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A NOTE ON A LEMMA OF J. W. ADDISON

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In [2] J. W. Addison, under the assumption of the axiom of constructibility, proved the following proposition:[1]

(C^1) *There exists an* ω_1 *well-ordering* $<$ *of* N^N *such that for any subset C of* N^N *and any predicate R recursive in functions in C, the set* $\hat{\alpha}\hat{\beta}(E\beta_1)_{\beta_1<\beta}(E\alpha)(x)R(\alpha, \beta_1, \alpha, x)$ *is in* $\Sigma_2^1[C] \cap \Pi_2^1[C]$.

In his proof of (C^1) Addison used $V = L$ only in the proof of the following lemma (1.3). If we define:

(1.0) $W(\phi) \equiv \phi(i,j) = 0$ well orders N,
(1.1) ϕ_i = the ordinal number corresponding to i in the well-ordering $\phi(i,j) = 0$,
(1.2) $M(\phi, \varepsilon) \equiv W(\phi) \,\&\, \varepsilon(i,j) = 0 \equiv F'\phi_i \epsilon F'\phi_j$,

and if we let $<$ be the ω_1 well-ordering of N^N defined by

$\alpha < \beta$ if and only if the least ordinal ν such that $\omega \times \omega \cdot F'\nu = \alpha$ is less than the first ordinal μ such that $\omega \times \omega \cdot F'\mu = \beta$,

then we have:

$$\begin{aligned} L(1.3)\quad & (E\beta_1)_{\beta_1<\beta}(E\alpha)(x)R(\alpha, \beta_1, \alpha, x) \\ & \equiv (E\beta_1)[(E\phi)(E\varepsilon)[M(\phi,\varepsilon) \,\&\, (Ei)[\omega \times \omega \cdot F'\phi_i = \beta_1] \\ & \quad \&\sim (Ei)[\omega \times \omega \cdot F'\phi_i = \beta]] \,\&\, (E\alpha)(x)R(\alpha, \beta_1, \alpha, x)] \\ & \equiv (\phi)(\varepsilon)[M(\phi, \varepsilon) \,\&\, (Ei)[\omega \times \omega \cdot F'\phi_i = \beta] \rightarrow \\ & \quad (E\beta_1)[\beta_1 \neq \beta \,\&\, (Ei)[\omega \times \omega \cdot F'\phi_i = \beta_1] \\ & \quad \&(E\alpha)(x)R(\alpha, \beta_1, \alpha, x)]]. \end{aligned}$$

We will show that L(1.3) can be proved under the weaker assumption that all real numbers are constructible ($N^N \subset L$) and that in fact L(1.3) is equivalent to $N^N \subset L$.[2] Thus we have the weakest assumption under which Addison's method can be used to prove (C^1).

Theorem $N^N \subset L \equiv$ *there exists an* ω_1 *well-ordering* $<$ *of* N^N *such that* L(1.3) *holds*.

Proof: The last two formulas of L(1.3) are equivalent by logic, so it

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suffices to show that $N^N \subset \mathsf{L}$ is equivalent to the fact that the first two formulas of L(1.3) are equivalent. We also note that

$$(E\beta_1)_{\beta_1 < \beta}(E\alpha)(x)R(\alpha, \beta_1, \alpha, x) \\ \equiv (E\beta_1)[(E\phi)(E\varepsilon)[M(\phi, \varepsilon) \& (Ei)[\omega \times \omega \cdot \mathrm{F}'\phi_i = \beta_1] \\ \& \sim(Ei)[\omega \times \omega \cdot \mathrm{F}'\phi_i = \beta]] \& (E\alpha)(x)R(\alpha, \beta_1, \alpha, x)]$$

says exactly the same thing as

$$\beta_1 < \beta \equiv (E\phi)(E\varepsilon)[M(\phi, \varepsilon) \& (Ei)[\omega \times \omega \cdot \mathrm{F}'\phi_i = \beta_1] \& \sim(Ei)[\omega \times \omega \cdot \mathrm{F}'\phi_i = \beta]].$$

Using (1.2) our statement becomes

$$\beta_1 < \beta \equiv (E\phi)(E\varepsilon)[W(\phi) \& [\varepsilon(i,j) = 0 \equiv \mathrm{F}'\phi_i \in \mathrm{F}'\phi_j] \\ \& (Ei)[\omega \times \omega \cdot \mathrm{F}'\phi_i = \beta_1] \& \sim(Ei)[\omega \times \omega \cdot \mathrm{F}'\phi_i = \beta]].$$

We will now show that this statement is equivalent to $N^N \subset \mathsf{L}$.

Suppose $N^N \subset \mathsf{L}$. Then let $<$ be the ω_1 well-ordering on N^N defined above. Then we have:

$\beta_1 < \beta$
$\equiv$ *the first ordinal number* ν *such that* $\omega \times \omega \cdot \mathrm{F}'\nu = \beta_1$ *is less than the first ordinal number* μ *such that* $\omega \times \omega \cdot \mathrm{F}'\mu = \beta$
$\equiv$ *there exists a well-ordering relation* ϕ *on* N *such that* $\nu \in \phi' N$ *and* $\sim(\mu \in \phi'' N)$

and then we can define $\varepsilon(i,j)$ such that

$$\varepsilon(i,j) = 0 \equiv \mathrm{F}'\phi_i \in \mathrm{F}'\phi_j$$

and $\varepsilon(i,j)$ depends only on our ϕ. Hence we have

$$\beta_1 < \beta \equiv (E\phi)(E\varepsilon)[W(\phi) \& \varepsilon(i,j) = 0 \equiv \mathrm{F}'\phi_i \in \mathrm{F}'\phi_j] \\ \& (Ei)[\omega \times \omega \cdot \mathrm{F}'\phi_i = \beta_1 \& \sim (Ei)[\omega \times \omega \cdot \mathrm{F}'\phi_i = \beta]].$$

Now suppose we have the above condition. Assume further that β is any non-constructible real number. By definition of an ω_1 ordering, there is a real number γ such that $\beta < \gamma$. But by the above condition, there exists $i \in N$ such that

$$\omega \times \omega \cdot \mathrm{F}'\phi_i = \beta.$$

But $\omega \times \omega$ is constructible ([3], 9.88 and 9.27) and $\mathrm{F}'\phi_i$ is constructible by definition. Therefore β is constructible ([3], 9.84). Hence we have a contradiction and every real number is constructible. The reader will note that this last implication could result from an application of (6.0) of [2]. In fact the proof of (6.0) is very similar to that which was used here, however by using the equivalence of formulas rather than the equality of sets we were able to get information about the well-ordering that was otherwise unavailable.

NOTES

1. We will use the notation of [1] and [2] except that we will use V = L instead of (A) for the axiom of constructibility.

2. The fact that $N^N \subset L$ is weaker than V = L can be derived from a model whose existence is stated in [4], on page 21 (attributed to K. Prikry). We will present a proof of this and related facts in a later paper.

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