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COMPLETENESS OF THE GENERALIZED PROPOSITIONAL CALCULUS

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By a *Generalized Propositional Calculus* we mean the Classical Propositional Calculus with *any number* (countable or uncountable) of atomic formulas (propositions) $p, q, r, s, \ldots$.

In this paper we prove that the Completeness theorem for the Generalized Propositional Calculus, i.e., the statement: "*A formula of the Generalized Propositional Calculus is a theorem if and only if it is a tautology*", is equivalent to the Prime Ideal theorem for Boolean rings.

By a Boolean ring we mean a Boolean ring with more than one element and by the Prime Ideal theorem for Boolean rings we mean any of the following pairwise equivalent statements.

(1) *Every Boolean ring has a proper prime ideal.*

(2) *For every element P^* of a Boolean ring Γ such that P^* is not the multiplicative unit of Γ there exists a nontrivial homomorphism from Γ onto the two-element Boolean ring $\{0, 1\}$ which maps P^* into* 0.

(3) *Every Boolean ring with a multiplicative unit has a proper prime ideal.*

For the Generalized Propositional Calculus we choose as the primitive logical connectives the *negation* denoted by "$\sim$" and the *disjunction* denoted by "$\vee$". These primitive connectives together with the grouping symbols i.e., the parentheses "(" and ")" are used in the usual manner for forming *formulas* (propositions).

The logical connectives $\wedge, \oplus, \rightarrow$ and $\leftrightarrow$ are introduced as abbreviations given by:

$P \wedge Q$	for	$\sim(\sim P \vee \sim Q)$
$P \oplus Q$	for	$(P \wedge \sim Q) \vee (\sim P \wedge Q)$
$P \rightarrow Q$	for	$\sim P \vee Q$
$P \leftrightarrow Q$	for	$(P \rightarrow Q) \wedge (Q \rightarrow P)$

where P and Q are metalinguistic symbols (formula schemes) standing for formulas.

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The axiom schemes for the Generalized Propositional Calculus (as in the case of the Classical Propositional Calculus) are given by:

A1. $P \to (Q \to P)$
A2. $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$
A3. $(\sim Q \to \sim P) \to ((\sim Q \to P) \to Q)$

where P, Q and R are formula schemes.

The only rule of inference is *Modus Ponens* (i.e., Q is deduced from P and $P \to Q$).

The usual definition of a *formal proof* is assumed and a *theorem* is (as usual) the last formula of a formal proof.

A *tautology* is a formula whose truth value is 1 for any assignment of truth values (0 or 1) to the atomic formulas which form that formula, where the truth values of $\sim P$, $P \vee Q$, $P \wedge Q$, $P \oplus Q$, $P \to Q$ and $P \leftrightarrow Q$ are defined by the following table (in terms of the truth values of P and Q).

P	Q	$\sim P$	$P \vee Q$	$P \wedge Q$	$P \oplus Q$	$P \to Q$	$P \leftrightarrow Q$
1	1	0	1	1	0	1	1
1	0	0	1	0	1	0	0
0	1	1	1	0	1	1	0
0	0	1	0	0	0	1	1

Let Σ be the set of all formulas. A function f from Σ into $\{0,1\}$ is called a *truth function* if and only if for every formula P and Q

(i) $f(P) \neq f(\sim P)$
(ii) $f(P \vee Q) = 1$ *if and only if* $f(P) = 1$ *or* $f(Q) = 1$

In terms of truth functions Lemma 1 follows from the definition of a tautology.

Lemma 1. *A formula P is a tautology if and only if $f(P) = 1$ for every truth function f.*

Lemma 2. *Every theorem of the Generalized Propositional Calculus is a tautology.*

Proof. The axioms A1, A2, A3 are readily shown to be tautologies. On the other hand, if P and $P \to Q$ are tautologies then for every truth function f we see that Lemma 1 asserts $f(P) = f(P \to Q) = 1$. Hence $f(Q) = 1$ for every truth function f and thus Q is a tautology. Now since every theorem is an axiom or is deduced from the axioms by Modus Ponens, we see that every formula of a formal proof (and hence, every theorem) is a tautology.

For every formula P and Q we write $P \equiv Q$ if and only if $P \leftrightarrow Q$ is a theorem. Then, clearly, $\equiv$ is an equivalence relation in the set Σ of all formulas and $\equiv$ partitions Σ. Let Γ be the set of the resulting equivalence classes. As usual, for every formula P, we let $[P]$ denote the equivalence class such that $P \in [P]$.

We define addition "+" and multiplication "·" in Γ as follows:

$$[P]+[Q]=[P\oplus Q] \quad \text{and} \quad [P]\cdot[Q]=[P\wedge Q]$$

The above operations are well defined since if $P \equiv P'$ and $Q \equiv Q'$ then $(P\oplus Q) \equiv (P'\oplus Q')$ and $(P\cdot Q) = (P'\cdot Q')$.

Lemma 3. *$\langle \Gamma, +, \cdot\rangle$ is a Boolean ring with unit.*
Proof. Clearly, the properties of $\oplus$ and $\wedge$ imply that $\langle \Gamma, +, \cdot\rangle$ is a ring. Moreover since $(P\wedge P) \equiv P$ for every formula P, it follows that $\langle \Gamma, +, \cdot\rangle$ is a Boolean ring.

It is easy to verify that for every formula P the equivalence classes $[P\vee{\sim}P]$ and $[P\wedge{\sim}P]$ are respectively the multiplicative unit I^* and the additive zero 0^* of $\langle \Gamma, +, \cdot\rangle$. Then Γ is a Boolean ring with unit.

In what follows $\{0,1\}$ will denote the two-element Boolean ring.

Lemma 4. *If η is a nontrivial Boolean homomorphism from Γ onto $\{0,1\}$ then the function f_η from Σ onto $\{0,1\}$ given by*

$$f_\eta(P)=\eta([P])$$

is a truth function.
Proof. Clearly, $\eta(0^*)=0$ and since η is a nontrivial homomorphism $\eta([R])=1$ for some formula R. But then $\eta([R])=\eta([R]\cdot I^*)=\eta([R])\cdot\eta(I^*)=1$ implying that $\eta(I^*)=1$. But then since $[P]+[{\sim}P]=I^*$ we see that $\eta([P])+\eta([{\sim}P])=1$ and consequently, $\eta([P])\neq\eta([{\sim}P])$ for every formula P. Thus, $f_\eta(P)\neq f_\eta({\sim}P)$, as required by (i). On the other hand, $[P\vee Q]=[P]+[Q]+[P\cdot Q]$ and thus, $\eta([P\vee Q])=\eta([P])+\eta([Q])+\eta([P])\cdot\eta([Q])$. However, clearly, $\eta([P])+\eta([Q])+\eta[P])\cdot\eta([Q])=1$ if and only if $\eta([P])=1$ or $\eta([Q])=1$. Thus, $\eta([P\vee Q])=1$ if and only if $\eta([P])=1$ or $\eta([Q])=1$. Consequently, $f_\eta(P\vee Q)=1$ if and only if $f_\eta(P)=1$ or $f_\eta(Q)=1$, as required by (ii). Hence, indeed f_η is a truth function, as desired.

Proposition 1. *The Prime Ideal theorem for Boolean rings implies that every tautology is a theorem of the Generalized Propositional Calculus.*
Proof. Let P be any formula of the Generalized Propositional Calculus Σ such that P is not a theorem. Clearly, $[P]\neq I^*$. Hence, by (2) there exists a nontrivial Boolean homomorphism η from Γ onto $\{0,1\}$ such that $\eta([P])=0$. But then by Lemma 4 there exists a truth function f_η such that $f_\eta(P)=0$. Thus, P is not a tautology.

From Lemma 2 and Proposition 1 it follows:

Proposition 2. *The Prime Ideal theorem for Boolean rings implies that a formula of the Generalized Propositional Calculus is a theorem if and only if it is a tautology.*

Next, let us observe that every Boolean ring $\langle \Gamma, +, \cdot\rangle$ with a multiplicative unit I^* and additive zero 0^* gives rise to a Propositional Calculus where for every element P and Q of Γ the disjunction $P\vee Q$ and the negation ${\sim}P$ are defined respectively by:

$$P+Q+P\cdot Q \quad \text{and} \quad I^*+P$$

But then the Completeness theorem for Generalized Propositional Calculus implies that there exists always a homomorphism f from Γ onto $\{0, 1\}$ such that $f(I^*) = 1$ and $f(0^*) = 0$. Thus, f is a nontrivial homomorphism and the kernel of f is a proper prime ideal of Γ.

In view of the above, Proposition 2 and (3) we have:

Proposition 3. *The Completeness theorem for the Generalized Propositional Calculus is equivalent to the Prime Ideal theorem for Boolean rings.*

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