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CREATIVE SEQUENCES AND DOUBLE SEQUENCES

Sr. M. ADRIAN CARPENTIER, O.S.U.

*Introduction.** Creative sets, creative pairs of sets, creative k-tuples for any finite k, and creative sequences of sets have been treated already by J. Myhill [8], R. M. Smullyan [11], A. H. Lachlan [4, 5], V. Vŭcković [12, 13], and J. P. Cleave [1], among others. This paper presents first of all a complete mathematical theory for sequences using the methodology of Smullyan and Vŭcković. Definitions of effective inseparability, creativity, and universality are given, and for disjoint recursively enumerable sequences these concepts are shown to be equivalent. Isomorphism of creative sequences follows immediately from universality as in previous literature.

The second part of this paper is a development of analogous theories for double sequences. Four cases arise from considering a double sequence as a square array:

$$\begin{array}{c}
A_0^0\ A_1^0\ A_2^0\ A_3^0\ A_4^0 \ldots A_i^0 \ldots \\
A_0^1\ A_1^1\ A_2^1\ A_3^1\ A_4^1 \ldots A_i^1 \ldots \\
\ldots \\
A_0^n\ A_1^n\ A_2^n\ A_3^n\ A_4^n \ldots A_i^n \ldots \\
\ldots
\end{array}$$

and viewing it from different aspects. This is best explained by considering the property of disjointness. A double sequence is: (1) **h**-*disjoint* or pairwise disjoint within each row if for each $n \varepsilon N$, $A_i^n \cap A_j^n = \phi$ whenever $i \neq j$, (2) **v**-*disjoint* or pairwise disjoint within each column if for each $i \varepsilon N$, $A_i^n \cap A_i^m = \phi$ for $n \neq m$, (3) **t**-*disjoint* or totally pairwise disjoint if $A_i^n \cap A_j^m = \phi$

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whenever $\langle n,i\rangle \neq \langle m,j\rangle$, (4) **v-h**-*disjoint* or pairwise disjoint within each column and each row if $A_j^n \cap A_j^m = \phi$ whenever ($n = m$ and $i \neq j$) or ($n \neq m$ and $i = j$). The first or **h**-case, where all conditions are stated for sets in the same row, gives uniform notions for a sequence of sequences. The **v**-case easily reduces to the **h**-case, and the **t**-case to that of a single sequence. Separate considerations for the **v-h**-case are given with some questions left open. Finally, that these cases are distinct is shown by theorem 2.7.1.

This exposition is given in the theory of general recursive functions. Throughout this paper N is the set of natural numbers with zero, and all sets are subsets of N unless otherwise specified. A sequence of sets, denoted $\langle A_i\rangle_{i\varepsilon N}$, will often be written $\langle A_i\rangle_i$ with the index set N understood; likewise $\langle\langle A_i^n\rangle_{i\varepsilon N}\rangle_{n\varepsilon N}$, meaning the sequence of sequences whose n^{th} sequence is $\langle A_i^n\rangle_i$, will be written $\langle A_i^n\rangle_{i,n}$. For the enumeration of all recursively enumerable (abbreviated r.e.) sets we use $w_0, w_1, w_2, \ldots$, where $x \varepsilon w_i \longleftrightarrow \underset{y}{\exists}\, T_1(i,x,y)$. A sequence $\langle A_i\rangle_i$ of subsets of N is said to be r.e. if there is a recursive function h such that $A_i = w_{h(i)}$ for every $i \varepsilon N$. This is equivalent to saying that the binary predicate "$x \varepsilon A_i$" is r.e. An enumeration of all r.e. sequences is obtained as follows: choose a recursive function γ by the iteration theorem such that

$$\underset{y}{\exists}\, T_2(u,v,x,y) \longleftrightarrow \underset{y}{\exists}\, T_1(\gamma(v,u),x,y); \tag{0.1}$$

then for each $e \varepsilon N$ set $W_i^e = w_{\gamma(i,e)}$, giving:

$$x \varepsilon W_i^e \longleftrightarrow \underset{y}{\exists}\, T_1(\gamma(i,e),x,y) \longleftrightarrow \underset{y}{\exists}\, T_2(e,i,x,y);$$

then $\langle W_i^0\rangle_i$, $\langle W_i^1\rangle_i$, $\langle W_i^2\rangle_i$, ... is an enumeration of all r.e. sequences. Similarly, a double sequence, $\langle A_i^n\rangle_{i,n}$, is r.e. if there is a recursive function h such that for each $n \varepsilon N$, $\langle A_i^n\rangle_i = \langle W_i^{h(n)}\rangle_i$, and this is true if and only if the predicate "$x \varepsilon A_i^n$" is r.e. Let ϕ be a recursive function satisfying

$$\underset{y}{\exists}\, T_3(u,v,w,x,y) \longleftrightarrow \underset{y}{\exists}\, T_2(\phi(u,v),w,x,y) \tag{0.2}$$

and write $W_i^{e,n}$ for $W_i^{\phi(e,n)}$. Then $\langle W_i^{0,n}\rangle_{i,n}$, $\langle W_i^{1,n}\rangle_{i,n}$, $\langle W_i^{2,n}\rangle_{i,n}$, ... is an enumeration of all r.e. double sequences.

A set A is reducible to a set B if there is a recursive function f such that $x \varepsilon A \longleftrightarrow f(x) \varepsilon B$, or equivalently, $A = f^{-1}(B)$. A function, f, reduces a sequence $\langle A_i\rangle_i$ to a sequence $\langle B_i\rangle_i$ if for each $i \varepsilon N$, f reduces A_i to B_i, and similarly for double sequences. A reduction is said to be 1-1 if the reduction function is 1-1.

Certain Greek letters will be used only for the following operators and functions: μ is the standard minimalization operator, $\mu z(P(z))$ = the minimal z such that $P(z)$; γ and ϕ will be used only for the functions defined in (0.1) and (0.2); ψ and χ are defined in lemma 1.1.1 and lemma 2.6.1, respectively, and give indices of disjoint sequences or double sequences with certain properties. It is well-known that there exist bijective primitive recursive correspondences between N and N^s for any $0 \neq s \varepsilon N$ (Ouspenski [9], thm. 19, p. 103). We will use the following notation for such a correspondence: $\tau^s : N^s \to N$, and $\tau_i^s : N \to N$ for $i = 1, 2, \ldots, s$, with $\tau^s(\tau_1^s(t), \ldots, \tau_s^s(t)) = t$ and $\tau_i^s(\tau^s(x_1, \ldots, x_s)) = x_i$ for $i = 1, \ldots, s$.

Finally, we conclude these preliminary remarks by quoting two theorems from the theory of recursive functions, which we shall use frequently.

Iteration Theorem. Given any recursively enumerable predicate Q, there is a primitive recursive 1-1 *function h such that*

$$Q(y_1, \ldots, y_m, x_1, \ldots, x_p) \longleftrightarrow \underset{y}{\exists}\, T_p(h(y_1, \ldots, y_m), x_1, \ldots, x_p, y).$$

(Kleene [3], thm. XXIII, p. 342.)

Myhill's Fixed Point Theorem. For any r.e. predicate Q of p + 2 arguments there is a 1-1 *primitive recursive function f such that*

$$Q(z, x_1, \ldots, x_p, f(z)) \longleftrightarrow \underset{y}{\exists}\, T_p(f(z), x_1, \ldots, x_p, y).$$

(Smullyan [11], p. 72.)

Part 1. **Sequences of Sets.**

1.1 Effective Inseparability.

Definition 1.1.1 A sequence $\langle A_i\rangle_i$ of sets is *effectively inseparable* (abbreviated **E.I.**) if there is a recursive function f such that $f(e) \notin \bigcup_{i=0}^{\infty} W_i^e$, whenever the r.e. sequence $\langle W_i^e\rangle_i$ satisfies:

E.I. 1) $W_i^e \cap A_i = \phi$, all $i \varepsilon N$;

E.I. 2) $W_k^e \cap W_j^e \subset \bigcup_{i=0}^{\infty} A_i$, for $k \neq j$, $k, j \varepsilon N$;

and

E.I. 3) $W_i^e \supset A_j$, for $i \neq j$, $i, j \varepsilon N$.

Theorem 1.1.1 *If a recursive function f reduces the disjoint sequence* $\langle A_i\rangle_i$ *to* $\langle B_i\rangle_i$ *and if* $\langle A_i\rangle_i$ *is* **E.I.**, *then* $\langle B_i\rangle_i$ *is also* **E.I.**

Proof. Let $\langle A_i\rangle_i$ be **E.I.** under g and let h be a recursive function such that for each $e \varepsilon N$, $\langle W_i^{h(e)}\rangle_i = \langle f^{-1}(W_i^e)\rangle_i$. Then it is easily verified that $\langle B_i\rangle_i$ is **E.I.** under $k(x) = f(g(h(x)))$.

Since effective inseparability for pairs of sets was classically considered in relation to recursive inseparability, we give the following definition and theorem.

Definition 1.1.2 A sequence $\langle A_i\rangle_i$ is *recursively separable* if there exists a r.e. sequence $\langle B_i\rangle_i$ of recursive sets, pairwise disjoint, such that $A_i \subset B_i$ for each $i \varepsilon N$ and $\bigcup_{i=0}^{\infty} B_i = N$. $\langle A_i\rangle_i$ is *recursively inseparable* (**R.I.**) if it is not recursively separable.

Theorem 1.1.2 *If a r.e. sequence* $\langle A_i\rangle_i$ *is* **E.I.**, *then it is also* **R.I.**

Proof. Let $\langle A_i\rangle_i$ be **E.I.** under the recursive function f, and suppose $\langle B_i\rangle_i$ recursively separates $\langle A_i\rangle_i$. Construct a sequence $\langle C_i\rangle_i$ with:

$$C_{2n} = A_0 \cup \ldots \cup A_{2n-1} \cup B_{2n+1} \cup A_{2n+2} \cup A_{2n+3} \cup \ldots$$
$$C_{2n+1} = A_0 \cup \ldots \cup A_{2n-1} \cup B_{2n} \cup A_{2n+2} \cup A_{2n+3} \cup \ldots;$$

that is, if r and s are indices of $\langle A_i \rangle_i$ and $\langle B_i \rangle_i$, respectively, then

$$\begin{aligned} x \varepsilon C_i \longleftrightarrow &\ \underset{n}{\exists}[i = 2n \wedge \{\underset{m}{\exists}(m \neq 2n \wedge m \neq 2n+1 \wedge x \varepsilon W_m^r) \vee x \varepsilon W_{2n+1}^s\}] \wedge \\ &\ \underset{n}{\exists}[i = 2n+1 \wedge \{\underset{m}{\exists}(m \neq 2n \wedge m \neq 2n+1 \wedge x \varepsilon W_m^r) \vee x \varepsilon W_{2n}^s\}] \\ \longleftrightarrow &\ Q(r,s,i,x) \longleftrightarrow \underset{y}{\exists}\, T_2(g(r,s),i,x,y) \longleftrightarrow x \varepsilon W_i^{g(r,s)}, \end{aligned}$$

for some recursive function g. Then by the construction of $\langle C_i \rangle_i$ and the disjointness of $\langle B_i \rangle_i$, $g(r,s)$ satisfies **E.I.** 1)-3) w.r.t. $\langle A_i \rangle_i$; hence $f(g(r,s)) \notin \bigcup_{i=0}^{\infty} W_i^{g(r,s)} = \bigcup_{i=0}^{\infty} B_i = N$, a contradiction.

Lemma 1.1.1 *There exists a primitive recursive function, ψ which has the following three properties:*

a) $\langle W_i^{\psi(e)} \rangle_i$ *is pairwise disjoint for any* $e \varepsilon N$;

b) $\bigcup_{i=0}^{\infty} W_i^{\psi(e)} = \bigcup_{i=0}^{\infty} W_i^e$, *for any* $e \varepsilon N$;

and

c) $W_i^e - \bigcup_{j \neq i} W_j^e \subset W_i^{\psi(e)} \subset W_i^e$.

Proof. Consider the predicate:

$$Q(e,i,x) \longleftrightarrow \underset{y}{\exists} \left[T_2(e,i,x,y) \wedge \overset{\max(i,y) \dot{-} 1}{\underset{j=0}{\forall}} \ \overset{\max(i,y) \dot{-} 1}{\underset{z=0}{\forall}} \sim T_2(e,j,x,z) \wedge \left\{ \overset{i \dot{-} 1}{\underset{j=0}{\forall}} \sim T_2(e,j,x,\max(i,y)) \vee i = 0 \right\} \right].$$

Choose ψ by the iteration theorem such that

$$Q(e,i,x) \longleftrightarrow \underset{y}{\exists}\, T_2(\psi(e),i,x,y) \longleftrightarrow x \varepsilon W_i^{\psi(e)}.$$

Examination of the predicate Q shows that it gives specific meaning to: "$x \varepsilon W_i^e$ before being in W_j^e for $j \neq i$," namely, for a sequence with index e and a particular $x \varepsilon N$, we have $Q(e,i,x)$ if and only if $x \varepsilon W_i^e$ and among pairs (i,y) satisfying $T_2(e,i,x,y)$ one is chosen such that $\max(i,y)$ is minimal; if more than one such minimal pair exists, the one with smallest i is chosen. Verification of a), b), and c) is straightforward.

Theorem 1.1.3 *There exists a r.e. sequence $\langle B_i \rangle_i$ of disjoint sets which is effectively inseparable.*

Proof. Consider the sequence $\langle A_i \rangle_i$ defined by: $x \varepsilon A_i \longleftrightarrow x \varepsilon W_i^x$. Since this predicate, and hence the sequence, is r.e., there is an $n \varepsilon N$ such that $\langle A_i \rangle_i = \langle W_i^n \rangle_i$. Define $\langle B_i \rangle_i$ by $B_i = W_i^{\psi(n)}$ for $i \varepsilon N$, where ψ is the function of lemma 1.1.1. Now suppose $e \varepsilon N$ satisfies **E.I.** 1)-3) w.r.t. $\langle B_i \rangle_i$ and suppose $e \varepsilon W_i^e$ for some $i \varepsilon N$. There are two cases:

a) $e \varepsilon W_i^e - \bigcup_{j \neq i} W_j^e \longleftrightarrow e \varepsilon A_i - \bigcup_{j \neq i} A_j \rightarrow e \varepsilon B_i$. Then $e \varepsilon W_i^e \cap B_i$, contradicting **E.I.** 1).

b) $e \not\in W_i^e - \bigcup_{j \neq i} W_j^e$ implies by **E.I.** 2) that $e \varepsilon B_k$ for some k. But then $B_k \subset A_k \rightarrow e \varepsilon A_k \rightarrow e \not\in W_k^e$, whereas **E.I.** 1) implies $e \varepsilon W_k^e$. Hence, if e satisfies **E.I.** 1)-3) w.r.t. $\langle B_i \rangle_i$, then $e \not\in \bigcup_{i=0}^{\infty} W_i^e$, which proves that $\langle B_i \rangle_i$ is **E.I.** under the identity.

1.2 Creative Sequences. In the classical theory, creative sets are defined as r.e. sets with productive complements; the complement of any productive set is called coproductive. To avoid repeated use of notation for complements, we phrase the definition for coproductive sequences.

Definition 1.2.1 The sequence $\langle A_i \rangle_i$ of sets is *coproductive* under the recursive function f if for any pairwise disjoint r.e. sequence $\langle W_i^e \rangle_i$ with $W_i^e \cap A_i = \phi$ for all $i \varepsilon N$, we have $f(e) \not\in \bigcup_{i=0}^{\infty} (A_i \cup W_i^e)$.

Definition 1.2.2 A sequence of sets is called *creative* if it is r.e. and coproductive.

Theorem 1.2.1 *A necessary and sufficient condition for a disjoint r.e. sequence to be* **E.I.** *is that it be creative.*

Proof. a) Assume $\langle A_i \rangle_i$ is creative under the recursive function f; let $e \varepsilon N$ satisfy **E.I.** 1)-3) w.r.t. $\langle A_i \rangle_i$. Consider the disjoint sequence $\langle W_i^{\psi(e)} \rangle_i$, where ψ is as in lemma 1.1.1. By **E.I.** 1) and lemma 1.1.1.c, we have $W_i^{\psi(e)} \cap A_i = \phi$ for all $i \varepsilon N$; hence by creativity of $\langle A_i \rangle_i$, $f(\psi(e)) \not\in \bigcup_{i=0}^{\infty} (A_i \cup W_i^{\psi(e)})$. Then property b) of lemma 1.1.1 gives $f(\psi(e)) \not\in \bigcup_{i=0}^{\infty} W_i^e$, and this proves that $\langle A_i \rangle_i$ is **E.I.** under $g = f \circ \psi$. b) Assume now that $\langle A_i \rangle_i$ is **E.I.** under a recursive function f. Let n be an index of $\langle A_i \rangle_i$, and consider the predicate:

$$Q(n,e,i,x) \longleftrightarrow x \varepsilon \left(W_i^e \cup \bigcup_{j \neq i} W_j^n \right).$$

By the iteration theorem there is a recursive function g such that

$$Q(n,e,i,x) \longleftrightarrow \underset{y}{\exists} T_2(g(n,e),i,x,y) \longleftrightarrow x \varepsilon W_i^{g(n,e)}.$$

Now for any r.e. sequence $\langle W_i^e \rangle_i$ which is disjoint and satisfies:

$$W_i^e \cap A_i = \phi \text{ for all } i \varepsilon N,$$

we will show that $g(n,e)$ satisfies **E.I.** 1)-3) w.r.t. $\langle A_i \rangle_i$, from which it follows by the assumption that $\langle A_i \rangle_i$ is **E.I.** under f that $f(g(n,e)) \not\in \bigcup_{i=0}^{\infty} W_i^{g(n,e)}$; then since $\bigcup_{i=0}^{\infty} W_i^{g(n,e)} = \bigcup_{i=0}^{\infty} (W_i^e \cup A_i)$, we have $f(g(n,e)) \not\in \bigcup_{i=0}^{\infty} (A_i \cup W_i^e)$, which proves that the sequence $\langle A_i \rangle_i$ is creative under the function h defined by $h(x) = f(g(n,x))$.

The verification of **E.I.** 1)-3) follows:

E.I. 1) $W_i^{g(n,e)} \cap A_i = \phi$ for all $i \varepsilon N$ because $W_i^e \cap A_i = \phi$ and $A_i \cap A_j = \phi$ for $i \neq j$.

E.I. 2) $W_i^{g(n,e)} \cap W_j^{g(n,e)} \subset \bigcup_{i=0}^{\infty} A_i$, for $i \neq j$, because $W_i^e \cap W_j^e = \phi$ implies

$$\left(W_i^e \cup \bigcup_{k \neq i} A_k\right) \cap \left(W_j^e \cup \bigcup_{k \neq j} A_k\right) \subset \bigcup_{i=0}^{\infty} A_i.$$

E.I. 3) $W_i^{g(n,e)} \supset A_j$ for $i \neq j$ by construction.

Observe that part a) of this proof demonstrates, in fact, the stronger statement: A coproductive sequence is strongly effectively inseparable, where we use:

Definition 1.2.3 A sequence $\langle A_i \rangle_i$ is *strongly effectively inseparable* if there is a recursive function f such that $f(e) \notin \bigcup_{i=0}^{\infty} W_i^e$ whenever the r.e. sequence $\langle W_i^e \rangle_i$ satisfies **E.I.** 1) and **E.I.** 2).

We see that a disjoint r.e. sequence is **E.I.** if and only if it is strongly **E.I.** In later sections we shall use only conditions **E.I.** 1) and 2).

Corollary 1.2.1.1 *There exists a creative sequence.*

Proof. The **E.I.** sequence of theorem 1.1.3 is disjoint and r.e.; hence by theorem 1.2.1 it is creative.

Theorem 1.2.2 *If $\langle A_i \rangle_i$ is reducible to $\langle B_i \rangle_i$ and if $\langle A_i \rangle_i$ is coproductive, then $\langle B_i \rangle_i$ is also coproductive.*

Proof. Entirely similar to theorem 1.1.1.

Theorem 1.2.3 *If a sequence $\langle A_i \rangle_i$ is coproductive under a recursive function f, then it is coproductive under a monotone increasing function f^*.*

Proof. There is a recursive function g such that for every $e \varepsilon N$

$$W_i^{g(e)} = \begin{cases} W_0^e \cup \{f(e)\}, \text{ if } i = 0 \\ W_i^e, \text{ if } i \neq 0. \end{cases}$$

(g can be obtained by applying the iteration theorem to:

$$Q(e,i,x) \longleftrightarrow (i = 0 \wedge x = f(e)) \vee x \varepsilon W_i^e.)$$

Let h be the recursive function obtained by iteration of g: $h(0,x) = x$ and $h(y+1,x) = g(h(y,x))$. Denote by A the set of all $e \varepsilon N$ such that $\langle W_i^e \rangle_i$ is disjoint and $W_i^e \cap A_i = \phi$ for all $i \varepsilon N$. Then for $e \varepsilon A$:

a) $f(e) \notin \bigcup_{i=0}^{\infty} (A_i \cup W_i^e)$;

b) $h(y,e) \varepsilon A$ for all $y \varepsilon N$; and

c) the sequence: $f(h(0,e))$, $f(h(1,e))$, $f(h(2,e))$, . . . is a non-repeating sequence of numbers all of which are outside of $\bigcup_{i=0}^{\infty} (A_i \cup W_i^e)$.

Proof of a) is immediate from the definition of coproductive. Proof of b) is by induction: $h(0,e) = e \varepsilon A$; suppose $r = h(y,e) \varepsilon A$ and let $s = h(y+1,e)$; then $s = g(r)$ and both conditions of A follow for s because they are true for r, and the only element added to any set of the sequence, namely $f(r)$, satisfies $f(r) \notin \bigcup_{i=0}^{\infty} (A_i \cup W_i^r)$. Proof of c) follows from a) and b), for if $u < v$, then $f(h(u,e)) \varepsilon W_0^{h(v,e)}$, but $f(h(v,e)) \notin \bigcup_{i=0}^{\infty} W_i^{h(v,e)}$. Now define the partial recursive functions s and t by:

$s(x,0) = f(0)$ and $s(x,y+1) = \mu z\{P(x,y+1,z)\}$

where $P(x,y,z) \longleftrightarrow f(h(z,y)) > x \wedge \overset{z}{\underset{u=0}{\forall}} \overset{z}{\underset{v=0}{\forall}} [u \neq v \rightarrow f(h(u,y)) \neq f(h(v,y))]$;

$t(x,0)$ is undefined and $t(x,y+1) = \mu z\{Q(x,y+1,z)\}$

where $Q(x,y,z) \longleftrightarrow \overset{z \dot{-} 1}{\underset{v=0}{\exists}} f(h(z,y)) = f(h(v,y)) \wedge z \neq 0 \wedge \overset{z}{\underset{v=0}{\forall}} f(h(v,y)) \leq x$.

By inspecting the defining equations for s and t we see for every $(x,y) \varepsilon N^2$ exactly one of $s(x,y)$ and $t(x,y)$ is defined. (For a given pair (x,y), s or t is defined depending on whether in generating the values of $f(h(z,y))$ for $z = 0,1,2,\ldots$ we first obtain a value greater than x or a repetition of values.) Finally, define f^* by

$$f^*(0) = f(0)$$

$$f^*(y+1) = \begin{cases} f(h(s(f^*(y),y+1),y+1)), & \text{if } s(f^*(y),\ y+1) \text{ is defined;} \\ f^*(y)+1, & \text{if } t(f^*(y),\ y+1) \text{ is defined.} \end{cases}$$

f^* is monotone increasing, for if $s(f^*(y),\ y+1) = z$, then $f^*(y+1) = f(h(z,y+1))$ and this is greater than $f^*(y)$ by definition of the function s, and in the other case the monotonicity is obvious. f^* is recursive since it is partial recursive and total. The implication:

$$e \varepsilon A \rightarrow f^*(e) \notin \bigcup_{i=0}^{\infty} (A_i \cup W_i^e) \tag{1.2.1}$$

must now be proved to show that f^* is a coproductive function for $\langle A_i \rangle_i$. If $e = 0$, then $f^*(0) = f(0)$, and (1.2.1) follows by hypothesis. If $e = y+1$ and $s(f^*(y),\ y+1)$ is defined, then $f^*(e) = f(h(s(f^*(y),\ y+1),\ y+1))$ and by property b) of the set A, $h(s(f^*(y),\ y+1),\ y+1) \varepsilon A$ whenever $y+1 \varepsilon A$; (1.2.1) then follows by property c) of the set A. If $e = y+1$ and $t(f^*(y),\ y+1)$ is defined then the sequence $f(h(0,e)),\ f(h(1,e)),\ \ldots$ has at least one repetition and hence by property c) $e \notin A$.

We give here two definitions and two theorems to show that k-tuples of sets may be treated as a special case of our theory for sequences.

Definition 1.2.4 A k-tuple of sets $\langle A_1, \ldots, A_k \rangle$ is *coproductive* under the recursive function f of k arguments if for any disjoint k-tuple $\langle w_{x_1}, \ldots, w_{x_k} \rangle$ satisfying

$$w_{x_i} \cap A_i = \phi \text{ for } 1 \leq i \leq k,$$

we have

$$f(x_1, \ldots, x_k) \notin \bigcup_{i=0}^{k} (A_i \cup w_{x_i}).$$

Definition 1.2.5 A sequence $\langle A_i \rangle_i$ is said to be coproductive w.r.t. disjoint sequences $\langle W_i^n \rangle_i$ having $W_i^n = \phi$ if $i = 0$ or $i > k$, if there is a recursive function f such that for such sequences

$$\bigcup_{i \varepsilon N} (W_i^n \cap A_i) = \phi \rightarrow f(n) \notin \bigcup_{i \varepsilon N} (A_i \cup W_i^n).$$

Theorem 1.2.4 *If a sequence $\langle A_i \rangle_i$ is coproductive, then $\langle A_{n_1}, \ldots, A_{n_k} \rangle$ is a coproductive k-tuple for any $k \geq 1$ and any $n_1, \ldots, n_k$ with $n_i \neq n_j$ for $i \neq j$.*

Proof. Without loss of generality we can assume that $(n_1, \ldots, n_k) = (1, \ldots, k)$ since we can define a recursive permutation rearranging the sequence $\langle A_i \rangle_i$ thus. Define a r.e. predicate R by: $R(r,k,i,x) \longleftrightarrow 1 \leq i \leq k \wedge x \varepsilon w_{\tau_i^k(r)}$, with τ_i^k as defined in the introduction. By the iteration theorem there exists a recursive function h satisfying: $R(r,k,i,x) \longleftrightarrow \underset{y}{\exists} T_2(h(r,k),i,x,y)$. Thus, for a k-tuple, $\langle w_{i_1}, \ldots, w_{i_k} \rangle$ and $r = \tau^k(i_1, \ldots, i_k)$, we have an index of a sequence $\langle W_i^{h(r,k)} \rangle_i$, where

$$W_i^{h(r,k)} = \begin{cases} w_{\tau_i^k(r)}, & \text{if } 1 \leq i \leq k; \\ \phi \text{ if } i = 0 \text{ or } i > k. \end{cases}$$

Now let f be a coproductive function for $\langle A_i \rangle_i$ and let $\langle w_{i_1}, \ldots, w_{i_k} \rangle$ be any disjoint k-tuple satisfying $w_{i_n} \cap A_n = \phi$ for $n = 1,2, \ldots, k$. Then, again taking $r = \tau^k(i_1, \ldots, i_k)$, the sequence $\langle W_i^{h(r,k)} \rangle_i$ is disjoint and $W_i^{h(r,k)} \cap A_i = \phi$ for all $i \varepsilon N$; hence, $f(h(r,k)) \notin \bigcup_{i=0}^{\infty} (A_i \cup W_i^{h(r,k)})$. Then for the recursive function g defined by $g(x_1, \ldots, x_k) = f(h(\tau^k(x_1, \ldots, x_k),k))$, we have $g(i_1, \ldots, i_k) \notin \bigcup_{n=1}^{k} (A_n \cup w_{i_n})$, which proves that $\langle A_1, \ldots, A_k \rangle$ is coproductive under g.

Theorem 1.2.5 *$\langle A_1, \ldots, A_k \rangle$ is a coproductive k-tuple if and only if the infinite sequence $\langle B_i \rangle_i$ where*

$$B_i = \begin{cases} A_i \text{ if } 1 \leq i \leq k \\ \phi \text{ if } i = 0 \text{ or } i > k, \end{cases}$$

is coproductive w.r.t. sequences $\langle W_i^n \rangle_i$ for which $W_i^n = \phi$ if $i = 0$ or $i > k$.

Proof. Given a coproductive function f for $\langle A_1, \ldots, A_k \rangle$ and using the function γ of (0.1), define $g(x) = f(\gamma(1,n), \ldots, \gamma(k,n))$. Then g is coproductive for $\langle B_i \rangle_i$. In the other direction, the proof is similar to that for theorem 1.2.4.

1.3 Universal Sequences. In this section we define what is meant by a universal sequence and show that for disjoint r.e. sequences universality is equivalent to creativity. We also use the property of universality to prove that any two disjoint r.e. creative sequences are isomorphic in the usual sense (definition 1.3.3).

Definition 1.3.1 A sequence $\langle A_i \rangle_i$ is *many-one universal* if every disjoint r.e. sequence is reducible to it; the sequence is 1-1 *universal*, or simply *universal*, if the reduction function can always be chosen 1-1.

Lemma 1.3.1 *Every many-one universal sequence is coproductive.*

Proof. By corollary 1.2.1.1 there exists a disjoint r.e. creative sequence; hence by theorem 1.2.2 a many-one universal sequence is coproductive.

Lemma 1.3.2 *Every disjoint coproductive sequence is* 1-1 *universal.*

Proof. Given by Vučković in [13], theorem 3.1.

Theorem 1.3.1 *A disjoint sequence is coproductive if and only if it is universal.*

Proof. Lemmas 1.3.1 and 1.3.2.

Corollary 1.3.1.1 *A disjoint sequence is* 1-1 *universal if and only if it is many-one universal.*

Corollary 1.3.1.2 *There exists a disjoint r.e. sequence which is universal.*

Proof. Theorem 1.3.1 and corollary 1.2.1.1.

Theorem 1.3.2 *Let* $\langle A_i \rangle_i$ *be a disjoint r.e. sequence. Then the following are equivalent*:

a) $\langle A_i \rangle_i$ *is creative*;
b) $\langle A_i \rangle_i$ *is* **E.I.**;
c) $\langle A_i \rangle_i$ *is universal.*

Proof. Theorems 1.2.1 and 1.3.1.

The following definitions are analogous to those of Myhill for sets in [8].

Definition 1.3.2 Two sequences are *equivalent* to each other if each is 1-1 reducible to the other.

Definition 1.3.3 Two sequences $\langle A_i \rangle_i$ and $\langle B_i \rangle_i$ are *isomorphic* if there is a recursive permutation, p, such that $p(A_i) = B_i$ for each $i \varepsilon N$.

Lemma 1.3.3 *Let f and g be* 1-1 *(primitive) recursive functions. Then there exists a* 1-1 *(primitive) recursive function h, having the following property: For any two sets $D \subset N$ and $E \subset N$, if $D = f^{-1}(E)$ and $E = g^{-1}(D)$ then $h(D) = E$ and $h^{-1}(E) = D$.*

Proof. Myhill, theorem 17 in [8].

Theorem 1.3.3 *Up to isomorphism there is only one disjoint r.e. sequence which is universal, hence also creative and* **E.I.**

Proof. By definition of universal, two disjoint r.e. sequences which are universal are equivalent. Applying lemma 1.3.3 to the reduction functions f and g, we obtain the desired isomorphism.

Part 2. **Double Sequences of Sets**

The first three sections of part 2 constitute a complete exposition of h-creative double sequences. It will be recalled from the introduction that this is a study of uniform notions for a sequence of sequences.

2.1 Uniformly Effectively Inseparable Double Sequences.

Definition 2.1.1 A double sequence $\langle A_i^n\rangle_{i,n}$ is *uniformly effectively inseparable* (abbr. **U.E.I.**) if there is a recursive function f such that $f(e)\notin \bigcup_{i=0}^{\infty} W_i^{e,n}$ for each $e,n\,\varepsilon\, N$ satisfying:

U.E.I.1) $W_i^{e,n} \cap A_i^n = \phi$ for all $i\,\varepsilon\, N$,

and

U.E.I.2) $W_k^{e,n} \cap W_j^{e,n} \subset \bigcup_{i=0}^{\infty} A_i^n$, for $k \neq j,\ k, j\,\varepsilon\, N$.

Observe that we have omitted a third condition analogous to **E.I.3)** of part 1. This definition corresponds to that of strongly effectively inseparable (def. 1.2.3); as in the case of a single sequence we shall prove that uniformly creative implies **U.E.I.** as defined here and that a weaker **U.E.I.** with the third condition on e and n is sufficient for uniformly creative. This differs from the literature (Smullyan [11] and Lachlan [4] and [5]); it is preferable for its simplicity.

Theorem 2.1.1 *There exists an* h-*disjoint r.e. double sequence which is* **U.E.I.**

Proof. Define a double sequence $\langle A_i^n\rangle_{i,n}$ by: $x\,\varepsilon\, A_i^n \longleftrightarrow x\,\varepsilon\, W_i^{x,n}$. Since this predicate is r.e., $\langle A_i^n\rangle_{i,n}$ is an r.e. double sequence; hence there is an $r\,\varepsilon\, N$ such that $\langle A_i^n\rangle_{i,n} = \langle W_i^{r,n}\rangle_{i,n}$. Let $\langle B_i^n\rangle_{i,n}$ be the h-disjoint r.e. double sequence defined by $x\,\varepsilon\, B_i^n \longleftrightarrow x\,\varepsilon\, W_i^{\psi(\phi(r,n))}$. Claim: $\langle B_i^n\rangle_{i,n}$ is **U.E.I.** under f = indentity. Assume $e,n\,\varepsilon\, N$ satisfy **U.E.I.** 1)-2), and suppose $e\,\varepsilon\, W_j^{e,n}$ for some $j\,\varepsilon\, N$. There are two cases:

a) $e\,\varepsilon\, W_j^{e,n} - \bigcup_{i\neq j} W_i^{e,n} \longleftrightarrow e\,\varepsilon\, A_j^n - \bigcup_{i\neq j} A_i^n \rightarrow e\,\varepsilon\, B_j^n$;

this contradicts **U.E.I.1)**.

b) $e\notin W_j^{e,n} - \bigcup_{i\neq j} W_i^{e,n} \rightarrow e\,\varepsilon\, B_k^n$ for some $k\,\varepsilon\, N$ by **U.E.I.2)**; then $B_k^n \subset A_k^n \rightarrow e\,\varepsilon\, A_k^n \longleftrightarrow e\,\varepsilon\, W_k^{e,n}$ and **U.E.I.1)** $\rightarrow e\notin W_k^{e,n}$; again we arrive at a contradiction; hence the supposition is false and the claim is established.

Theorem 2.1.2 *If a double sequence* $\langle A_i^n\rangle_{i,n}$ *is* **U.E.I.**, *then for each* $n \varepsilon N$ *the sequence* $\langle A_i^n\rangle_i$ *is* **E.I.**

Proof. Define a predicate R by: $R(n,k,m,i,x) \longleftrightarrow m = n \wedge x \varepsilon w_{\gamma(i,k)}$ and choose by the iteration theorem a rec. function, h, such that

$$R(n,k,m,i,x) \longleftrightarrow \underset{y}{\exists} T_3(h(n,k),m,i,x,y) \longleftrightarrow x \varepsilon w_{\gamma(i,\phi(h(n,k),m))}.$$

Thus for a given r.e. sequence with index k, $\langle W_i^k\rangle_i$, (we use again our convention of suppressing γ and ϕ), $h(n,k)$ is an index of the double sequence, $\langle W_i^{h(n,k),m}\rangle_{i,m}$, in which the n^{th} sequence is $\langle W_i^k\rangle_i$ and all other sets are empty. Now let $\langle A_i^n\rangle_{i,n}$ be **U.E.I.** under f, and for a given n let g be the rec. function given by: $g(x) = f(h(n,x))$. Claim: $\langle A_i^n\rangle_i$ is **E.I.** under g. For, suppose k satisfies **E.I.** 1)-2) w.r.t $\langle A_i^n\rangle_i$; then $h(n,k)$ satisfies **U.E.I.** 1)-2) w.r.t. $\langle A_i^n\rangle_{i,n}$; it follows that

$$g(k) = f(h(n,k)) \notin \bigcup_{i=0}^{\infty} W_i^{h(n,k),n} = \bigcup_{i=0}^{\infty} W_i^k.$$

Definition 2.1.2 A double sequence $\langle A_i^n\rangle_{i,n}$ is *recursively separable* if there exists an r.e. h-disjoint double sequence $\langle B_i^n\rangle_{i,n}$ of recursive sets such that for each $n \varepsilon N$ the sequence $\langle B_i^n\rangle_i$ recursively separates $\langle A_i^n\rangle_i$. A double sequence is *recursively inseparable* (**R.I.**) if it is not recursively separable.

Theorem 2.1.3 *If a r.e. double sequence* $\langle A_i^n\rangle_{i,n}$ *is* **U.E.I.**, *then it is also* **R.I.**

Proof. By theorem 2.1.2, for each $n \varepsilon N$, $\langle A_i^n\rangle_i$ is **E.I.**, hence also **R.I.** by theorem 1.1.2. Clearly, then, $\langle A_i^n\rangle_{i,n}$ is **R.I.**

2.2 Uniformly Creative Double Sequences.

Definition 2.2.1 The double sequence $\langle A_i^n\rangle_{i,n}$ is *uniformly coproductive* if there is a recursive function, f, such that $f(e) \notin \bigcup_{i \varepsilon N} (A_i^n \cup W_i^{e,n})$ whenever e and n satisfy:

U.C.1) $W_i^{e,n} \cap W_j^{e,n} = \phi$ for $i \neq j$, $i,j \varepsilon N$

and

U.C.2) $W_i^{e,n} \cap A_i^n = \phi$ for all $i \varepsilon N$.

A double sequence is *uniformly productive* if the double sequence of complements is uniformly coproductive.

Definition 2.2.2 A double sequence is *uniformly creative* if it is recursively enumerable and uniformly coproductive.

Theorem 2.2.1 *A necessary and sufficient condition for an* h*-disjoint r.e. double sequence to be* **U.E.I.** *is that it be uniformly creative.*

Proof. a) Assume $\langle A_i^n\rangle_{i,n}$ is uniformly creative under the rec. function f, and let $\langle W_i^{e,n}\rangle_{i,n}$ be any r.e. double sequence. Consider the predicate:

$$Q(e,n,i,x) \longleftrightarrow x \varepsilon W_{\gamma(i,\psi(\phi(e,n)))}.$$

By the iteration theorem there exists a rec. function g such that

$$Q(e,n,i,x) \longleftrightarrow \underset{y}{\exists} T_3(g(e),n,i,x,y) \longleftrightarrow x \varepsilon W_i^{g(e),n}.$$

Thus, $g(e)$ is an index of an **h**-disjoint r.e. double sequence such that for each $n \varepsilon N$:

(2.2.1) $\bigcup_{i \varepsilon N} W_i^{g(e),n} = \bigcup_{i \varepsilon N} W_i^{e,n}$, and

(2.2.2) $W_i^{e,n} - \bigcup_{j \neq i} W_j^{e,n} \subset W_i^{g(e),n} \subset W_i^{e,n}$.

(*cf.* lemma 1.1.1 where the function ψ was introduced.)
Now, for any $e,n \varepsilon N$ satisfying **U.E.I.** 1)-2):

$$W_i^{g(e),n} \cap A_i^n = \phi \text{ for all } i \varepsilon N,$$

by **U.E.I.** 1) and (2.2.2). Hence by uniform creativity, $f(g(e)) \not\varepsilon \bigcup_{i=0}^{\infty} (A_i^n \cup W_i^{g(e),n})$.
Then by (2.2.1) we have $f(g(e)) \not\varepsilon \bigcup_{i=0}^{\infty} W_i^{e,n}$, which concludes the proof that $\langle A_i^n \rangle_{i,n}$ is **U.E.I.** under the rec. function h defined by $h(x) = f(g(x))$.
b) Assume, now, $\langle A_i^n \rangle_{i,n}$ is **U.E.I.** under the rec. function f. Define a r.e. predicate Q as follows:

$$Q(u,v,n,i,x) \longleftrightarrow x \varepsilon W_i^{u,n} \vee \underset{j}{\exists}(j \neq i \wedge x \varepsilon W_j^{v,n}).$$

By the iteration theorem there is a rec. function g such that

$$Q(u,v,n,i,x) \longleftrightarrow \underset{y}{\exists} T_3(g(u,v),n,i,x,y),$$

so that for any indices $u,v \varepsilon N$: $W_i^{g(u,v),n} = W_i^{u,n} \cup \bigcup_{j \neq i} W_j^{v,n}$.
Let r be an index of $\langle A_i^n \rangle_{i,n}$. Then for any $e,n \varepsilon N$ satisfying **U.C.** 1)-2), we will show that $g(e,r),n$ satisfy **U.E.I.** 1)-2) w.r.t. $\langle A_i^n \rangle_{i,n}$:

1) $W_i^{g(e,r),n} \cap A_i^n = \phi$ for all $i \varepsilon N$ since $\langle A_i^n \rangle_{i,n}$ is **h**-disjoint and $W_i^{e,n} \cap A_i^n = \phi$ for all $i \varepsilon N$;

2) $W_i^{g(e,r),n} \cap W_j^{g(e,r),n} \subset \bigcup_{i \varepsilon N} A_i^n$, for $i \neq j$ since $W_j^{e,n} \cap W_i^{e,n} = \phi$ and the construction shows that

$$W_i^{g(e,r),n} \cap W_j^{g(e,r),n} \subset [(W_i^{e,n} \cap W_j^{e,n}) \cup \bigcup_{i \varepsilon N} A_i^n].$$

As $\langle A_i^n \rangle_{i,n}$ is **U.E.I.** under f,

$$f(g(e,r)) \not\varepsilon \bigcup_{i \varepsilon N} W_i^{g(e,r),n} = \bigcup_{i \varepsilon N} (A_i^n \cup W_i^{e,n});$$

thus $\langle A_i^n \rangle_{i,n}$ is uniformly creative under the recursive function h defined by $h(x) = f(g(x,r))$.

Corollary 2.2.1.1 *There exists an* h*-disjoint creative double sequence.*

Proof. The h-disjoint **U.E.I.** double sequence of theorem 2.1.1 is uniformly creative.

Notice that in this proof the double sequence with index $g(e,r)$ used in part b) satisfies also what we would define as **U.E.I.** 3), namely:

U.E.I. 3) $W_i^{g(e,r),n} \supset A_j^n$, for $i \neq j$.

Thus as stated at the beginning of section 2.1, a weaker form of uniform effective inseparability with **U.E.I.** 1)-3) is sufficient to imply uniform creativity and, hence, is equivalent to the definition we have given.

Theorem 2.2.2 *If a double sequence* $\langle A_i^n\rangle_{i,n}$ *is reducible to* $\langle B_i^n\rangle_{i,n}$ *and the former is uniformly coproductive, then the latter is also.*

Proof. Let f reduce $\langle A_i^n\rangle_{i,n}$ to $\langle B_i^n\rangle_{i,n}$, and let $\langle A_i^n\rangle_{i,n}$ be uniformly coproductive under g. By the iteration theorem applied to: $Q(e,n,i,x) \longleftrightarrow f(x) \varepsilon W_i^{e,n}$ we can find a rec. function k such that $W_i^{k(e),n} = f^{-1}(W_i^{e,n})$.
Define the rec. function h by: $h(x) = f(g(k(x)))$. Claim: $\langle B_i^n\rangle_{i,n}$ is uniformly coproductive under h. To prove this, let $e,n \varepsilon N$ satisfy **U.C.** 1)-2) w.r.t. $\langle A_i^n\rangle_{i,n}$. Since for any function f: $A \cap B = \phi \rightarrow f^{-1}(A) \cap f^{-1}(B) = \phi$, we have $W_i^{k(e),n} \cap W_j^{k(e),n} = \phi$ for $i \neq j$ and $W_i^{k(e),n} \cap A_i^n = \phi$ for all $i \varepsilon N$.

Then by the hypothesis that $\langle A_i^n\rangle_{i,n}$ is uniformly coproductive:

$$g(k(e)) \notin \bigcup_{i=0}^{\infty} (A_i^n \cup W_i^{k(e),n}),$$

and hence

$$h(e) = f(g(k(e))) \notin \bigcup_{i=0}^{\infty} (B_i^n \cup W_i^{e,n}).$$

Theorem 2.2.3 *If a double sequence* $\langle A_i^n\rangle_{i,n}$ *is uniformly coproductive under a recursive function* f*, then it is uniformly coproductive under a monotone increasing function* f^*.

Proof. The reasoning is the same as in part 1, theorem 1.2.3. We will give only the technical and notational changes necessary. Obtain the rec. function g by applying the iteration theorem to:

$$Q(u,n,i,x) \longleftrightarrow (i = 0 \wedge x = f(u)) \vee x \varepsilon W_i^{u,n}$$

so that

$$W_i^{g(e),n} = \begin{cases} W_0^{e,n} \cup \{f(e)\}, \text{ if } i = 0; \\ W_i^{e,n}, \text{ otherwise.} \end{cases}$$

The functions h,s,t, and f^* are defined exactly as before. For each $n \varepsilon N$ define a set A_n by:

$$A_n = \{e \varepsilon N \mid e,n \text{ satisfy U.C. 1)-2) w.r.t. } \langle A_i^n\rangle_{i,n}\}.$$

Properties a), b), and c) are essentially the same as in theorem 1.2.3 and the proofs are the same; however, there are notational differences. For any $e \varepsilon A_n$:

a) $f(e) \notin \bigcup_{i \varepsilon N} (A_i^n \cup W_i^{e,n})$;

b) $h(y,e) \varepsilon A_n$ for all $y \varepsilon N$;

c) the sequence: $f(h(0,e))$, $f(h(1,e))$, . . . is a non-repeating sequence of numbers all of which are outside of $\bigcup_{i \varepsilon N} (A_i^n \cup W_i^{e,n})$.

We must show:

(2.2.3) $e \varepsilon A_n \rightarrow f^*(e) \notin \bigcup_{i \varepsilon N} (A_i^n \cup W_i^{e,n})$.

If $e = 0$, $f^*(e) = f(e)$, and (2.2.3) follows from a). If $e = y + 1$ and $s(f^*(y), y + 1)$ is defined, then $f^*(e) = f(h(s(f^*(y), y + 1), y + 1))$; by property b), $e \varepsilon A_n \rightarrow h(s(f^*(y), y + 1), e) \varepsilon A_n$; (2.2.3) follows then from a). If $e = y + 1$ and $t(f^*(y), y + 1)$ is defined, then the sequence $f(h(0,e)), f(h(1,e)), \ldots$ has at least one repetition and hence by c), $e \notin A_n$. This completes the proof.

2.3 Uniformly Universal Double Sequences.

Definition 2.3.1 A double sequence $\langle A_i^n \rangle_{i,n}$ is said to be *uniformly universal* if every h-disjoint r.e. double sequence is 1-1 reducible to it.

Lemma 2.3.1 *Every* h-*disjoint uniformly coproductive double sequence is uniformly universal.*

Proof. Let $\langle A_i^n \rangle_{i,n}$ be uniformly coproductive under the 1-1 rec. function f. (By theorem 2.2.3 we know such a 1-1 coproductive function exists.) Let $\langle B_i^n \rangle_{i,n}$ be an h-disjoint r.e. double sequence. Applying Myhill's fixed point theorem to the predicate: $Q(v,n,i,x,u) \longleftrightarrow v \varepsilon B_i^n \wedge x = f(u)$, we obtain a 1-1 recursive function g such that:

$$Q(e,n,i,x,g(e)) \longleftrightarrow \exists_y T_3(g(e),n,i,x,y).$$

Thus,

$$e \varepsilon B_i^n \wedge x = f(g(e)) \longleftrightarrow x \varepsilon W_i^{g(e),n}.$$

Then for any $x \varepsilon N$, $g(x)$ is an index of a r.e. double sequence which is h-disjoint, in fact, $W_i^{g(x),n} = \phi$ unless $x \varepsilon B_i^n$, in which case $W_i^{g(x),n} = \{f(g(x))\}$. We shall prove that the rec. function $h = f \circ g$ reduces $\langle B_i^n \rangle_{i,n}$ to $\langle A_i^n \rangle_{i,n}$:

$$x \varepsilon B_i^n \longleftrightarrow h(x) = f(g(x)) \varepsilon A_i^n.$$

1) Suppose $x \varepsilon B_j^n$, for some $n,j \varepsilon N$. Then $W_j^{g(x),n} = \{ f(g(x))\}$, and $W_i^{g(x),n} = \phi$ for $i \neq j$. Thus

$$\bigcup_{i=0}^{\infty} (W_i^{g(x),n} \cap A_i^n) = W_j^{g(x),n} \cap A_j^n.$$

As the double sequence is uniformly coproductive,

$$\bigcup_{i=0}^{\infty} (W_i^{g(x),n} \cap A_i^n) = \phi \rightarrow f(g(x)) \notin \bigcup_{i=0}^{\infty} (A_i^n \cup W_i^{g(x),n}).$$

From these two facts follows:

$$\{f(g(x))\} \cap A_j^n = \phi \rightarrow f(g(x)) \notin W_j^{g(x),n},$$

which would contradict the supposition. We have shown: $x \varepsilon B_j^n \rightarrow f(g(x)) \varepsilon A_j^n$.

2) Now suppose $f(g(x)) \varepsilon A_j^n$ for some $n, j \varepsilon N$. If $x \notin \bigcup_{i \varepsilon N} B_i^n$, then $W_i^{g(x),n} = \phi$ for all $i \varepsilon N$, so that $W_i^{g(x),n} \cap A_i^n = \phi$, all $i \varepsilon N$ and hence by the uniform coproductivity of $\langle A_i^n \rangle_{i,n}$,

$$f(g(x)) \notin \bigcup_{i=0}^{\infty} A_i^n;$$

in particular, $f(g(x)) \notin A_j^n$, contrary to the supposition. Hence, there is an $s \varepsilon N$ such that $x \varepsilon B_s^n$; but by the previous part this implies $f(g(x)) \varepsilon A_s^n$ from which it follows that $s = j$, since $A_j^n \cap A_s^n \neq \phi$ implies $j = s$.
Thus $f(g(x)) \varepsilon A_j^n \rightarrow x \varepsilon B_j^n$.

Theorem 2.3.1 *An* **h**-*disjoint double sequence is uniformly coproductive if and only if it is uniformly universal.*

Proof. "if" by corollary 2.2.1.1 and theorem 2.2.2. "only if" by lemma 2.3.1.

Corollary 2.3.1.1 *There exists an* **h**-*disjoint r.e. double sequence which is uniformly universal.*

Definition 2.3.2 Two double sequences are *equivalent* to each other if each is 1-1 reducible to the other.

Definition 2.3.3 Two double sequences, $\langle A_i^n \rangle_{i,n}$ and $\langle B_i^n \rangle_{i,n}$ are *isomorphic* if there is a recursive permutation, p, such that $p(A_i^n) = B_i^n$, for all $i, n \varepsilon N$.

Theorem 2.3.2 *Up to isomorphism there is only one* **h**-*disjoint r.e. double sequence which is uniformly universal, hence also* **U.E.I.** *and uniformly creative.*

Proof. By lemma 1.3.3 as for theorem 1.3.3.

2.4 **v**-Creative Double Sequences.

Definition 2.4.1 For a double sequence $\langle A_i^n \rangle_{i,n}$ we define its *transpose* to be the sequence $\langle B_i^n \rangle_{i,n}$ where $B_i^n = A_n^i$, for $i, n \varepsilon N$.

Definition 2.4.2 A double sequence $\langle A_i^n \rangle_{i,n}$ is **v**-*disjoint* if for each $i \varepsilon N$, $A_i^n \cap A_i^m = \phi$ for $n \neq m$, $n, m \varepsilon N$.

Definition 2.4.3 A double sequence $\langle A_i^n \rangle_{i,n}$ is **v**-*effectively inseparable* (abbr. **v-E.I.**) if there is a recursive function f such that $f(e) \notin \bigcup_{n=0}^{\infty} W_i^{e,n}$ for each $e, i \varepsilon N$ satisfying

v-E.I. 1) $W_i^{e,n} \cap A_i^n = \phi$ for all $n \varepsilon N$, and

v-E.I. 2) $W_i^{e,n} \cap W_i^{e,m} \subset \bigcup_{n=0}^{\infty} A_i^n$ for $n \neq m$, $n, m \varepsilon N$.

Definition 2.4.4 A double sequence $\langle A_i^n \rangle_{i,n}$ is **v**-*coproductive* if there is a recursive function f such that $f(e) \notin \bigcup_{n \varepsilon N} (A_i^n \cup W_i^{e,n})$, whenever e and i satisfy:

v-C. 1) $W_i^{e,n} \cap W_i^{e,m} = \phi$, for $n \neq m$, and

v-C. 2) $W_i^{e,n} \cap A_i^n = \phi$, for all $n \varepsilon N$.

Definition 2.4.5 A double sequence is **v**-*creative* if it is r.e. and **v**-coproductive.

Definition 2.4.6 A double sequence is **v**-universal if every **v**-disjoint r.e. double sequence is 1-1 reducible to it.

The theory for the **v**-case is obtained from that for the uniform or **h**-case by the following theorem.

Theorem 2.4.1 *A double sequence is r.e.,* **v**-*disjoint,* **v-E.I.**, **v**-*coproductive;* **v**-*universal if and only if the transposed double sequence is respectively r.e.,* **h**-*disjoint,* **U.E.I.**, *uniformly coproductive, uniformly universal.*

Proof. a) Transposing a double sequence preserves recursive enumerability since a permutation of variables is a primitive recursive function.

b) Transposition carries rows into columns and hence **h**-disjointness to **v**-disjointness, and conversely.

c) Effective inseparability and coproductivity will be demonstrated by lemmas 2.4.1-2.4.3.

d) Let $\langle A_i^n \rangle_{i,n}$ be a given double sequence with $\langle B_i^n \rangle_{i,n}$ its transposed sequence, and $\langle W_i^{e,n} \rangle_{i,n}$ a **v**-disjoint r.e. double sequence with $\langle C_i^n \rangle_{i,n}$ its transposed sequence. Then a recursive function f reduces $\langle C_i^n \rangle_{i,n}$ to $\langle B_i^n \rangle_{i,n}$ if and only if it reduces $\langle W_i^{e,n} \rangle_{i,n}$ to $\langle A_i^n \rangle_{i,n}$. Thus $\langle A_i^n \rangle_{i,n}$ is **v**-universal if and only if $\langle B_i^n \rangle_{i,n}$ is uniformly universal.

Lemma 2.4.1 *There exists a recursive function g such that for any index e of a r.e. double sequence, g(e) is an index of the transposed double sequence.*

Proof. Let Q be the r.e. predicate defined by: $Q(u,v,w,x) \longleftrightarrow \underset{y}{\exists}\, T_3(u,w,v,x,y)$. By the iteration theorem there exists a recursive function g such that:

$$Q(u,v,w,x) \longleftrightarrow \underset{y}{\exists}\, T_3(g(u),v,w,x,y).$$

Hence,

$$\underset{y}{\exists}\, T_3(u,w,v,x,y) \longleftrightarrow \underset{y}{\exists}\, T_3(g(u),v,w,x,y).$$

Now, for any index e:

$$x \varepsilon W_i^{e,n} \longleftrightarrow \underset{y}{\exists}\, T_3(e,n,i,x,y);$$

thus we have:

$$x \varepsilon W_i^{g(e),n} \longleftrightarrow \underset{y}{\exists}\, T_3(g(e),n,i,x,y) \longleftrightarrow \underset{y}{\exists}\, T_3(e,i,n,x,y) \longleftrightarrow x \varepsilon W_n^{e,i},$$

which shows that $g(e)$ is an index of the transposed double sequence.

Lemma 2.4.2 *A double sequence is* **v-E.I.** *if and only if the transposed double sequence is* **U.E.I.**

Proof. Let $\langle A_i^n\rangle_{i,n}$ and $\langle B_i^n\rangle_{i,n}$ be a double sequence and its transposed double sequence.
i.) Assume $\langle B_i^n\rangle_{i,n}$ is **U.E.I.** under the recursive function f, and let $e,r \varepsilon N$ satisfy **v-E.I.** 1)-2) w.r.t. $\langle A_i^n\rangle_{i,n}$. Then $g(e)$ and r satisfy **U.E.I.** 1)-2) w.r.t. $\langle B_i^n\rangle_{i,n}$, so that

$$f(g(e)) \notin \bigcup_{n=0}^{\infty} W_n^{g(e),r} = \bigcup_{n=0}^{\infty} W_r^{e,n}.$$

Thus $\langle A_i^n\rangle_{i,n}$ is **v-E.I.** under the rec. function $f \circ g$.
ii.) Assume $\langle A_i^n\rangle_{i,n}$ is **v-E.I.** under a recursive function f, and let $\langle W_i^{e,n}\rangle_{i,n}$ be a r.e. sequence with $e,r \varepsilon N$ satisfying **U.E.I.** 1)-2) w.r.t. $\langle B_i^n\rangle_{i,n}$. Then $g(e),r \varepsilon N$ satisfy **v-E.I.** 1)-2) w.r.t. $\langle A_i^n\rangle_{i,n}$; hence

$$f(g(e)) \notin \bigcup_{i=0}^{\infty} W_r^{g(e),i} = \bigcup_{i=0}^{\infty} W_i^{e,r}.$$

Thus, $\langle B_i^n\rangle_{i,n}$ is **U.E.I.** under the rec. function $f \circ g$.

Lemma 2.4.3 *A double sequence is* **v**-*coproductive if and only if the transposed sequence is uniformly coproductive.*

Proof. Analogous to that for lemma 2.4.2.

Theorem 2.4.2 *A r.e.* **v**-*disjoint double sequence is* **v-E.I.** *if and only if it is* **v**-*creative if and only if it is* **v**-*universal.*

Proof. Theorem 2.4.1 together with theorems 2.2.1 and 2.3.1.

Theorem 2.4.3 *Up to isomorphism there is only one* **v**-*disjoint r.e.* **v**-*universal double sequence.*

Proof. Theorem 2.4.1 and theorem 2.3.2.

2.5 t-Creative Double Sequences.

Definition 2.5.1 A double sequence $\langle A_i^n\rangle_{i,n}$ is said to be **t**-*disjoint* if $A_i^n \cap A_j^m = \phi$ for $\langle i,n\rangle \neq \langle j,m\rangle$.

Lemma 2.5.1 *Any double sequence can be written as a single sequence and conversely. Further, there exist recursive functions h and k such that* $h(e)$ *is an index of the r.e. sequence obtained from the r.e. double sequence with index e and* $k(e)$ *is an index of the r.e. double sequence obtained from the r.e. sequence with index e.*

Proof. Let $\langle A_i^n\rangle_{i,n}$ be any double sequence. Define a sequence $\langle B_i\rangle_i$ by: $B_i = A_j^m$ with $j = \tau_1^2(i)$ and $m = \tau_2^2(i)$. Conversely, for any sequence $\langle B_i\rangle_i$, there is the double sequence $\langle A_i^n\rangle_{i,n}$ defined by: $A_i^n = B_{\tau^2(i,n)}$. By the iteration theorem we obtain a rec. function h such that

$$\underset{y}{\exists}\, T_3(e, \tau_2^2(i), \tau_1^2(i), x, y) \longleftrightarrow \underset{y}{\exists}\, T_2(h(e), i, x, y);$$

that is,

$$x \varepsilon W_j^{e,m} \wedge j = \tau_1^2(i) \wedge m = \tau_2^2(i) \longleftrightarrow x \varepsilon W_i^{h(e)}.$$

Likewise, there is a rec. function k such that

$$\underset{y}{\exists}\, T_2(e, \tau^2(i,n), x, y) \longleftrightarrow \underset{y}{\exists}\, T_3(k(e), n, i, x, y);$$

which means $x \varepsilon W^e_{\tau^2(i,n)} \longleftrightarrow x \varepsilon W_i^{k(e),n}$.

Definition 2.5.2 A double sequence is **t**-*effectively inseparable* (**t-E.I.**) if there is a recursive function f such that

$$f(e) \notin \bigcup_{n=0}^{\infty} \bigcup_{i=0}^{\infty} W_i^{e,n}$$

whenever e is an index of a r.e. double sequence satisfying:

t-E.I. 1) $W_i^{e,n} \cap A_i^n = \phi$, all $i,n \varepsilon N$,

and

t-E.I. 2) $W_i^{e,n} \cap W_j^{e,m} \subset \bigcup_{n=0}^{\infty} \bigcup_{i=0}^{\infty} A_i^n$, $\langle i,n\rangle \neq \langle j,m\rangle$.

Definition 2.5.3 A double sequence $\langle A_i^n\rangle_{i,n}$ is **t**-*coproductive* if there is a recursive function f such that for any **t**-disjoint r.e. double sequence $\langle W_i^{e,n}\rangle_{i,n}$:

$$\bigcup_{n=0}^{\infty} \bigcup_{i=0}^{\infty} (W_i^{e,n} \cap A_i^n) = \phi \rightarrow f(e) \notin \bigcup_{n=0}^{\infty} \bigcup_{i=0}^{\infty} (A_i^n \cup W_i^{e,n}).$$

Definition 2.5.4 A double sequence is **t**-*creative* if it is r.e. and **t**-coproductive.

Definition 2.5.5 A double sequence is **t**-*universal* if every **t**-disjoint r.e. double sequence is 1-1 reducible to it.

Theorem 2.5.1 *A double sequence is r.e.,* **t**-*disjoint,* **t-E.I.**, **t**-*coproductive, or* **t**-*universal if and only if the related single sequence of lemma* 2.5.1 *is, respectively, r.e., disjoint,* **E.I.**, *coproductive, or universal.*

Proof. Let $\langle A_i^n\rangle_{i,n}$ be a double sequence and $\langle B_i\rangle_i$ the related single sequence. Since the functions τ^2, τ_1^2, and τ_2^2 are primitive recursive, either sequence is r.e. if and only if the other is. The disjointness is straightforward, as is universality. For **E.I.**:

i) Assume $\langle B_i\rangle_i$ is **E.I.** under the rec. function f and let $\langle W_i^{e,n}\rangle_{i,n}$ be a r.e. double sequence satisfying **t-E.I.** 1)-2) w.r.t. $\langle A_i^n\rangle_{i,n}$. Then $\langle W_i^{h(e)}\rangle_i$, where h is the function from lemma 2.5.1, satisfies:

E.I. 1) $W_i^{h(e)} \cap B_i = W_j^{e,m} \cap A_j^m = \phi$, where $j = \tau_1^2(i)$, $m = \tau_2^2(i)$;

E.I. 2) $W_i^{h(e)} \cap W_j^{h(e)} = W_k^{e,m} \cap W_q^{e,p} \subset \bigcup_{n=0}^{\infty} \bigcup_{i=0}^{\infty} A_i^n = \bigcup_{i=0}^{\infty} B_i$,

for $i \neq j$, $k = \tau_1^2(i)$, $m = \tau_2^2(i)$, $q = \tau_1^2(j)$, $p = \tau_2^2(j)$.

Hence, $f(h(e)) \notin \bigcup_{i=0}^{\infty} W_i^{h(e)} = \bigcup_{n=0}^{\infty} \bigcup_{i=0}^{\infty} W_i^{e,n}$.

It follows that $\langle A_i^n \rangle_{i,n}$ is **t-E.I.** under the function $f \circ h$.

ii) Assume $\langle A_i^n \rangle_{i,n}$ is **t-E.I.** under the recursive function f, and let $\langle W_i^e \rangle_i$ be any r.e. sequence satisfying **E.I.** 1)-2) w.r.t. $\langle B_i \rangle_i$. Then $\langle W_i^{k(e),n} \rangle_{i,n}$ satisfies **t-E.I.** 1)-2) w.r.t. $\langle A_i^n \rangle_{i,n}$, so that

$$f(k(e)) \notin \bigcup_{n=0}^{\infty} \bigcup_{i=0}^{\infty} W_i^{k(e),n} = \bigcup_{i=0}^{\infty} W_i^e;$$

hence $\langle B_i \rangle_i$ is **E.I.** under the function $f \circ k$. An analogous proof shows that $\langle A_i^n \rangle_{i,n}$ is **t**-coproductive if and only if $\langle B_i \rangle_i$ is coproductive.

Theorem 2.5.2 *Let $\langle A_i^n \rangle_{i,n}$ be a r.e.* **t**-*disjoint double sequence. Then $\langle A_i^n \rangle_{i,n}$ is* **t-E.I.** *if and only if it is* **t**-*creative, if and only if it is* **t**-*universal.*

Proof. Theorem 2.5.1 and theorem 1.3.2.

Theorem 2.5.3 *Up to isomorphism there is only one* **t**-*disjoint r.e. double sequence which is* **t**-*universal.*

Proof. Theorem 1.3.3 and theorem 2.5.1.

2.6 **v-h**-Creative Double Sequences. The following development of the theory for the **v-h**-case is not so complete as that for the other cases. However, the results do include the existence of **v-h**-universal double sequences and of **v-h**-creative ones as well as the equivalence of **v-h**-creative with many-one **v-h**-universal. Isomorphism of any two **v-h**-universal double sequences follows as in the other cases.

Definition 2.6.1 A double sequence $\langle A_i^n \rangle_{i,n}$ is **v-h**-*disjoint* if for every $i \varepsilon N$, $n \varepsilon N$: $A_i^n \cap A_i^m = \phi$, for $n \neq m$, $m \varepsilon N$, and $A_i^n \cap A_j^n = \phi$, for $i \neq j$, $j \varepsilon N$.

Lemma 2.6.1 *There exists a recursive function, χ, such that for any $e \varepsilon N$, $\chi(e)$ is an index of a* **v-h**-*disjoint double sequence such that for all $\langle i,n \rangle \varepsilon N^2$:*

(2.6.1) $W_i^{e,n} - \left(\bigcup_{k \neq i} W_k^{e,n} \cup \bigcup_{k \neq n} W_i^{e,k} \right) \subset W_i^{\chi(e),n} \subset W_i^{e,n}$

(2.6.2) $\bigcup_{i \varepsilon N} \bigcup_{n \varepsilon N} W_i^{e,n} = \bigcup_{i \varepsilon N} \bigcup_{n \varepsilon N} W_i^{\chi(e),n}$.

Proof. Let Q be the r.e. predicate defined by:

$$Q(e,n,i,x) \longleftrightarrow \exists_y \Big[T_3(e,n,i,x,y) \wedge \forall_{m=0}^{\max(n,y)\dot{-}1} \forall_{z=0}^{\max(n,y)\dot{-}1} \{\sim T_3(e,m,i,x,z)\} \wedge \forall_{m=0}^{n \dot{-} 1} \{\sim T_3(e,m,i,x,\max(n,y)) \vee n = 0\} \Big].$$

By the iteration theorem there exists a rec. function h such that:

$$Q(e,n,i,x) \longleftrightarrow \underset{y}{\exists}\, T_3(h(e),n,i,x,y).$$

For a particular (e,i,x) this predicate determines the meaning of "$x \varepsilon W_i^{e,n}$ before being in $W_i^{e,m}$ for any $n \neq m$." The function h gives an index of a **v**-disjoint double sequence satisfying:

(2.6.3) $\bigcup_{n\varepsilon N} W_i^{h(e),n} = \bigcup_{n\varepsilon N} W_i^{e,n}$, for each $i\varepsilon N$,

and

(2.6.4) $W_i^{e,n} - \bigcup_{k\neq n} W_i^{e,k} \subset W_i^{h(e),n} \subset W_i^{e,n}$, for all $i\varepsilon N$.

(cf. the predicate used in lemma 1.1.1.) From theorem 2.2.1 we have the function g which from an index of a double sequence gives an index of an **h**-disjoint double sequence with properties (2.2.1) and (2.2.2) analogous to (2.6.3) and (2.6.4). Define χ by: $\chi(x) = h(g(x))$. Then for any $e\varepsilon N$, $\chi(e)$ is an index of a **v-h**-disjoint double sequence. Moreover, letting $s = g(e)$ and $\tilde{e} = \chi(e)$,

$$W_i^{e,n} - \bigcup_{k\neq i} W_k^{e,n} \subset W_i^{s,n} \subset W_i^{e,n},$$

by (2.2.2);

$$W_i^{e,n} - \left(\bigcup_{k\neq i} W_k^{e,n} \cup \bigcup_{k\neq n} W_i^{e,k}\right) \subset W_i^{s,n} - \bigcup_{k\neq n} W_i^{e,k} \subset$$
$$\subset W_i^{s,n} - \bigcup_{k\neq n} W_i^{s,k} \subset W_i^{\tilde{e},n} \subset W_i^{s,n} \subset W_i^{e,n}$$

by (2.6.4) and the first line. This proves (2.6.1). Then by (2.2.1) and (2.6.3):

$$x \varepsilon W_i^{e,n} \rightarrow \underset{j}{\exists}\, x \varepsilon W_j^{s,n} \rightarrow \underset{m}{\exists}\,\underset{j}{\exists}\, x \varepsilon W_j^{\tilde{e},m};$$

hence

$$\bigcup_{i\varepsilon N}\bigcup_{n\varepsilon N} W_i^{e,n} \subset \bigcup_{i\varepsilon N}\bigcup_{n\varepsilon N} W_i^{\tilde{e},n},$$

inclusion the other way follows from (2.6.1) so that we have now verified (2.6.2).

Definition 2.6.2 A double sequence is *many-one* **v-h**-*universal* if every **v-h**-disjoint r.e. double sequence is reducible to it. If every such reduction can be chosen 1-1, if is said to be 1-1 **v-h**-universal or, simply, **v-h**-*universal*.

Theorem 2.6.1 *There exists a* **v-h**-*universal,* **v-h**-*disjoint r.e. double sequence.*

Proof. Let f be the bijective primitive recursive function τ^2: $N^2 \to N$. Define a double sequence $\langle B_i^n \rangle_{i,n}$ by:

$$y \,\varepsilon\, B_i^n \longleftrightarrow \underset{x}{\exists}\, \underset{e}{\exists}\, [x \,\varepsilon\, W_i^{\chi(e),n} \wedge y = f(e,x)].$$

Clearly $\langle B_i^n \rangle_{i,n}$ is r.e. since the defining predicate is r.e. That it is **v-h**-disjoint follows from the injectivity of f and the **v-h**-disjointness of $\langle W_i^{\chi(e),n} \rangle_{i,n}$. We have only to prove it is **v-h**-universal. For this, let $e \varepsilon N$ be an index of a **v-h**-disjoint r.e. double sequence. Then for each $i,n \varepsilon N$: $W_i^{\chi(e),n} = W_i^{e,n}$. Let f_e be the recursive function defined by: $f_e(x) = f(e,n)$. It follows that:

$$f_e(x) \,\varepsilon\, B_i^n \longleftrightarrow x \,\varepsilon\, W_i^{\chi(e),n} \longleftrightarrow x \,\varepsilon\, W_i^{e,n},$$

which shows that f_e reduces $\langle W_i^{e,n} \rangle_{i,n}$ to $\langle B_i^n \rangle_{i,n}$ and completes the proof.

Definition 2.6.3 A double sequence $\langle A_i^n \rangle_{i,n}$ is **v-h**-*coproductive* if there is a recursive function f such that for $e \varepsilon N$ an index of a **v-h**-disjoint r.e. double sequence and $\langle j,m \rangle \varepsilon N^2$: $\bigcup_{k \varepsilon N} [(W_k^{e,m} \cap A_k^m) \cup (W_j^{e,k} \cap A_j^k)] = \phi \to f(e) \notin \bigcup_{k \varepsilon N} (A_k^m \cup W_k^{e,m} \cup A_j^k \cup W_j^{e,k})$.

Definition 2.6.4 A double sequence is **v-h**-*creative* if it is r.e. and **v-h**-coproductive.

Theorem 2.6.2 *There exists a* **v-h**-*creative,* **v-h**-*disjoint r.e. double sequence.*

Proof. Define a sequence $\langle A_i^n \rangle_{i,n}$ by: $x \varepsilon A_i^n \longleftrightarrow x \varepsilon W_i^{x,n}$. As the right hand side is r.e., there is an $r \varepsilon N$ such that $\langle A_i^n \rangle_{i,n} = \langle W_i^{r,n} \rangle_{i,n}$. Let $\langle B_i^n \rangle_{i,n}$ be the r.e. double sequence with index $\chi(r)$. Claim: $\langle B_i^n \rangle_{i,n}$ is **v-h**-coproductive under the identity function. (Then being r.e. and **v-h**-disjoint, it is the **v-h**-creative double sequence we seek.) Let $\langle W_i^{e,n} \rangle_{i,n}$ be a **v-h**-disjoint r.e. double sequence and $\langle j,m \rangle \varepsilon N^2$ such that $\bigcup_{k \varepsilon N} [(W_k^{e,m} \cap B_k^m) \cup (W_j^{e,k} \cap B_j^k)] = \phi$. We must show:

$$e \notin \bigcup_{k \varepsilon N} (B_k^m \cup W_k^{e,m} \cup B_j^k \cup W_j^{e,k}).$$

Recall that by definition of $\langle A_i^n \rangle_{i,n}$: $e \varepsilon A_i^n \longleftrightarrow e \varepsilon W_i^{e,n}$. Thus, if $e \varepsilon B_i^n \subset A_i^n$, with $n = m$ or $i = j$, it follows also that $e \varepsilon W_i^{e,n}$ contrary to the assumption that $W_i^{e,n} \cap B_i^n = \phi$. On the other hand, since $\langle W_i^{e,n} \rangle_{i,n}$ is **v-h**-disjoint,

$$e \,\varepsilon\, W_i^{e,n} \to e \,\varepsilon\, W_i^{e,n} - \left(\bigcup_{k \neq i} W_i^{e,n} \cup \bigcup_{k \neq n} W_i^{e,k}\right),$$

which in turn implies: $e \varepsilon A_i^n - \left(\bigcup_{k \neq i} A_k^n \cup \bigcup_{k \neq n} A_i^k\right) \subset B_i^n$;
again, for $i = j$ or $n = m$ we have a contradiction.

Lemma 2.6.2 *If a* **v-h**-*coproductive double sequence* $\langle A_i^n \rangle_{i,n}$ *is reducible to* $\langle B_i^n \rangle_{i,n}$*, then* $\langle B_i^n \rangle_{i,n}$ *is also* **v-h**-*coproductive.*

Proof. (*cf.* theorem 2.2.2) Let f reduce $\langle A_i^n\rangle_{i,n}$ to $\langle B_i^n\rangle_{i,n}$ and let g be the **v-h-**coproductive function for $\langle A_i^n\rangle_{i,n}$. Let k be the recursive function such that $W_i^{k(e),n} = f^{-1}(W_i^{e,n})$, all $i,n,e \varepsilon N$. Claim: $\langle B_i^n\rangle_{i,n}$ is **v-h-**coproductive under the recursive function h defined by $h(x) = f(g(k(x)))$. To prove this, let $\langle W_i^{e,n}\rangle_{i,n}$ be a **v-h-**disjoint r.e. double sequence with $\langle j,m\rangle \varepsilon N^2$ such that $W_i^{e,n} \cap B_i^n = \phi$ for $n = m$ or $i = j$. Since for any function f and sets A, B, and C:

$$A \cap B \subset C \rightarrow f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(C),$$

and since in this case $A_i^n = f^{-1}(B_i^n)$ and $W_i^{k(e),n} = f^{-1}(W_i^{e,n})$, we have $\langle W_i^{k(e),n}\rangle_{i,n}$ is **v-h-**disjoint and $W_i^{k(e),n} \cap A_i^n = \phi$ for $n = m$ or $i = j$; hence by the **v-h-**coproductivity of $\langle A_i^n\rangle_{i,n}$:

$$g(k(e)) \notin \bigcup_{r \varepsilon N} [A_r^m \cup W_r^{k(e),m} \cup A_j^r \cup W_j^{k(e),r}].$$

Then using again the properties of inverse image sets:

$$h(e) = f(g(k(e))) \notin \bigcup_{r \varepsilon N} [B_r^m \cup W_r^{e,m} \cup B_j^r \cup W_j^{e,r}].$$

Lemma 2.6.3 *If* $\langle A_i^n\rangle_{i,n}$ *is a* **v-h-***disjoint,* **v-h-***coproductive double sequence, then it is many-one* **v-h-***universal.*

Proof. Let $\langle A_i^n\rangle_{i,n}$ be a **v-h-**coproductive double sequence under the rec. function f. For any r.e. double sequence $\langle B_i^n\rangle_{i,n}$ we can obtain, as shown in the proof of lemma 2.3.1, a rec. 1-1 function g such that

$$y \varepsilon W_i^{g(x),n} \longleftrightarrow x \varepsilon B_i^n \wedge y = f(g(x)).$$

Then if $\langle B_i^n\rangle_{i,n}$ is **v-h-**disjoint, $g(x)$ for any $x \varepsilon N$ is an index of a r.e. double sequence which is **v-h-**disjoint, in fact, $W_i^{g(x),n} = \phi$ unless $x \varepsilon B_i^n$, in which case $W_i^{g(x),n} = \{f(g(x))\}$. We shall prove that the rec. function $h = f \circ g$ reduces $\langle B_i^n\rangle_{i,n}$ to $\langle A_i^n\rangle_{i,n}$, i.e.:

$$x \varepsilon B_i^n \longleftrightarrow h(x) = f(g(x)) \varepsilon A_i^n.$$

1) Suppose $x \varepsilon B_j^m$ but $h(x) \notin A_j^m$ for some $\langle j, m\rangle \varepsilon N^2$. Then

$$\bigcup_{k \varepsilon N} [(W_k^{g(x),m} \cap A_k^m) \cup (W_j^{g(x),k} \cap A_j^k)] = W_j^{g(x),m} \cap A_j^m = \phi,$$

so that by the coproductivity of $\langle A_i^n\rangle_{i,n}$:

$$f(g(x)) \notin \bigcup_{k \varepsilon N} [A_k^m \cup W_k^{g(x),m} \cup A_j^k \cup W_j^{g(x),k}].$$

But this means, in particular, that $f(g(x)) \notin W_j^{g(x),m} = \{f(g(x))\}$, a contradiction.

2) Suppose $h(x) \varepsilon A_j^m$. If $x \notin \bigcup_{k \varepsilon N} (B_k^m \cup B_j^k)$, then $W_i^{g(x),n} = \phi$ if $i = j$ or $n = m$ and by **v-h-**coproductivity $f(g(x)) = h(x) \notin A_j^m$, contradicting the supposition. Hence, there is some $\langle i,n\rangle$ with $i = j$ or $n = m$ such that $x \varepsilon B_i^n$. By part 1)

this implies $h(x) \varepsilon A_i^n$. Then $h(x) \varepsilon A_j^m \cap A_i^n$ and v-h-disjointness yield $n = m$ and $i = j$.
From 1) and 2): $x \varepsilon B_i^n \longleftrightarrow h(x) \varepsilon A_i^n$.

Theorem 2.6.3 *A* v-h-*disjoint double sequence,* $\langle A_i^n \rangle_{i,n}$, *is many-one* v-h-*universal if and only if it is* v-h-*coproductive.*

Proof. "if" by lemma 2.6.3. "only if" by theorem 2.6.2 and lemma 2.6.2.

Definition 2.6.5 A double sequence $\langle A_i^n \rangle_{i,n}$ is v-h-*effectively inseparable* (abbr. v-h-E.I.) if there is a recursive function f such that $f(e) \notin \bigcup_{k \varepsilon N} (W_k^{e,m} \cup W_j^{e,k})$ for each $e \varepsilon N$, $\langle j,m \rangle \varepsilon N^2$ satisfying:
v-h-E.I. 1) $W_i^{e,n} \cap A_i^n = \phi$ if $i = j$ or $n = m$;
v-h-E.I. 2) $W_k^{e,m} \cap W_i^{e,m} \subset \bigcup_{i \varepsilon N} A_i^m$ if $i \neq k$, and

$$W_j^{e,n} \cap W_j^{e,k} \subset \bigcup_{n \varepsilon N} A_j^n \text{ if } n \neq k.$$

Theorem 2.6.4 *If a recursive function f reduces a* v-h-E.I. *double sequence* $\langle A_i^n \rangle_{i,n}$ *to* $\langle B_i^n \rangle_{i,n}$, *then* $\langle B_i^n \rangle_{i,n}$ *is also* v-h-E.I.

Proof. Let $\langle A_i^n \rangle_{i,n}$ be v-h-E.I. under the rec. function g, and let h be a rec. function such that $W_i^{h(e),n} = f^{-1}(W_i^{e,n})$. Claim: $\langle B_i^n \rangle_{i,n}$ is v-h-E.I. under $f \circ g \circ h$. Suppose $e \varepsilon N$, $\langle j,m \rangle \varepsilon N^2$ satisfy v-h-E.I. 1)-2) w.r.t. $\langle B_i^n \rangle_{i,n}$. Then $h(e)$ and $\langle j,m \rangle$ satisfy v-h-E.I. 1)-2) w.r.t. $\langle A_i^n \rangle_{i,n}$ so that $g(h(e)) \notin \bigcup_{k \varepsilon N} (W_k^{h(e),m} \cup W_j^{h(e),k})$. By the properties of inverse image sets, we then have:

$$f(g(h(e))) \notin \bigcup_{k \varepsilon N} (W_k^{e,m} \cup W_j^{e,k}).$$

Corollary 2.6.4.1 *If there exists a* v-h-*disjoint,* v-h-E.I. *r.e. double sequence, then any* v-h-*universal double sequence is* v-h-E.I.

Theorem 2.6.5 *Up to isomorphism there is only one* v-h-*disjoint r.e. double sequence which is* v-h-*universal.*

Proof. By lemma 1.3.3.

At present, attempts to prove a) that v-h-E.I. implies v-h-universal, b) the existence of a v-h-disjoint v-h-E.I. r.e. double sequence, and c) the equivalence of v-h-E.I. and v-h-creativity for v-h-disjoint r.e. double sequences have failed. It would be sufficient to prove either a) and b) or c): c) would follow from a) and b) by theorem 2.6.3 and corollary 2.6.4.1, and from c) the others would follow by theorems 2.6.2 and 2.6.3.

It will also be noted that we have shown that v-h-coproductive implies many-one v-h-universal only. This is because the proof for the existence of a monotone coproductive function does not hold as in the other cases. It would be possible to prove this theorem with a stronger definition of v-h-coproductive, but we have been unable to prove existence for this definition. The definition is as follows:

Definition 2.6.6 A double sequence $\langle A_i^n\rangle_{i,n}$ is *strongly* **v-h**-*coproductive* if there is a recursive function f such that $f(e) \notin \bigcup_{k \varepsilon N} [A_k^m \cup W_k^{e,m} \cup A_j^k \cup W_j^{e,k}]$, whenever $e \varepsilon N$ and $\langle j,m\rangle \varepsilon N^2$ satisfy:

v-h-C 1) $W_i^{e,m} \cap W_k^{e,m} = \phi$ for $i \neq k$ and $W_j^{e,n} \cap W_j^{e,k} = \phi$ for $n \neq k$;
v-h-C 2) $W_i^{e,n} \cap A_i^n = \phi$ if $n = m$ or $i = j$.

2.7 Classifications of Recursively Enumerable Double Sequences. We are now ready to show that the four cases of double sequences which we have considered are really distinct cases. To obtain similarity of notation we replace "uniformly" for the first case by "**h**": e.g. "uniformly universal" will be "**h**-universal". All double sequences will be recursively enumerable in this section.

Notation. For X = **h,v,v-h,** or **t,** let
$\mathfrak{D}_X$ = the class of X-disjoint r.e. double sequences;
$\mathfrak{U}_X$ = the class of X-universal r.e. double sequences.

It is clear from the definitions that:

$$\mathfrak{D}_{\mathbf{t}} \subset \mathfrak{D}_{\mathbf{v\text{-}h}} = (\mathfrak{D}_{\mathbf{v}} \cap \mathfrak{D}_{\mathbf{h}}), \text{ and } \mathfrak{U}_{\mathbf{h}} \cup \mathfrak{U}_{\mathbf{v}} \subset \mathfrak{U}_{\mathbf{v\text{-}h}} \subset \mathfrak{U}_{\mathbf{t}}.$$

Now, let

$\mathfrak{C}_X = \mathfrak{D}_X \cap \mathfrak{U}_X$ = the class of X-disjoint X-universal r.e. double sequences.

Theorem 2.7.1 $\mathfrak{C}_X \cap \mathfrak{C}_Y = \phi$ *for* X $\neq$ Y, X,Y = **h,v,v-h,** *or* **t.**

Proof. Take X = **v** and Y = **h**, and suppose $\langle C_i^n\rangle_{i,n} \varepsilon \mathfrak{C}_X \cap \mathfrak{C}_Y$. Let $\langle A_i^n\rangle_{i,n}$ be r.e. and **v**-disjoint but not **h**-disjoint, for example, define $x \varepsilon A_i^n \longleftrightarrow x = 1 \wedge n = 1$. Then, since $\langle C_i^n\rangle_{i,n} \varepsilon \mathfrak{U}_{\mathbf{v}}$, there is a rec. function f such that $x \varepsilon A_i^n \longleftrightarrow f(x) \varepsilon C_i^n$. Since $\langle A_i^n\rangle_{i,n}$ is not **h**-disjoint, there exist $y,n,i,j \varepsilon N$, with $i \neq j$ and $y \varepsilon A_i^n \cap A_j^n$. We then have $f(y) \varepsilon C_i^n \cap C_j^n$, contradicting $\langle C_i^n\rangle_{i,n} \varepsilon \mathfrak{D}_{\mathbf{h}}$. Other cases for X and Y are proved similarly.

Theorem 2.7.2 *Each class* $\mathfrak{C}_X$, X = **h,v,v-h,** *or* **t,** *is nonempty and consists of isomorphic double sequences which are also* X-*creative. For* X = **h, v,** *or* **t,** *the members of* $\mathfrak{C}_X$ *are also* X-**E.I.**

Proof. X = **h**: corollary 2.3.1.1, theorems 2.3.1 and 2.2.1. X = **v**: theorems 2.4.1, 2.4.2, and 2.4.3, with corollary 2.3.1.1 for the existence of one **v**-disjoint, **v**-universal r.e. double sequence. X = **t**: theorems 2.5.1, 2.5.2, and 2.5.3, with corollary 1.3.1.2 to show $\mathfrak{C}_{\mathbf{t}} \neq \phi$. X = **v-h**: theorems 2.6.1 and 2.6.5.

2.8 k-tuples of Sequences. We have developed the theory for double infinite sequences. That this theory includes that of k-tuples of sequences is the content of this section. We assume $k \geq 1$.

Definiton 2.8.1 Let $\langle A_i^n\rangle_{i \varepsilon N, 0 \leqslant n < k}$ be any k-tuple of sequences. We will say that $\langle B_i^n\rangle_{i \varepsilon N, n \varepsilon N}$ is the *double sequence associated* with $\langle A_i^n\rangle_{i \varepsilon N, 0 \leqslant n < k}$ if for all $i \varepsilon N$:

$$B_i^n = \begin{cases} A_i^n, \text{ for } 0 \le n < k \\ \phi, \text{ for } n \ge k. \end{cases}$$

Definition 2.8.2 A double sequence $\langle A_i^n \rangle_{i,n}$ is said to have *length* $\le k$ if for all $i \varepsilon N$:

$$A_i^n = \phi \text{ if } n \ge k.$$

That a k-tuple of sequences is universal for k-tuples of r.e. sequences if and only if the associated double sequence is universal for r.e. double sequences having length $\le k$, is straightforward, whichever type of disjointness is considered. To obtain the analogous theorems for effective inseparability and creativity, we must be able from the indices of k r.e. sequences to give an index of the associated r.e. double sequence and conversely. This is accomplished in the first direction by the recursive function h defined as follows:

1) Let R be the recursively enumerable predicate:

$$R(r,k,n,i,x) \longleftrightarrow \underset{m}{\exists} [0 \le n < k \wedge x \varepsilon W_i^m \wedge m = \tau_{n+1}^k(r)].$$

2) By the iteration theorem there is a rec. h such that:

$$R(r,k,n,i,x) \longleftrightarrow \underset{y}{\exists} T_3(h(r,k),n,i,x,y).$$

3) Then $h(r,k)$ is an index of the double sequence:

$$W_i^{h(r,k),n} = \begin{cases} W_i^m \text{ with } m = \tau_{n+1}^k(r), \text{ if } 0 \le n < k \\ \phi \text{ if } n \ge k. \end{cases}$$

Thus, given indices $x_0, x_1, \ldots, x_{k-1}$, $h(\tau^k(x_0, \ldots, x_{k-1}), k)$ is an index of the double sequence associated with the given k-tuple of sequences. For the converse, given any index e of a r.e. double sequence, $\phi(e,n)$ is the index of the n^{th} sequence.

Definition 2.8.3 A k-tuple of sequences $\langle A_i^n \rangle_{i \varepsilon N, 0 \leqslant n < k}$ is **U.E.I.** if there is a rec. function f such that $f(x_0, x_1, \ldots, x_{k-1}) \notin \bigcup_{i=0}^{\infty} W_i^{x_n}$ whenever the n^{th} sequence satisfies:

U.E.I. 1) $W_i^{x_n} \cap A_i^n = \phi$ for all $i \varepsilon N$,

and

U.E.I. 2) $W_i^{x_n} \cap W_j^{x_n} \subset \bigcup_{i=0}^{\infty} A_i^n$, for $i \neq j$, $i,j \varepsilon N$.

Definition 2.8.4 A double sequence $\langle A_i^n \rangle_{i,n}$ is **U.E.I.** for double sequences of length $\le k$ if there is a recursive function f such that

$$f(e) \notin \bigcup_{i=0}^{\infty} W_i^{e,n}$$

whenever e is an index of a r.e. double sequence of length $\leq k$ of which the n^{th} sequence, $0 \leq n < k$, satisfies:

U.E.I. 1) $W_i^{e,n} \cap A_i^n = \phi$, for all $i \varepsilon N$,

and

U.E.I. 2) $W_i^{e,n} \cap W_j^{e,n} \subset \bigcup_{i=0}^{\infty} A_i^n$, for $i \neq j, i, j \varepsilon N$.

Theorem 2.8.1 *A k-tuple of sequences $\langle A_i^n \rangle_{i\varepsilon N, 0 \leqslant n < k}$ is* **U.E.I.** *if and only if its associated double sequence is* **U.E.I.** *for r.e. double sequences of length $\leq k$.*

Proof. Let $\langle B_i^n \rangle_{i \varepsilon N, n \varepsilon N}$ be the associated double sequence.
a) Assume $\langle A_i^n \rangle_{i\varepsilon N, 0 \leqslant n<k}$ is **U.E.I.** under f, and let $\langle W_i^{e,n} \rangle_{i,n}$ be a double sequence of length $\leq k$ whose n^{th} sequence, $0 \leq n < k$, satisfies **U.E.I.** 1)-2). Then the n^{th} sequence of the k-tuple $\langle W_i^{\phi(e,n)} \rangle_{i\varepsilon N, 0 \leqslant n<k}$ satisfies **U.E.I.** 1)-2) w.r.t. $\langle A_i^n \rangle_{i\varepsilon N, 0 \leqslant n<k}$, so that

$$f(\phi(e,0), \phi(e,1), \ldots, \phi(e,k \dot{-} 1)) \notin \bigcup_{i=0}^{\infty} W_i^{e,n}.$$

Hence, $\langle B_i^n \rangle_{i,n}$ is **U.E.I.** under the rec. function g:

$$g(x) = f(\phi(x,0), \ldots, \phi(x,k \dot{-} 1)).$$

b) Assume $\langle B_i^n \rangle_{i\varepsilon N, n\varepsilon N}$ is **U.E.I.** for double sequences of length $\leq k$ under the rec. function f, and let $\langle W_i^{x_n} \rangle_{i\varepsilon N, 0 \leqslant n<k}$ be a k-tuple of sequences whose n^{th} sequence, $0 \leq n < k$, satisfies **U.E.I.** 1)-2). Then the associated double sequence with index $h(\tau^k(x_0, \ldots, x_{k-1}), k)$ is such that its n^{th} sequence satisfies **U.E.I.** 1)-2) w.r.t. $\langle B_i^n \rangle_{i,n}$ so that $f(h(\tau^k(x_0, \ldots, x_{k-1}), k)) \notin \bigcup_{i=0}^{\infty} W_i^{x_n}$.
Hence, $\langle A_i^n \rangle_{i\varepsilon N, 0 \leqslant n<k}$ is **U.E.I.** under the rec. function g:

$$g(x_0, x_1, \ldots, x_{k-1}) = f(h(\tau^k(x_0, \ldots, x_{k-1}), k)).$$

Analogous definitions and proofs can be given for uniformly coproductive and for both effective inseparability and coproductivity in the **v-**, **t-**, and **v-h-**cases. It remains, then, only to give a treatment of the theory for double sequences of length $\leq k$. This can be obtained by re-writing sections 2.1-2.7 with all double sequences assumed to be of length $\leq k$. A few changes need to be made for the **v-** and **t-**cases. For the **v-**case, the transpose of a double sequence of length $\leq k$ becomes a double sequence each of whose sequences has length $\leq k$. As the uniform theory is valid also when restricted to sequences of sequences of length $\leq k$, the analog of theorem 2.4.1 is true. In the **t-**case to obtain the analog of theorem 2.5.1 we use the following map from double sequences of length $\leq k$, $\langle A_i^n \rangle_{i\varepsilon N, 0 \leqslant n<k}$, to single sequences $\langle B_i \rangle_{i\varepsilon N}$:

$$B_i = A_r^s \text{ for } i = rk + s.$$

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Springfield Junior College
Springfield, Illinois