \varepsilon \mathbf{at}(A)$ [DII; $A2$]

$Z2$ $[A] :: A \varepsilon A : [B]: B \varepsilon \mathbf{el}(A) . \supset . [\exists C] . C \varepsilon \mathbf{el}(B) . \sim (C = B): \supset .\therefore [\exists B] .\because B \varepsilon \mathbf{el}(A): [C]: C \varepsilon \mathbf{el}(B) . \supset . C = B$

PR $[A] :: \mathrm{Hp}(2): \supset .\therefore$

3. $A \varepsilon \mathbf{el}(A) .\therefore$ [$A5$; 1]

$[\exists C] .\because$

4. $C \varepsilon \mathbf{el}(A) .$

5. $\sim (C = A):$ [2; 3]

$[\exists B]:$

6. $B \varepsilon \mathbf{at}(A) . \sim (B \varepsilon \mathbf{at}(C)) . \vee . \sim (B \varepsilon \mathbf{at}(A)) . B \varepsilon \mathbf{at}(C):$ [$T1$; $V5$; 1; 4; 5]

7. $B\varepsilon \mathbf{at}(A) .\because.$ [6; $Z1$; 4]
$[\exists B] .\because. B\varepsilon \mathbf{el}(A) : [C] : C\varepsilon \mathbf{el}(B) . \supset . C = B$ [DII; DI; 7]

$V1$ $[A] : A\varepsilon A . \supset . [\exists B] . B\varepsilon \mathbf{at}(A)$ [$Z2$; DII; DI]

Thus, $\{V5\} \rightarrow \{V1\}$. Therefore, by (a) and (b), $\{V1\} \rightleftarrows \{V4\} \rightleftarrows \{V5\}$.

3.5.5 $\{V3\} \rightleftarrows \{V6\}$. (a) Assume $V3$. Then:

$V6$ $[A] :: A\varepsilon A . \supset .\because. [\exists a] .\because. A\varepsilon \mathbf{Kl}(a) : [BC] : B\varepsilon a . C\varepsilon \mathbf{el}(B) . \supset . C = B$ [$V3$; DII; DI]

Thus, $\{V3\} \rightarrow \{V6\}$. (b) Assume $V6$. Then:

$V1$ $[A] : A\varepsilon A . \supset . [\exists B] . B\varepsilon \mathbf{at}(A)$

PR $[A] :\cdot: \mathrm{Hp}(1) . \supset ::$

$[\exists a] ::$

2. $A\varepsilon \mathbf{Kl}(a) :$
3. $[BC] : B\varepsilon a . C\varepsilon \mathbf{el}(B) . \supset . C = B .\because.$ } [$V6$; 1]

$[\exists B] .\because.$

4. $B\varepsilon a .$ [$T1$; $A5$; $D1$; 2]
5. $B\varepsilon \mathbf{el}(A) :$ [$D1$; 2; 4]
6. $[C] : C\varepsilon \mathbf{el}(B) . \supset . C = B ::$ [3; 4]

$[\exists B] . B\varepsilon \mathbf{at}(A)$ [$T1$; DI; DII; 4; 6; 5]

Thus, $\{V6\} \rightarrow \{V1\}$. Hence, by **3.5.3.** and (a),

$\{V3\} \rightleftarrows \{V6\}$.

3.5.6 Points **3.5.1 - 3.5.5** imply at once that in the field of mereology taken together with definitions DI and DII: $\{V\} \rightleftarrows \{V1\} \rightleftarrows \{V2\} \rightleftarrows \{V3\} \rightleftarrows \{V4\} \rightleftarrows \{V5\} \rightleftarrows \{V6\}$.

BIBLIOGRAPHY

[1] Canty, J. T., *Leśniewski's ontology and Gödel's incompleteness theorem*, Ph.D. thesis in Philosophy, University of Notre Dame, Notre Dame, Indiana (June 1967).

[2] Canty, J. T., "Leśniewski's terminological explanations as recursive concepts," *Notre Dame Journal of Formal Logic*, vol. X, (1969), pp. 337-369.

[3] Canty, J. T., "Ontology: Leśniewski's logical language," *Foundation of Language*, vol. 5 (1969), pp. 237-257.

[4] Canty, J. T., "The numerical epsilon," *Notre Dame Journal of Formal Logic*, vol. X (1969), pp. 47-63.

[5] Clay, R. E., *Contributions to mereology*, Ph.D. thesis in Mathematics, University of Notre Dame, Notre Dame, Indiana (August 1961).

[6] Clay, R. E., "The dependence of a mereological axiom," *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 471-472.

[7] Lejewski, C., "A note on a problem concerning the axiomatic foundations of mereology," *Notre Dame Journal of Formal Logic*, vol. IV (1962), pp. 135-139.

[8] Lejewski, C., "Consistency of Leśniewski's mereology," *The Journal of Symbolic Logic*, vol. 34 (1969), pp. 321-328.

[9] Leśniewski, S., "O podstawach matematyki," (On the foundations of mathematics), *Przegląd Filozoficzny* (*Philosophical Review*), vol. 30 (1927), pp. 164-206; vol. 31 (1928), pp. 261-301; vol. 32 (1929), pp. 60-101; vol. 33 (1930), pp. 75-105; vol. 34 (1931), pp. 142-170.

[10] Leśniewski, S., "Über die Grundlagen der Ontologie," *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie*, Classe III, vol. 23 (1930), pp. 111-132.

[11] Luschei, E. C., *The logical system of Leśniewski*, North-Holland Publishing Company, Amsterdam (1962).

[12] Schröder, E., *Vorlesungen über die Algebra der Logik* (Zweiter Band), Volume I, part 1, Leipzig, (1891), and Volume I, part 2, Leipzig (1909), Reprinted by Chelsea Publishing Company, Bronx, New York (1963).

[13] Słupecki, J., "Leśniewski's calculus of Names," *Studia Logica*, vol. 3 (1955), pp. 7-76.

[14] Sobociński, B., "L'analyse de l'antinomie russellienne par Leśniewski," *Methodos*, vol. 1 (1949), pp. 94-107, 220-228, 308-316; vol. 2 (1950), pp. 237-257.

[15] Sobociński, B., "Lattice-theoretical and mereological forms of Hauber's law," *Notre Dame Journal of Formal Logic*, vol. XII (1971), pp. 81-85.

[16] Sobociński, B., "Studies in Leśniewski's mereology," *Year book for 1954-55 of Polish Society of Arts and Sciences Abroad*, vol. 5 (1955), pp. 34-48, London (1955).

[17] Sobociński, B., "Successive Simplifications of the Axiom-System of Leśniewski's Ontology," in *Polish Logic*, 1920-1939, edited by S. McCall, Clarendon Press, Oxford (1967), pp. 188-200.

[18] Tarski, A., "On the foundations of Boolean algebra," in A. Tarski, *Logic, Semantics, Metamathematics*, Clarendon Press, Oxford (1956), pp. 320-341.

To be continued.

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