Notre Dame Journal of Formal Logic
Volume IX, Number 4, October 1968

INCOMPLETENESS OF A LOGIC OF ROUTLEY'S

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In 'Some things do not exist', *Notre Dame Journal of Formal Logic*, v. VII (1966), pp. 251-276, Routley examines the relations between certain predicate logics. His system **R*** differs from the usual restricted predicate logic only in having added to it individual constants and a predicate constant 'E', read 'exist(s)', and in having assigned to its individual variables, a domain consisting of all possible things, in place of the usual domain consisting of all existing things. **R*** has a standard theory for its quantifiers, (π, Σ). Within **R***, 'E' can be used to define restricted quantifiers, $(\forall, \exists)$, in the following way.

$$(\exists x) A(x) =_{df} (\Sigma x)\ (A(x)\ \&\ E(x))$$
$$(\forall x) A(x) =_{df} \neg(\exists x) \neg A(x)$$

From these definitions, follows, $(\forall x) A(x) \equiv (\pi x)\ (E(x) \supset A(x))$.

Routley characterizes **FR*** as that subsystem of **R***, the wff of which consist of all wff (or definitional contractions of wff) which contain no quantifiers other than '$\forall$' or '$\exists$', and no individual constants, and says that **FR*** can be axiomatised by the following postulate set:

R0: *If A is truth-functionally valid, then A.*
R1': $(\forall x)\ (A \supset B) \supset .\ A \supset (\forall x)B$, *provided that individual variable x is not free in A.*
R2': $(\forall x)A\ \&\ E(y) \supset .\ \check{\mathsf{S}}^{x}_{y} A \Big| .$
RR1: $A,\ A \supset B \rightarrow B$ (modus ponens).
RR2': $A \rightarrow (\forall x)A$ (generalisation).

Routley asserts as a metatheorem: *Every theorem of* **R*** *which is (or the definitional abbreviation of which is) a wff of* **FR***, *is a theorem of* **FR***. The metatheorem does not hold for this axiomatisation of **FR***, and the attempted proof of the metatheorem is incorrect. Consider (a):

$$(\forall x)\ (E(x) \supset A(x)) \supset .\ (\forall x) A(x).$$

Received April 8, 1968

(a) is provable in **R***, and is a wff of **FR***, but it is independent of the postulate set above. (a) is proved in **R*** by generalising the truth-functionally valid $E(x) \supset (E(x) \supset A(x)) \supset. E(x) \supset A(x)$, and then distributing the quantifier to get $(\pi x)\ (E(x) \supset (E(x) \supset A(x)) \supset. (\pi x)\ (E(x) + A(x))$, which abbreviates to (a). That (a) is independent of the postulate set given above, is shown by taking each axiom or rule, and erasing any quantifiers, associated brackets and individual variables, and replacing predicate letters by statement letters, occurrences of the same predicate letter being replaced by occurrences of the same statement letter. The results of this operation are, in the case of **R0**, **R1'**, and **R2'**, truth-functionally valid, and in the case of **RR1** and **RR2'** the results of the operation preserve the property of being truth-functionally valid. But the result of this operation on (a) is $B \supset A \supset. A$, which is not truth-functionally valid.

$(\forall x) E(x)$ is also provable in **R***, (by generalising the tautology $E(x) \supset E(x)$, and abbreviating), and is also independent of the given postulate set, (since the result of applying the operation described above is A, which is not truth-functionally valid).

Adding (a) to the postulates given, results in a complete postulate set for **FR***, on the basis of which Routley's metatheorem can be proved.

The metatheorem is to be proved by showing that for a theorem B of **R***, which is a wff of **FR***, if there is a sequence of wffs of **R***, $B_1, B_2, \ldots B_n = B$, representing a proof of B, then, given this sequence, a new sequence can be constructed which constitutes a proof of B in **FR***. Alternatively stated, for every π-Σ free theorem of **R***, there is a π-Σ free proof of it, where a wff is π-Σ free if every occurrence of 'π' or 'Σ' can be definitionally replaced by an occurrence of '$\forall$' or '$\exists$' respectively.

Routley replaces **R*** and **FR*** by equivalent Gentzen sequents systems, to prove the metatheorem. **R*** is replaced by **G1R***, obtained from Kleene's **G1** by suppressing all individual constants, adding predicate constant 'E', and replacing '$\forall$', '$\exists$' by 'π', 'Σ'. **FR*** is replaced by **G1FR***, obtained from **G1** by suppressing all individual constants, adding predicate constant 'E', and replacing $\forall\rightarrow$ and $\rightarrow\exists$ by:

$$\forall\rightarrow \frac{\Gamma \rightarrow E(y), \theta \quad A(y)\ \Gamma \rightarrow \theta}{(\forall x)A(x),\ \Gamma \rightarrow \theta} \qquad \rightarrow\exists\ \frac{\Gamma \rightarrow E(y), \theta \quad \Gamma \rightarrow \theta, A(y)}{\Gamma \rightarrow (\exists x)A(x),\ \theta}$$

The 'proof' is by induction over the number of occurrences of $\forall, \exists$ in B, showing that any proof of a π-Σ free sequent B in **G1R*** can be transformed into a proof in **G1FR***. The metatheorem clearly holds for quantifier-free sequents, and it need only be shown that the last introduction of $\forall$ or $\exists$ in a proof in **G1R*** can always be achieved in **G1FR***. Considering only the last introduction of $\forall$, since the cases of $\forall$ and $\exists$ are similar, Routley correctly shows that if $\forall$ is introduced into the antecedent in the last introduction, by $\pi\rightarrow$ and definitional abbreviation, then to get something of the form $(\pi x)\ (E(x) \supset A(x))$ to abbreviate to $(\forall x)A(x)$, an extra premiss of the form $\Lambda,\ \Gamma \rightarrow E(y),\ \theta,\ \Delta$ must be supplied; and given this extra premiss, it can be used in the rule $\forall\rightarrow$ in **G1FR***.

Routley then says, 'If the last introduction of $\forall$ is introduced in the succedent, the same step will suffice in **G1FR***, and if in fact π was introduced in the succedent in **G1R***, $\forall$ could equally well have been introduced.' This is incorrect. It is true that $\rightarrow\forall$ is a derived rule in **G1R***, derived as follows:

$$
\begin{array}{ll}
1. & \Gamma \rightarrow \theta, A(b) \\
\hline
2. & \Gamma, E(b) \rightarrow \theta, A(b) \quad \text{Thinning} \\
\hline
3. & \Gamma \rightarrow \theta, E(b) \supset A(b) \quad \rightarrow \supset \\
\hline
4. & \Gamma \rightarrow \theta, (\pi x)(E(x) \supset A(x)) \quad \rightarrow \pi \\
\hline
5. & \Gamma \rightarrow \theta, (\forall x) A(x) \quad \text{Abbrev. } \forall \\
\hline
\end{array}
$$

But of course there is no π-free derivation of $\rightarrow\forall$ in **G1R***, so that no proof using the derived rule is a π-free proof.

What is needed is a Gentzen sequents system equivalent to the postulate set obtained by adding (a) to the original postulate set. This is achieved by replacing $\rightarrow\forall$ and $\exists\rightarrow$ in **G1FR*** by:

$$
(\rightarrow \forall)^* \; \frac{E(b),\ \Gamma \rightarrow \forall(b),\ \theta}{\Gamma \rightarrow (\forall x)A(x),\ \theta} \qquad (\exists \rightarrow)^* \; \frac{E(b),\ A(b),\ \Gamma \rightarrow \theta}{(\exists x)A(x) \rightarrow \theta} \quad \begin{array}{l}(b \text{ not free} \\ \text{in conclusion)}\end{array}
$$

Given these rules, the metatheorem follows: for if the last introduction of $\forall$ in the succedent is by $\rightarrow \pi$ and definitional abbreviation, then the sequent to which $\rightarrow \pi$ is applied, must be of the form $\Gamma \rightarrow E(b) \supset A(b)$, which must have come by $\rightarrow \supset$ from a sequent of the form $\Gamma, E(b) \rightarrow A(b)$, and this last sequent can be used as premiss in the rule $(\rightarrow\forall)^*$ in **G1FR***. This is illustrated in the derivation of $\rightarrow\forall$ in **G1R***, above: from line 2, line 5 follows by $(\rightarrow\forall)^*$ in **G1FR***.

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