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FREGEAN SEMANTICS FOR A REALIST ONTOLOGY

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T* is a logistic system[1] designed to represent the original ontological context behind Russell's paradox of predication. It encompasses standard second order logic, hereafter referred to as **T**, but goes beyond it by allowing predicate variables to occupy subject positions in its formulas. Because of a violation of the restrictions imposed for the proper substitution of a formula for a predicate variable, Russell's argument fails in **T***. Indeed, not only is **T*** consistent but it is also a conservative extension of **T**.[2]

Nevertheless, **T*** is not without its oddities. E.g., although "the Russell property" of *being a property which does not possess itself* does not exist in the ontology of **T***, the modified Russell property of *being an individual which is indiscernible* (in the sense of having all properties in common) *with a property which that individual does not possess* does exist in this ontology. Instead of leading to a contradiction, Russell's argument applied to the modified Russell property shows that the principle that properties which are indiscernible are co-extensive is disprovable in **T***, i.e., according to the ontology of **T***, there are properties which are indiscernible (in the sense indicated above) but which nevertheless are not co-extensive.[3]

It has been suggested that one way of understanding this result is to construe occurrences of predicates in subject positions as referring, not to the properties which occurrences of the same predicates in predicate positions designate, but instead, to individual objects associated with these

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1. The author was partially supported by NSF grant GS-28605.

2. *Cf.* [2], §6.

3. *Ibid.*, §5. We should avoid using 'identical' in place of 'indiscernible' here. In [3], Meyer has shown that according to **T*** there exists no relation which satisfies full substitutivity, and, accordingly, insofar as full substitutivity is taken to be a necessary feature of identity, there is and can be no *identity relation* in the ontology of **T***.

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properties.[4] Such a suggestion of course is reminiscent of Frege's ontology. And were it not that Frege is quite insistent in viewing predicates as "unsaturated" expressions and therefore not qualified as substituends for subject positions which can be occupied only by "saturated" expressions, it might be tempting to construe **T***as representative of Frege's ontology. Be that as it may, the disproof of the principle that indiscernible properties are co-extensive, which is all that Russell's paradox comes to in **T***, is reinterpreted according to this suggestion so as to show merely a variant of Cantor's theorem. And that after all is rather appropriate, since Russell's argument for his supposed paradox is really but a variant of Cantor's argument for his theorem.

In what follows we formulate the suggestion semantically and show that although the semantics thus provided does not characterize **T***, it does characterize a certain rather interesting subsystem **T**** of **T*** supplemented by the extensionality principle that co-extensive properties are indiscernible.[5] The supplemented system, **T****+(Ext*), no doubt appears bizarre from the point of view of the original ontological background represented by **T***—since in this ontology not all indiscernible properties are co-extensive whereas, according to the supplement, all co-extensive properties are indiscernible, thus suggesting co-extensiveness to be a stronger connection between properties than is the indiscernibility relation. On the other hand, from the point of view of its quasi-Fregean semantics, the supplement seems rather natural—for according to this semantics the supplement amounts to the stipulation that the same individual object is to be associated with co-extensive properties. Fregean naturalness aside, it should perhaps be noted that the existence of a model-set-theoretic semantics characteristic for **T***—or of **T*** supplemented with principles natural to the ontology of **T***—remains yet an open problem.

Incidentally, it is noteworthy that the axiom schemas for **T*** described in [2] were in general determined through a natural extension of the axiom schemas for **T** as described in [1]. The naturalness of the extension was determined by the following simple rule: utilize the same axiom schemas, only extended so as to apply to the wider notion of formulahood found in **T*** as compared with that found in **T**. Accordingly, as the principle of universal instantiation of a formula for a predicate variable is an axiom schema of **T**, its extension in the above sense was taken as an axiom schema of **T***. Naturally, such an axiom set requires in its characterization the rather complex syntactical notion of proper substitution.

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4. This suggestion is implicit, though only in a partial way, in the argument independently arrived at by Zorn and Meyer that **T*** is a conservative extension of **T**. It is explicit in the type of model defined below as quasi-Fregean and first recommended to the author as characteristic of **T*** by N. Belnap.

5. It is easily seen from the proof in [2] that **T*** is a conservative extension of **T**, that this extensionality principle is not a theorem of **T***—nor for that matter is its negation.

Now in a footnote of [2] it was claimed that a certain substitution free axiom set, which we shall hereafter refer to as T**, was equivalent to T*. This claim is false, as we shall show in the present paper, i.e., although T** is a subsystem of T*, the two formulations are not equivalent. Nevertheless, T** retains all the ontological oddities described above and is similarly determined through a natural extension of the axiom schemas of a substitution free axiom set for T. In this regard T** represents a viable alternative to T* as a formalization of the original ontological context behind Russell's paradox of predication.

1 Terminology We specify the grammar of T* as follows. We assume there are enumerably infinite and pairwise disjoint sets of variables: individual variables and, for each natural number n, n-place predicate variables. (Propositional variables are 0-place predicate variables.) We shall use 'α', 'β', 'γ' to refer to individual variables, also called *subject terms,* and 'π', 'ρ', 'σ', 'τ' to refer to predicate variables (of arbitrary many places), also called *predicate terms.* We shall use 'μ' and 'ν' to refer to individual and predicate variables collectively, referring to them simply as variables. As logical particles we shall use $\sim$, the negation sign, $\rightarrow$, the conditional sign, and $\wedge$, the universal quantifier. Other logical particles, such as $\leftrightarrow$, the biconditional sign, and $\vee$, the existential quantifier, are assumed to be defined (as syntactical abbreviations in the metalanguage) in the usual manner. An *atomic formula* is, for some natural number n, the result of applying an n-place predicate variable π to n variables $\mu_0, \ldots, \mu_{n-1}$: $\pi(\mu_0, \ldots, \mu_{n-1})$. If $n = 0$, this result is understood to be π itself. (Note that though a predicate term is not a subject term, a predicate term may occupy a subject position in an atomic formula. A subject term on the other hand, is not allowed to occupy a predicate position.) A *formula* is any member of the intersection of those sets K containing the atomic formulas and such that $\sim\phi$, $(\phi \rightarrow \psi)$, $\wedge\mu\phi$ are in K whenever ϕ, ψ are in K and μ is a variable. We shall use 'ϕ', 'ψ', 'χ' to refer to formulas and 'Γ' and 'K' to sets of formulas. Bondage and freedom of (occurrences of) variables is understood in the usual manner, except of course that predicate variables may have bound or free occurrences in subject as well as in predicate position. Accordingly, we assume as understood the obvious distinction between a free (bound) occurrence of a predicate variable *in predicate position* as opposed to a free (bound) occurrence of the same predicate variable *in subject position.*

We shall understand two variables to be of *the same type* if either both are individual variables or, for some natural number n, both are n-place predicate variables. Where μ, ν are variables, whether of the same type or not, we take $\phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$ to be the result of replacing each free occurrence of μ in ϕ by a free occurrence of ν, if such a formula exists; otherwise $\phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$ is understood to be ϕ itself. (It should be remembered in this context that the free occurrences of a predicate variable include those in subject positions as well as those in predicate positions.) We shall understand $\phi\begin{bmatrix}\mu_0 \ldots \mu_{n-1} \\ \nu_0 \ldots \nu_{n-1}\end{bmatrix}$ to be the result of *simultaneously* replacing all the free occurrences of $\mu_0, \ldots, \mu_{n-1}$ in ϕ by free occurrences of $\nu_0, \ldots, \nu_{n-1}$, respectively, if such a formula exists; otherwise it is ϕ itself. Where π is an n-place predicate variable and $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables, then

$$\check{S}\,\pi(\alpha_0, \ldots, \alpha_{n-1})\,\psi\Big|_{\phi}$$

shall be ψ unless the following conditions are satisfied: (1) no free occurrence of π in predicate position in ψ occurs within a subformula of ψ of the form $\wedge\mu\chi$, where μ is a variable distinct from $\alpha_0, \ldots, \alpha_{n-1}$ and occurring free in ϕ and (2) for all variables $\mu_0, \ldots, \mu_{n-1}$, if $\pi(\mu_0, \ldots, \mu_{n-1})$ occurs in ψ in such a way that the occurrence of π is a free occurrence (in predicate position), then for each $i < n$, there is no subformula of ϕ of the form $\wedge\mu_i\chi$ in which α_i has a free occurrence. If these two conditions are satisfied, then

$$\check{S}\,\pi(\alpha_0, \ldots, \alpha_{n-1})\,\psi\Big|_{\phi}$$

is the result of replacing, for arbitrary variables $\mu_0, \ldots, \mu_{n-1}$, each occurrence of $\pi(\mu_0, \ldots, \mu_{n-1})$ in ψ at which π is free (in predicate position) by an occurrence of $\phi\begin{bmatrix}\alpha_0 \ldots \alpha_{n-1} \\ \mu_0 \ldots \mu_{n-1}\end{bmatrix}$.

It is noteworthy that the proper substitution of a formula for predicate position occurrences of a predicate variable is not defined when the specified "argument" positions are not subject terms (individual variables). Since a formula with n specified "argument" positions determines an n-ary relation only insofar as it is a substituend—relative to those "argument" positions—for a generalized n-place predicate variable, it follows that a formula determines such a relation only through those of its "argument" positions that are subject positions, i.e., only through its free individual variables. The reason for this, according to the ontology of both **T*** and **T****, is that in going beyond standard second order logic, through allowing predicate variables to occupy subject as well as predicate positions, we are construing predicate variables as substituends of individual variables and, accordingly, the values of predicate variables as values of individual variables, i.e., the values of predicate variables are being construed as a special breed of "individual". In other words, in extending the ontological framework of standard second order logic—where "being is not a genus"—to that of **T***, we are in effect extending the ontological category of individuals, i.e., the category of that type of entity for which it is ontologically significant that *it* be a subject of predication, or that type of entity which can be *referred to* through the subject expressions of the sentences of the ontological language in question. Accordingly, as the conditions under which a formula can be said to determine—relative to specified "argument" positions—a property or relation must be conditions comprehending *all the individuals*, it follows that these conditions can be fulfilled only when the specified "argument" positions are occupied by individual variables (subject terms).

2 A Substitution Free Axiom Set for T** By a *generalization* of a formula we understand any formula resulting from that formula by prefixing to it any finite number of universal quantifier phrases binding any predicate or individual variable. As *axioms* of T* we take all formulas that are generalizations of formulas of any one of the following forms:

(A1) $\phi \rightarrow (\psi \rightarrow \phi)$,
(A2) $(\phi \rightarrow [\psi \rightarrow \chi]) \rightarrow ([\phi \rightarrow \psi] \rightarrow [\phi \rightarrow \chi])$,
(A3) $(\sim\phi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \phi)$,
(A4) $\vee\pi\wedge\alpha_0 \ldots \wedge\alpha_{n-1} [\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \phi]$, where $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables among those occurring free in ϕ and π is an n-place predicate variable not occurring free in ϕ,
(A5) $\wedge\mu(\phi \rightarrow \psi) \rightarrow (\wedge\mu\phi \rightarrow \wedge\mu\psi)$,
(A6) $\phi \rightarrow \wedge\mu\phi$, where μ is a variable not occurring free in ϕ,
(A7) $\vee\alpha\wedge\sigma[\sigma(\alpha) \rightarrow \sigma(\mu)]$, where μ is a variable distinct from the individual variable α and the 1-place predicate variable σ,
(A8) $\vee\pi\wedge\sigma[\sigma(\pi) \rightarrow \sigma(\rho)]$, where π, ρ are distinct n-place predicate variables and each is distinct from σ if $n = 1$,
(A9) $\wedge\sigma[\sigma(\mu) \rightarrow \sigma(\nu)] \rightarrow (\phi \rightarrow \psi)$, where ϕ, ψ are atomic formulas and ψ is obtained from ϕ by replacing an occurrence of ν in *subject position* by an occurrence of μ.

We take *modus ponens* to be our only inference rule and understand ϕ to be a *theorem* of T**, in symbols $\vdash_{T^{**}} \phi$, if ϕ is the terminal formula of a finite sequence of formulas each constituent of which is either an axiom or is obtained by *modus ponens* from a pair of preceding formulas in the sequence. We say that ϕ is *derivable* in T** from a set Γ of formulas, in symbols $\Gamma \vdash_{T^{**}} \phi$, if for some natural number n there are $\psi_0, \ldots, \psi_{n-1}$, all members of Γ and such that $\vdash_{T^{**}} \psi_0 \rightarrow (\psi_1 \rightarrow \ldots \rightarrow (\psi_{n-1} \rightarrow \phi) \ldots)$. We take this last formula to be ϕ itself if $n = 0$.

In what follows we establish a few useful theorems and indicate which of the axioms and rules of T* as described in [2] are theorems and rules of T**. Because of (A1)-(A3), every tautologous formula is a theorem of T**. Moreover, because of (A5) and the fact that every generalization of an axiom is an axiom, we easily derive the rule of generalization (for predicate and individual variables), which was a primitive rule in the formulation of T* in [2]. Utilizing generalization, along with (A5) and (A6), the interchange law for provably equivalent formulas follows by a simple inductive argument on formulas. Where ψ is obtained from ϕ by replacing a free occurrence of ν *in subject position* by a free occurrence of μ, we are able to show the following restricted version[6] of Leibniz' law

6. *Cf.* footnote 3.

(L.L.*) $\vdash_{\mathbf{T}^{**}} \wedge\pi[\pi(\mu) \rightarrow \pi(\nu)] \rightarrow [\phi \longleftrightarrow \psi]$

by induction over the subformulas of ϕ. We note first that by (A9) $\vdash_{\mathbf{T}^{**}} \wedge\pi[\pi(\mu) \rightarrow \pi(\nu)] \rightarrow [\pi(\nu) \rightarrow \pi(\mu)]$, and therefore, by generalization, (A5) and (A6), $\vdash_{\mathbf{T}^{**}} \wedge\pi[\pi(\mu) \rightarrow \pi(\nu)] \rightarrow \wedge\pi[\pi(\nu) \rightarrow \pi(\mu)]$. Accordingly, if ϕ is atomic the above theorem is seen to hold by (A9) along with this last observation. Where ϕ is a negation or a conditional, the theorem follows by tautologous transformations on the inductive hypotheses. Similarly, where ϕ is a generalized formula, the theorem follows by generalization on the inductive hypotheses and by (A5) and (A6).

By repeated application of the above version of Leibniz' law, $\vdash_{\mathbf{T}^{**}} \wedge\pi[\pi(\alpha) \rightarrow \pi(\mu)] \rightarrow \left(\phi \rightarrow \phi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}\right)$. Therefore, by tautologous transformations, generalization, (A5) and (A6), $\vdash_{\mathbf{T}^{**}} \vee\alpha \wedge\pi[\pi(\alpha) \rightarrow \pi(\mu)] \rightarrow \left(\wedge\alpha\phi \rightarrow \phi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}\right)$, where μ can be properly substituted for α in ϕ; and, accordingly, by (A7), we have the following principle of universal instantiation (or specification)

(U.I.*) $\vdash_{\mathbf{T}^{**}} \wedge\alpha\phi \rightarrow \phi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}$

where μ is a variable distinct from α and which can be properly substituted for α in ϕ. We can dispense with the qualification that μ be distinct from α by generalizing on the qualified version, replacing μ by an individual variable β distinct from α and not occurring in ϕ, obtaining $\vdash_{\mathbf{T}^{**}} \wedge\beta\left(\wedge\alpha\phi \rightarrow \phi\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right)$, from which it follows by the qualified version applied now to β that $\vdash_{\mathbf{T}^{**}} \left(\wedge\alpha\phi \rightarrow \phi\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right)\begin{bmatrix}\beta\\ \alpha\end{bmatrix}$, i.e., $\vdash_{\mathbf{T}^{**}} \wedge\alpha\phi \rightarrow \phi$.

By (U.I.*), generalization, (A5) and (A6), $\vdash_{\mathbf{T}^{**}} \wedge\alpha\phi \rightarrow \wedge\pi\phi\begin{bmatrix}\alpha\\ \pi\end{bmatrix}$, where π is any predicate variable which does not occur in ϕ. This last theorem is an axiom of the formulation of $\mathbf{T}^*$ given in [2]. Moreover, by (U.I.*), generalization, (A5) and (A6), we easily show that $\vdash_{\mathbf{T}^{**}} \wedge\alpha\phi \longleftrightarrow \wedge\beta\phi\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$, where β has no free occurrences in ϕ. This last theorem, together with the interchange law yields the rule of alphabetic change of bound individual variables, one of the primitive rules of the formulation of $\mathbf{T}^*$ in [2]. The final primitive rule of that formulation, viz., the rule of substitution for individual variables, is readily derivable by generalization and (U.I.*).

Where $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables, we have the following restricted form of Leibniz' law for predicates occurring in predicate position:

(L.L.*$_2$) $\vdash_{\mathbf{T}^{**}} \wedge\alpha_0 \ldots \wedge\alpha_{n-1}[\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \psi] \rightarrow \left(\phi \longleftrightarrow \check{\mathbf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\psi}\phi\Big|\right)$.

The proof is by induction on the subformulas of ϕ. If the substitution is not proper, the theorem holds trivially by tautologous transformations. We assume therefore that the substitution is proper. Where ϕ is atomic, (L.L.$*_2$) holds by n applications of (U.I.*).[7] Where ϕ is a negation, conditional or generalized formula, (L.L.$*_2$) follows from the inductive hypothesis utilizing obvious axioms and theorems already established.

Because of the alphabetic rewrite law for bound variables we can abbreviate our notation for indiscernibility as follows:

$$\alpha \equiv \beta =_{df} \wedge\pi[\pi(\alpha) \longleftrightarrow \pi(\beta)].$$

Indiscernibility of course is a congruence or equivalence relation, and, moreover, by (L.L.*), it is the strongest congruence relation expressible in **T****. But it must be remembered that *full* substitutivity is not a consequence of indiscernibility (*cf.* footnote 3).

We take (Ext*) to be the set of formulas of the following form

$$(\text{Ext}*_n) \quad \wedge\pi\, \wedge\sigma\Big(\wedge\alpha_0 \ldots \wedge\alpha_{n-1}[\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1})] \to \pi \equiv \sigma\Big)$$

where n is a natural number, π, σ are distinct n-place predicate variables and $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables. Hereafter we shall write "$\Gamma \vdash_{\mathbf{T}^{**}+(\text{Ext}*)} \phi$" in place of "$\Gamma \cup (\text{Ext}*) \vdash_{\mathbf{T}^{**}} \phi$".

Utilizing (Ext*) we can prove in **T****+(Ext*) another of the axioms of the formulation of **T*** in [2], viz.,

$$(\text{U.I.}*_2) \vdash_{\mathbf{T}^{**}+(\text{Ext}*)} \wedge\mu\phi \to \phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$$

where μ, ν are variables of the same type. Where μ, ν are individual variables, (U.I.$*_2$) follows from (U.I.*). Suppose then that μ, ν are distinct n-place predicate variables and that ψ is obtained from ϕ by replacing each free occurrence of μ in ϕ in subject position by a free occurrence of ν in subject position. Then by (L.L.*), $\vdash_{\mathbf{T}^{**}} \mu \equiv \nu \to [\phi \longleftrightarrow \psi]$, and (L.L.$*_2$), $\vdash_{\mathbf{T}^{**}} \wedge\alpha_0 \ldots \wedge\alpha_{n-1}[\mu(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \nu(\alpha_0, \ldots, \alpha_{n-1})] \to \left[\psi \longleftrightarrow \check{S}^{\mu(\alpha_0, \ldots, \alpha_{n-1})}_{\nu(\alpha_0, \ldots, \alpha_{n-1})}\psi\Big|\right]$. But as $\check{S}^{\mu(\alpha_0, \ldots, \alpha_{n-1})}_{\nu(\alpha_0, \ldots, \alpha_{n-1})}\psi\Big|$ is just $\phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$, then $\vdash_{\mathbf{T}^{**}+(\text{Ext}*)} \wedge\alpha_0 \ldots \wedge\alpha_{n-1}[\mu(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \nu(\alpha_0, \ldots, \alpha_{n-1})] \to \left(\phi \longleftrightarrow \phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\right)$. Therefore by tautologous transformations, generalization, (A5) and (A6), $\vdash_{\mathbf{T}^{**}+(\text{Ext}*)} \vee\mu\, \wedge\alpha_0 \ldots \wedge\alpha_{n-1}[\mu(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \nu(\alpha_0, \ldots, \alpha_{n-1})] \to \left(\wedge\mu\phi \to \phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\right)$, from which (U.I.$*_2$) follows by (A4).

Incidentally, observe that if (A4) were supplemented with

$$(\text{A4}') \quad \vee\pi\Big(\pi \equiv \sigma \wedge \wedge\alpha_0 \ldots \wedge\alpha_{n-1}[\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1})]\Big)$$

where π, σ are distinct n-place predicate variables, then the above use of

7. In case there is a conflict for simultaneous substitution, apply (U.I.*) n times to n new individual variables, generalize on these, use (A5) and (A6), and apply (U.I.*) to these new individual variables.

(Ext*) becomes unnecessary, i.e., (U.I.$*_2$) is a theorem schema of $\mathbf{T}^{**}$ + (A4′).

If in addition to being distinct individual variables, $\alpha_0, \ldots, \alpha_{n-1}$ are among the variables occurring free in ψ and π is an n-place predicate variable not occurring free in ψ nor in any subject position in ϕ, then, by an argument similar to that for (U.I.*), except for utilizing (L.L.$*_2$) and (A4) in place of (L.L.*) and (A7),

$$(\text{U.I.}*_3)\ \vdash_{\mathbf{T}^{**}} \bigwedge \pi\phi \rightarrow \check{\mathsf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\psi}\ \phi\Big| .$$

This proof for (U.I.$*_3$) has the qualification that the substitution is proper. We can dispense with this qualification by generalizing on the qualified version where ψ is replaced by $\rho(\alpha_0, \ldots, \alpha_{n-1})$, ρ being an n-place predicate variable not occurring in ϕ. Thus, $\vdash_{\mathbf{T}^{**}} \bigwedge\rho\left(\bigwedge\pi\phi \rightarrow \check{\mathsf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\rho(\alpha_0, \ldots, \alpha_{n-1})}\ \phi\Big|\right)$, from which it follows by the qualified version, only substituting now $\pi(\alpha_0, \ldots, \alpha_{n-1})$ for ρ, that $\vdash_{\mathbf{T}^{**}} \bigwedge\pi\phi \rightarrow \phi$, which is what (U.I.$*_3$) comes to when the substitution is not proper. Moreover, by utilizing this same replacement of the free predicate position occurrences of π in ϕ by ρ we can also dispense with the qualification that π not occur free in ψ—through substituting ψ for ρ in $\check{\mathsf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\rho(\alpha_0, \ldots, \alpha_{n-1})}\ \phi\Big|$, where we assume that ρ does not occur in ψ as well. Finally, we can also dispense with the qualification that $\alpha_0, \ldots, \alpha_{n-1}$ be among the free variables of ψ. For by a simple inductive argument

$$\vdash_{\mathbf{T}^{**}} \left(\check{\mathsf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\chi}\ \phi\Big| \longleftrightarrow \check{\mathsf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\psi}\ \phi\Big|\right)$$

where $\chi = \left([\pi(\alpha_0, \ldots, \alpha_{n-1}) \rightarrow \pi(\alpha_0, \ldots, \alpha_{n-1})] \rightarrow \psi\right)$. And since $\alpha_0, \ldots, \alpha_{n-1}$ are among the free variables of χ, $\vdash_{\mathbf{T}^{**}} \bigwedge\pi\phi \rightarrow \check{\mathsf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\chi}\ \phi\Big|$ by the qualified version of (U.I.$*_3$), from which the unqualified version follows by the interchange law. In its final form (U.I.$*_3$) has as its only qualifications that π not occur free in ϕ in subject position and that $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables.

It is noteworthy that (U.I.$*_3$), without the qualification that π not have any free subject position occurrences in ϕ, was taken as an axiom schema of $\mathbf{T}^*$ in [2]. Moreover, it is the one remaining axiom schema of $\mathbf{T}^*$ that needs to be accounted for here. Of course this stronger version of (U.I.$*_3$) is not provable in $\mathbf{T}^{**}$—nor, for that matter, in $\mathbf{T}^{**}$ + (Ext*)—since otherwise $\mathbf{T}^{**}$ would be equivalent to $\mathbf{T}^*$. However, $\mathbf{T}^{**}$ supplemented with an especially simple form of the stronger version of (U.I.$*_3$), viz.,

$$(\text{U.I.}*_4)\quad \bigwedge\pi\phi \rightarrow \check{\mathsf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\sigma(\alpha_0, \ldots, \alpha_{n-1})}\ \phi\Big|$$

where π, σ are n-place predicate variables, does yield the general form of this stronger version of (U.I.$*_3$). The proof is straightforward utilizing the qualified version of (U.I.$*_3$). For by (U.I.$*_4$), generalization, (A5) and (A6) the free predicate position occurrences of π in ϕ in (U.I.$*_3$) can be rewritten to a predicate variable σ which does not occur in ϕ and which therefore in particular has no free subject position occurrences in ϕ. Then by the qualified version of (U.I.$*_3$), $\vdash_{\overline{\mathbf{T}^{**}}} \wedge\sigma \check{S}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\sigma(\alpha_0, \ldots, \alpha_{n-1})} \phi| \rightarrow \check{S}^{\sigma(\alpha_0, \ldots, \alpha_{n-1})}_{\psi} \check{S}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\sigma(\alpha_0, \ldots, \alpha_{n-1})} \phi||$. But as the consequent of this last formula is just $\check{S}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\psi} \phi|$, it follows that (U.I.$*_3$) without the qualification that π not have free subject position occurrences in ϕ is therefore provable in $\mathbf{T}^{**}$ + (U.I.$*_4$).

We might note that although (Ext*) was used in our earlier proof of (U.I.$*_2$), this use of extensionality is unnecessary in $\mathbf{T}^{**}$ + (U.I.$*_4$), i.e.,

$$\vdash_{\overline{\mathbf{T}^{**}+(\mathrm{U.I.}*_4)}} \wedge\mu\phi \rightarrow \phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$$

where μ, ν are variables of the same type. If μ, ν are individual variables, this result of course already holds in $\mathbf{T}^{**}$ by (U.I.*). Suppose then that μ and ν are distinct n-place predicate variables and that replacing free occurrences of μ in ϕ in both subject and predicate position results in free occurrences of ν in those same positions. Let α be an individual variable not occurring in ϕ and let ψ be the result obtained from ϕ by replacing all the free occurrences of μ in subject position by free occurrences of α. Then $\phi = \psi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}$ and $\phi\begin{bmatrix}\mu\\ \nu\end{bmatrix} = \check{S}^{\mu(\beta_0, \ldots, \beta_{n-1})}_{\nu(\beta_0, \ldots, \beta_{n-1})} \psi\begin{bmatrix}\alpha\\ \nu\end{bmatrix}|$. Then, by (L.L.*), $\vdash_{\overline{\mathbf{T}^{**}}} \mu \equiv \alpha \rightarrow (\phi \rightarrow \psi)$; and therefore by generalization twice, (A5), (A6) and tautologous transformations, $\vdash_{\overline{\mathbf{T}^{**}}} \wedge\mu\phi \rightarrow \wedge\mu \wedge\alpha(\mu \equiv \alpha \rightarrow \psi)$. To show (U.I.$*_2$) with the stated qualification on μ and ν it now suffices to show $\vdash_{\overline{\mathbf{T}^{**}+(\mathrm{U.I.}*_4)}} \wedge\mu \wedge\alpha(\mu \equiv \alpha \rightarrow \psi) \rightarrow \phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$. We observe in this regard that by (U.I.$*_4$) $\vdash_{\overline{\mathbf{T}^{**}+(\mathrm{U.I.}*_4)}} \wedge\mu \wedge\alpha(\mu \equiv \alpha \rightarrow \psi) \rightarrow \check{S}^{\mu(\beta_0, \ldots, \beta_{n-1})}_{\nu(\beta_0, \ldots, \beta_{n-1})} \wedge\alpha(\mu \equiv \alpha \rightarrow \psi)|$, i.e., since this form of substitution affects only predicate occurrences of μ, $\vdash_{\overline{\mathbf{T}^{**}+(\mathrm{U.I.}*_4)}} \wedge\mu \wedge\alpha(\mu \equiv \alpha \rightarrow \psi) \rightarrow \wedge\alpha\left(\mu \equiv \alpha \rightarrow \check{S}^{\mu(\beta_0, \ldots, \beta_{n-1})}_{\nu(\beta_0, \ldots, \beta_{n-1})} \psi|\right)$. Now by (U.I.*) $\vdash_{\overline{\mathbf{T}^{**}}} \wedge\alpha\left(\mu \equiv \alpha \rightarrow \check{S}^{\mu(\beta_0, \ldots, \beta_{n-1})}_{\nu(\beta_0, \ldots, \beta_{n-1})} \psi|\right) \rightarrow \left(\mu \equiv \nu \rightarrow \check{S}^{\mu(\beta_0, \ldots, \beta_{n-1})}_{\nu(\beta_0, \ldots, \beta_{n-1})} \psi\begin{bmatrix}\alpha\\ \nu\end{bmatrix}|\right)$; and, as the final consequent of this last formula is just $\phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$, we therefore have $\vdash_{\overline{\mathbf{T}^{**}+(\mathrm{U.I.}*_4)}} \wedge\mu \wedge\alpha(\mu \equiv \alpha \rightarrow \psi) \rightarrow \left(\mu \equiv \nu \rightarrow \phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\right)$, from which, by generalization (A5), (A6)—noting that μ is not free in $\phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$—and tautologous transformations $\vdash_{\overline{\mathbf{T}^{**}+(\mathrm{U.I.}*_4)}} \wedge\mu \wedge\alpha(\mu \equiv \alpha \rightarrow \psi) \rightarrow \left(\vee\mu\ \mu \equiv \nu \rightarrow \phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\right)$; and therefore, by (A8), $\vdash_{\overline{\mathbf{T}^{**}+(\mathrm{U.I.}*_4)}} \wedge\mu \wedge\alpha(\mu \equiv \alpha \rightarrow \psi) \rightarrow \phi\begin{bmatrix}\mu\\ \nu\end{bmatrix}$, which was to be shown in order to obtain the qualified version of (U.I.$*_2$). We can dispense in the usual manner with the qualifications by generalization upon the qualified form and applying the qualified form to the generalization, only now instantiating ν to μ, thereby obtaining $\bigwedge\mu\, \phi \rightarrow \phi$ as a theorem. But (U.I.$*_2$) is just this last formula if the qualifications fail.

As all the remaining axioms of the formulation of $\mathsf{T}*$ in [2] are either (A1)-(A3) or a confinement law trivially obtainable from (A5), (A6) and tautologous transformations, we have shown therefore that every axiom and primitive rule of $\mathsf{T}*$ is derivable in $\mathsf{T}** +$ (U.I.$*_4$). As the converse also clearly holds, we conclude then that $\mathsf{T}*$ and $\mathsf{T}** +$ (U.I.$*_4$) are equivalent.

We might note that though (Ext$*$) is consistent in $\mathsf{T}*$ and therefore also in its subsystem $\mathsf{T}**$, it implies in the former but not in the latter that no properties (relations) are discernible from one another, i.e.,

$$\vdash_{\overline{\mathsf{T}*}} (\mathsf{Ext}*_n) \rightarrow \bigwedge\pi\ \bigwedge\sigma\, \pi \equiv \sigma$$

where π, σ are n-place predicate variables. For by (U.I.$*_4$), $\vdash_{\overline{\mathsf{T}*}} \bigwedge\pi\bigwedge\sigma\big(\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}\,[\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1})] \rightarrow \pi \equiv \sigma\big) \rightarrow \big(\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}\,[\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \pi(\alpha_0, \ldots, \alpha_{n-1})] \rightarrow \pi \equiv \sigma\big)$, from which the above follows by tautologous transformations, generalization, (A5) and (A6). As we shall see from the semantics of section **3**, $\sim\bigwedge\pi\ \bigwedge\sigma\pi \equiv \sigma$ is consistent in $\mathsf{T}** +$ (Ext$*$), and therefore $\mathsf{T}**$ must be a *proper* subsystem of $\mathsf{T}*$. Let (I$*$) be the set of formulas of the following form

$$(\mathsf{I}*_n) \qquad \bigwedge\pi\ \bigwedge\sigma\,\pi \equiv \sigma$$

where π, σ are distinct n-place predicate variables. As noted above, (I$*$) is a theorem schema of $\mathsf{T}* +$ (Ext$*$). Therefore $\mathsf{T}** +$ (Ext$*$) $+$ (I$*$) is a subsystem of $\mathsf{T}* +$ (Ext$*$). However in this case the converse also holds. For by (U.I.$*_2$), $\vdash_{\overline{\mathsf{T}**+(\mathsf{Ext}*)}} \bigwedge\pi\phi \rightarrow \phi\begin{bmatrix}\pi\\ \sigma\end{bmatrix}$, and by (L.L.$*$),

$$\vdash_{\overline{\mathsf{T}**}} \pi \equiv \sigma \rightarrow \left(\phi\begin{bmatrix}\pi\\ \sigma\end{bmatrix} \longleftrightarrow \check{\mathsf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\sigma(\alpha_0, \ldots, \alpha_{n-1})}\, \phi\,\Big|\right),$$

where σ does not occur in ϕ, and therefore

$$\vdash_{\overline{\mathsf{T}**+(\mathsf{Ext}*)}} \pi \equiv \sigma \rightarrow \left(\bigwedge\pi\phi \rightarrow \check{\mathsf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\sigma(\alpha_0, \ldots, \alpha_{n-1})}\, \phi\,\Big|\right).$$

But by (I$*$) and (U.I.$*_2$), $\vdash_{\overline{\mathsf{T}**+(\mathsf{Ext}*)+(\mathsf{I}*)}} \pi \equiv \sigma$, and therefore

$$\vdash_{\overline{\mathsf{T}**+(\mathsf{Ext}*)+(\mathsf{I}*)}} \bigwedge\pi\phi \rightarrow \check{\mathsf{S}}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\sigma(\alpha_0, \ldots, \alpha_{n-1})}\, \phi\,\Big|.$$

This last formula is (U.I.$*_4$) with the qualification that σ does not occur in ϕ. This qualification can be dropped by generalization on the qualified form, (A5), (A6), and (U.I.$*_2$). Accordingly, (U.I.$*_4$) is a theorem schema of $\mathsf{T}** +$ (Ext$*$) $+$ (I$*$). But as $\mathsf{T}*$ is equivalent to $\mathsf{T}** +$ (U.I.$*_4$), it follows that $\mathsf{T}* +$ (Ext$*$) is equivalent to $\mathsf{T}** +$ (Ext$*$) $+$ (I$*$). This is noteworthy in that $\mathsf{T}** +$ (Ext$*$) $+$ (I$*$) suggests an alternative Fregean interpretation of $\mathsf{T}*$, viz., one in which subject position occurrences of all predicates of the same type are to denote the same object.

Let us observe that because of the stronger form of (U.I.$*_3$) in **T***, we are able to establish a stronger form of the comprehension principle in **T*** than we have assumed as an axiom of **T****. Specifically, in **T*** we have

(CP*) $$\vdash_{\overline{\mathbf{T}^*}} \bigvee\pi \bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1} [\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \phi]$$

where $\alpha_0, \ldots, \alpha_{n-1}$ are distinct individual variables and π is an n-place predicate variable which does not occur free in any predicate position in ϕ *though π may occur free in subject position in ϕ*. The proof is straightforward since by the stronger unqualified form of (U.I.$*_3$),

$$\vdash_{\overline{\mathbf{T}^*}} \bigwedge\pi \sim \bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1} [\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \phi] \rightarrow \sim \bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1} [\phi \longleftrightarrow \phi],$$

from which (CP*) follows by a tautologous transformation.

Question: Let **T***** be the system obtained from **T**** by replacing the comprehension axiom schema (A4) by the stronger comprehension principle (CP*) and (A4$'$). Is **T***** a proper subsystem of **T*** or is **T***** equivalent to **T***?

If **T***** is equivalent to **T***, this suggests an answer as to why two equivalent axiomatic formulations of **T**, standard second order logic, lead to the *different* systems **T*** and **T**** when extended in the same natural way by utilizing the same schemas respectively of the two formulations but applied to the wider notion of formulahood. For when the substitution free axiom set for **T** with the comprehension principle as a constituent axiom schema is extended in this way, we have two options in regard to the question whether to allow free subject position occurrences of the generalized predicate variable. The first option leads to (A4) and allows no free occurrences whether in subject or in predicate positions of the generalized predicate variable. The second option of course leads to the stronger (CP*).

We note that essentially the same question and issue applies if we consider instead the options of replacing (U.I.$*_3$) as an axiom schema of **T*** by either (A4) or (CP*). Replacing (U.I.$*_3$) by (A4) results in a weaker system, viz., one equivalent to **T**** + (A4$'$). It remains an open question whether replacing (U.I.$*_3$) by (CP*) also results in a weaker system.

Finally, in concluding this section on syntax we note the following to be theorems of **T**** as they are utilized in our proof of completeness in section **4**. (The first follows by using tautologous transformations on (A5) and (A7). The second follows similarly except for using (A4) in place of (A7).)

$$\vdash_{\overline{\mathbf{T}^{**}}} \bigwedge\alpha\,\phi \rightarrow \bigvee\alpha(\mu \equiv \alpha \wedge \phi)$$
$$\vdash_{\overline{\mathbf{T}^{**}}} \bigwedge\pi\,\phi \rightarrow \bigvee\pi\Big(\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1} [\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1})] \wedge \phi\Big).$$

3 A Quasi-Fregean Semantics for T* Where $\mathfrak{A} = \langle D, \langle \mathcal{F}_n \rangle_{n\in\omega}, f \rangle$, we shall say that $\mathfrak{A}$ is a *quasi-Fregean model* if (1) D is a non-empty set, (2) $\langle \mathcal{F}_n \rangle_{n\in\omega}$ is an ω-indexed family such that for all $n \in \omega$, every member of $\mathcal{F}_n$ is a subset of D^n, and (3) f is a function whose domain is $D \cup \bigcup_{n\in\omega} \mathcal{F}_n$ and such that for all $n \in \omega$, for all $X \in \mathcal{F}_n$, $f(X) \in D$, and, for all $x \in D$, $f(x) = x$. (We include in f the identity function on D only for convenience so as to simplify our definition below of satisfaction in $\mathfrak{A}$.)

In regard to assigning to variables values drawn from $\mathfrak{A}$ (defined as above), we say that $\mathfrak{a}$ is such an *assignment in* $\mathfrak{A}$ if $\mathfrak{a}$ is a function with the set of variables as domain and such that (1) for each individual variable α, $\mathfrak{a}(\alpha) \in D$, and, for $n \in \omega$, for each n-place predicate variable π, $\mathfrak{a}(\pi) \in \mathcal{F}_n$. We take $\mathfrak{a}\binom{\mu}{y}$ to be that assignment identical with $\mathfrak{a}$ in all respects except (at most) in its assigning y to μ.

If $\mathfrak{A}$ (defined as above) is a quasi-Fregean model and $\mathfrak{a}$ is an assignment in $\mathfrak{A}$ then we recursively define *satisfaction in* $\mathfrak{A}$ by $\mathfrak{a}$ of a formula as follows: (1) $\mathfrak{a}$ *satisfies* $\pi(\mu_0, \ldots, \mu_{n-1})$ *in* $\mathfrak{A}$ iff $\langle f(\mathfrak{a}(\mu_0)), \ldots, f(\mathfrak{a}(\mu_{n-1}))\rangle \in \mathfrak{a}(\pi)$; (2) $\mathfrak{a}$ *satisfies* $\sim\phi$ *in* $\mathfrak{A}$ iff $\mathfrak{a}$ does not satisfy ϕ in $\mathfrak{A}$; (3) $\mathfrak{a}$ *satisfies* $(\phi \rightarrow \psi)$ *in* $\mathfrak{A}$ iff either $\mathfrak{a}$ does not satisfy ϕ in $\mathfrak{A}$ or $\mathfrak{a}$ satisfies ψ in $\mathfrak{A}$; (4) $\mathfrak{a}$ *satisfies* $\bigwedge\alpha\phi$ *in* $\mathfrak{A}$ iff for all $x \in D$, $\mathfrak{a}\binom{\alpha}{x}$ satisfies ϕ in $\mathfrak{A}$; and (5) $\mathfrak{a}$ *satisfies* $\bigwedge\pi\phi$ *in* $\mathfrak{A}$ (where π is an n-place predicate variable) iff for all $X \in \mathcal{F}_n$, $\mathfrak{a}\binom{\pi}{X}$ satisfies ϕ in $\mathfrak{A}$.

We understand a formula to be *true in* a quasi-Fregean model if it is satisfied by every assignment in that model. A quasi-Fregean model is said to be *normal* if every instance of (A4), the comprehension principle for T^{**}, is true in that model. Finally, by a *valid* formula we understand a formula which is true in every normal quasi-Fregean model.

Soundness *If* $\vdash_{\mathsf{T}^{**}+(\mathsf{Ext}^*)} \phi$, *then* ϕ *is valid.*

Proof: From the definition of a quasi-Fregean model and of truth in such a model, it is clear that every member of (Ext*) as well as every axiom of any of the forms (A1)-(A3), (A5)-(A8) is true in any quasi-Fregean model. Moreover, if $\mathfrak{A}$ is a normal quasi-Fregean model, then by definition of normalcy, every instance of (A4) is true in $\mathfrak{A}$. In particular, $\bigvee\pi \bigwedge\alpha[\pi(\alpha) \longleftrightarrow \sigma(\beta_0, \ldots, \beta_{i-1}, \alpha, \beta_{i+1}, \ldots \beta_{n-1})]$ is true in $\mathfrak{A}$, and therefore if x and y belong to all the same members of $\mathcal{F}_1$, then they stand in the same way in all the relations that are in $\mathcal{F}_n$; from which it follows that every instance of (A9) is true in $\mathfrak{A}$. We conclude then that every theorem of T^{**} + (Ext*) is true in $\mathfrak{A}$.

The following semantic lemma will be found useful in our proof of completeness in section **4**.

Semantic Lemma *If* $\mathfrak{A} = \langle D, \langle \mathcal{F}_n\rangle_{n\in\omega}, f\rangle$ *is a quasi-Fregean model,* μ *is a variable which can be properly substituted for* α *in* ϕ, *and* π, σ *are n-place predicate variables such that* σ *can be properly substituted for* π *in* ϕ, *then for all assignments* $\mathfrak{a}$ *in* $\mathfrak{A}$*:*

(1) $\mathfrak{a}\binom{\alpha}{f(\mathfrak{a}(\mu))}$ *satisfies* ϕ *in* $\mathfrak{A}$ *iff* $\mathfrak{a}$ *satisfies* $\phi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}$ *in* $\mathfrak{A}$;

and

(2) $\mathfrak{a}\binom{\pi}{\mathfrak{a}(\sigma)}$ *satisfies* ϕ *in* $\mathfrak{A}$ *iff* $\mathfrak{a}$ *satisfies* $\phi\begin{bmatrix}\pi\\ \sigma\end{bmatrix}$ *in* $\mathfrak{A}$.

Proof: Assume the hypothesis. We establish the consequent by induction on the subformulas of ϕ. In regard to (1), if ϕ is atomic, the argument is easily seen to go through by observing that $f\left(\mathfrak{a}\binom{\alpha}{f(\mathfrak{a}(\mu))}(\nu)\right) = f(\mathfrak{a}(\nu))$ if ν is a variable other than α and that $f\left(\mathfrak{a}\binom{\alpha}{f(\mathfrak{a}(\mu))}(\alpha)\right) = f(f(\mathfrak{a}(\mu))) = f(\mathfrak{a}(\mu))$. If ϕ is of the form $\sim\psi$, $(\psi \rightarrow \chi)$, $\wedge\beta\chi$ or $\wedge\rho\psi$, then since $(\sim\psi)\begin{bmatrix}\alpha\\ \mu\end{bmatrix} = \sim\left(\psi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}\right)$ and, since by hypothesis μ can be properly substituted for α in ϕ, $(\psi \rightarrow \chi)\begin{bmatrix}\alpha\\ \mu\end{bmatrix} = \left(\psi\begin{bmatrix}\alpha\\ \mu\end{bmatrix} \rightarrow \chi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}\right)$, $(\wedge\beta\psi)\begin{bmatrix}\alpha\\ \mu\end{bmatrix} = \wedge\beta\left(\psi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}\right)$ and $(\wedge\rho\psi)\begin{bmatrix}\alpha\\ \mu\end{bmatrix} = \wedge\rho\left(\psi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}\right)$, (1) is seen to follow by application of the inductive hypothesis to ψ and χ. A similar argument holds in regard to (2).

If $\mathfrak{A} = \langle D, \langle\mathcal{F}_n\rangle_{n\in\omega}, f\rangle$ is a quasi-Fregean model such that $\mathcal{F}_n$ is the power set of D^n, for all $n \in \omega$, then $\mathfrak{A}$ is said to be a *standard* quasi-Fregean model. Now by a simple inductive argument it is easily seen that every standard quasi-Fregean model is normal. We note that if in addition to being a standard model, D is a finite set with at least two distinct members, say x and y, and $f(X) = x$ if X has an even number of members and otherwise $f(X) = y$, for all $X \in \mathcal{F}_n$ and for arbitrary $n \in \omega$, then $\sim \wedge\pi \wedge\sigma\pi \equiv \sigma$ is true in $\mathfrak{A}$, where π, σ are distinct n-place predicate variables. Accordingly, by our soundness theorem above and the fact that $\sim \wedge\pi \wedge\sigma\pi \equiv \sigma$ is true in a normal quasi-Fregean model, it follows that $\sim \wedge\pi \wedge\sigma\pi \equiv \sigma$ is consistent in $\mathbf{T}^{**}$ + (Ext*), as was claimed in section **2**.

Before concluding this section on semantics, let us briefly consider the following quasi-Fregean model $\mathfrak{A} = \langle D \cup \{D, \{D\}\}, \langle\mathcal{F}_n\rangle_{n\in\omega}, f\rangle$, where D is any set, empty or otherwise, $\mathcal{F}_n$ is the set of all subsets of $(D \cup \{D, \{D\}\})^n$, $f(0) = D$, $f(1) = \{D\}$, and if D is finite, then for some fixed $x \in D \cup \{D, \{D\}\}$, for all positive $n \in \omega$, for all $X \in \mathcal{F}_n$, $f(X) = x$, but if D is infinite and $\langle x_n\rangle_{n\in\omega}$ is an ω-indexed sequence of pairwise distinct members of D, then for all positive $n \in \omega$, for all $X \in \mathcal{F}_n$, $f(X) = x_n$. (We take each natural number to be the set of natural numbers less than it, and therefore 0 is understood to be the empty set and $1 = \{0\}$. As the empty set, 0 is the only 0-tuple, i.e., 0-place sequence, and consequently $\mathcal{F}_0 = \{0, 1\}$. Since this last identity holds for every normal model, there are according to our extensional quasi-Fregean semantics only two "propositions", a false and a true one, viz., 0 and 1 respectively, that are the values of the propositional variables in propositional position—i.e., propositional variables occurring as (sub)formulas. In the model presently being considered, these "propositions" have as their associated individuals D and $\{D\}$, respectively, which can intuitively be construed as the counter-parts in $\mathfrak{A}$ of the Fregean "individuals" referred to as "the False" and "the True".) Now since $\mathfrak{A}$ is a standard quasi-Fregean model, it follows that $\mathfrak{A}$ is normal. In addition, where π, σ are propositional (0-place predicate) variables,

$$(\breve{\mathrm{Ext}}_0*) \qquad \bigwedge\pi \bigwedge\sigma[\pi \equiv \sigma \rightarrow (\pi \longleftrightarrow \sigma)]$$

is also true in $\mathfrak{A}$. Thus although for each positive natural number n, ($\sim \breve{\mathrm{Ext}}_n*$) is a theorem of T**—which in the model $\mathfrak{A}$ is seen to result from Cantor's theorem—nevertheless this is not the case when $n = 0$, i.e., ($\breve{\mathrm{Ext}}_0*$) is consistent in T**. Moreover, if D is empty, then there exist no individuals in $\mathfrak{A}$ beyond the (counter-parts of the) Fregean individuals "the True" and "the False", i.e., then

$$(\sim \mathrm{E}\,!) \qquad \bigvee\pi \bigvee\sigma[\pi \wedge \sim\sigma \wedge \bigwedge\alpha(\alpha \equiv \pi \vee \alpha \equiv \sigma)]$$

where π, σ are distinct propositional variables, is true in $\mathfrak{A}$. On the other hand, where (Inf*) is the set of formulas of the form

$$\bigwedge\pi \bigwedge\sigma\pi \not\equiv \sigma$$

where for some natural numbers m, n such that $m \neq n$, π is an m-place and σ is an n-place predicate variable, then it follows from the way $\mathfrak{A}$ was constructed that D is infinite if every member of (Inf*) is true in $\mathfrak{A}$. From the point of view of the ontology of T**, (Inf*) is a readily acceptable ontological principle, and of course amounts to an axiom of infinity as well—which fact, on the other hand, renders it questionable from the point of view of our quasi-Fregean semantics. From these observations, we have the following result.

Consistency T** + (Ext*) + ($\breve{\mathrm{Ext}}_0*$) + ($\sim$ E !) *and* T** + (Ext*) + ($\breve{\mathrm{Ext}}_0*$) + (Inf*) *are consistent.*

4 A Completeness Theorem for T + (Ext*)** We understand a set K of formulas which is consistent in T** + (Ext*) to be *maximally consistent* in T** + (Ext*) if $\phi \in K$ whenever $K \cup \{\phi\}$ is consistent in T** + (Ext*). We shall say that K is ω-complete if for all formulas ϕ and variables μ: if $\phi\begin{bmatrix}\mu\\ \nu\end{bmatrix} \in K$ whenever ν is a variable of the same type as μ that can be properly substituted for μ in ϕ, then $\bigwedge\mu\phi \in K$.

Lemma $\vdash_{\mathrm{T^{**}+(Ext^*)}} \phi$ *iff* ϕ *is a member of every set of formulas which is* ω*-complete and maximally consistent relative to* T** + (Ext*).

Proof: The left to right direction of the lemma follows immediately from the definition of maximal consistency. Assume then that ϕ belongs to every maximally consistent and ω-complete set but that ϕ is not a theorem of T** + (Ext*). Then $\sim\phi$ is consistent in T** + (Ext*). Let $\Sigma_1, \ldots, \Sigma_n, \ldots$ ($n \in \omega$) be an ordering of the formulas of the form $\bigvee\mu\psi$. We recursively define the chain Γ as follows:

$$\Gamma_0 = \{\sim\phi\}$$
$$\Gamma_{n+1} = \Gamma_n \cup \left\{\bigvee\mu\psi \rightarrow \psi\begin{bmatrix}\mu\\ \nu\end{bmatrix}\right\},$$

where $\Sigma_{n+1} = \bigvee \mu\psi$ and ν is the first variable of the same type as μ which does not occur in any member of $\Gamma_n \cup \{\Sigma_{n+1}\}$. By generalization, (A5), (A6), tautologous transformations and (U.I.$*_2$), it is easily seen by the usual form of argument here that Γ_n is consistent in $\mathsf{T}^{**} + (\mathsf{Ext}^*)$, for all $n \epsilon \omega$. Accordingly, $\bigcup_{n\epsilon\omega} \Gamma_n$ is also consistent, and therefore by Lindenbaum's lemma—which is easily shown in the usual way to hold for $\mathsf{T}^{**} + (\mathsf{Ext}^*)$—there exists a maximally consistent set K such that $\bigcup_{n\epsilon\omega} \Gamma_n \subseteq K$. That K is ω-complete follows from the way the chain Γ was constructed.

Completeness *If ϕ is valid, then* $\vdash_{\overline{\mathsf{T}^{**}+(\mathsf{Ext}^*)}} \phi$.

Proof: Assume that ϕ is valid and that K is an ω-complete set which is maximally consistent in $\mathsf{T}^{**} + (\mathsf{Ext}^*)$. By the above lemma, it suffices to show that $\phi \epsilon K$.

For each variable μ, let $[\mu] = \{\nu : \mu \equiv \nu \epsilon K\}$, and let g be that function whose domain is the set of predicate variables and which is such that for $n \epsilon \omega$ and π an n-place predicate variable, $g(\pi) = \{\langle [\mu_0], \ldots, [\mu_{n-1}]\rangle : \pi(\mu_0, \ldots, \mu_{n-1}) \epsilon K\}$. We show first (1): for $n \epsilon \omega$ and π, σ n-place predicate variables, if $g(\pi) = g(\sigma)$, then $[\pi] = [\sigma]$. Suppose $g(\pi) = g(\sigma)$. Now for all variables $\mu_0, \ldots, \mu_{n-1}$, if either $\pi(\mu_0, \ldots, \mu_{n-1}) \epsilon K$ and $\sigma(\mu_0, \ldots, \mu_{n-1}) \notin K$ or $\pi(\mu_0, \ldots, \mu_{n-1}) \notin K$ and $\sigma(\mu_0, \ldots, \mu_{n-1}) \epsilon K$, then, by definition of g, $g(\pi) \neq g(\sigma)$. Therefore, since K is maximally consistent, $[\pi(\mu_0, \ldots, \mu_{n-1}) \longleftrightarrow \sigma(\mu_0, \ldots, \mu_{n-1})] \epsilon K$, for all variables $\mu_0, \ldots, \mu_{n-1}$. But then, since K is ω-complete, $\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}[\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1})] \epsilon K$; and, accordingly, by (Ext^*) and the fact that K is maximally consistent, $\pi \equiv \sigma \epsilon K$, and therefore, since indiscernibility is a congruence relation, $[\pi] = [\sigma]$.

Now, let $D = \{[\mu]: \mu$ is a variable$\}$, and for $n \epsilon \omega$, let $\mathscr{F}_n = \{g(\pi): \pi$ is an n-place predicate variable$\}$, and finally let f be that function whose domain is $D \cup \bigcup_{n\epsilon\omega} \mathscr{F}_n$ and which is such that for $x \epsilon D$, $f(x) = x$ but for $n \epsilon \omega$ and $X \epsilon \mathscr{F}_n$, $F(X) = [\pi]$, where π is the first n-place predicate variable such that $X = g(\pi)$. (Note that since $X \epsilon \mathscr{F}_n$, then $X = g(\sigma)$ for at least one n-place predicate variable σ. But then by (1) above, $[\sigma] = [\pi]$, where π is the first such n-place predicate variable.) Let $\mathfrak{A} = \langle D, \langle \mathscr{F}_n \rangle_{n\epsilon\omega}, f\rangle$. Then, from the way $\mathfrak{A}$ was constructed, $\mathfrak{A}$ is a quasi-Fregean model.

We take $\mathfrak{a}$ to be that assignment in $\mathfrak{A}$ which is such that for each individual variable α, $\mathfrak{a}(\alpha) = [\alpha]$ and for each n-place predicate variable π, $\mathfrak{a}(\pi) = g(\pi)$. We now show (2): for each variable μ, $f(\mathfrak{a}(\mu)) = [\mu]$. If μ is an individual variable, then by definition of $\mathfrak{a}$, $\mathfrak{a}(\mu) = [\mu] \epsilon D$, and therefore, by definition of f, $f(\mathfrak{a}(\mu)) = [\mu]$. If μ is an n-place predicate variable, then $\mathfrak{a}(\mu) = g(\mu)$, and, accordingly, $f(\mathfrak{a}(\mu)) = f(g(\mu)) = [\pi]$, where π is the first n-place predicate variable such that $g(\mu) = g(\pi)$, in which case, by (1) above, $[\mu] = [\pi]$, and therefore $f(\mathfrak{a}(\mu)) = [\mu]$.

Finally, we show (3): for all formulas ψ, $\mathfrak{a}$ satisfies ψ in $\mathfrak{A}$ iff $\psi \epsilon K$. We prove (3) by strong induction on the number n of logical constants occurring in ψ. If $n = 0$, then ψ is an atomic formula of the form

$\pi(\mu_0, \ldots, \mu_{n-1})$. But by the satisfaction clause for atomic formulas, $\mathfrak{a}$ satisfies $\pi(\mu_0, \ldots, \mu_{n-1})$ in $\mathfrak{A}$ iff $\langle f(\mathfrak{a}(\mu_0)), \ldots, f(\mathfrak{a}(\mu_{n-1}))\rangle \in \mathfrak{a}(\pi)$, i.e., by (2) above, iff $\langle [\mu_0], \ldots, [\mu_{n-1}]\rangle \in g(\pi)$, and thus by definition of g, iff $\pi(\mu_0, \ldots, \mu_{n-1}) \in K$. If $n \neq 0$ and ψ is of the form either of a negation or a conditional, then (3) is easily seen to hold by the inductive hypothesis. Suppose ψ is of the form $\wedge\alpha\chi$ and that $\wedge\alpha\chi \notin K$ even though $\mathfrak{a}$ satisfies $\wedge\alpha\chi$ in $\mathfrak{A}$. Then, since K is ω-complete, $\chi\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \notin K$, for some individual variable β which can be properly substituted for α in χ, from which it follows by the inductive hypothesis that $\mathfrak{a}$ does not satisfy $\chi\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ in $\mathfrak{A}$; and accordingly, by the semantic lemma of section 3 and (2) above, $\mathfrak{a}\begin{pmatrix}\alpha\\ [\beta]\end{pmatrix}$ does not satisfy χ in $\mathfrak{A}$, which is impossible since $\mathfrak{a}$ satisfies $\wedge\alpha\chi$ in $\mathfrak{A}$ and $[\beta] \in D$. Suppose on the other hand that $\wedge\alpha\chi \in K$ but that $\mathfrak{a}$ does not satisfy $\wedge\alpha\chi$ in $\mathfrak{A}$. Then, for some $[\mu] \in D$, $\mathfrak{a}\begin{pmatrix}\alpha\\ [\mu]\end{pmatrix}$ does not satisfy χ in $\mathfrak{A}$. Now since, as noted in section 2, $\vdash_{\mathbf{T}^{**}} \wedge\alpha\chi \rightarrow \vee\alpha(\mu \equiv \alpha \wedge \chi)$ and K is maximally consistent, then $\vee\alpha(\mu \equiv \alpha \wedge \chi) \in K$. But then, as K is ω-complete, there is an individual variable β which can be properly substituted for α in $(\mu \equiv \alpha \wedge \chi)$ such that $\left(\mu \equiv \beta \wedge \chi\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right) \in K$, and therefore $\mu \equiv \beta \in K$ and $\chi\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \in K$. But as indiscernibility is a congruence relation, $[\mu] = [\beta]$, and therefore $\mathfrak{a}\begin{pmatrix}\alpha\\ [\beta]\end{pmatrix}$ does not satisfy χ in $\mathfrak{A}$, from which it follows by the semantic lemma that $\mathfrak{a}$ does not satisfy $\chi\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ in $\mathfrak{A}$; and, accordingly, by the inductive hypothesis, $\chi\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \notin K$, which is impossible. Finally, if ψ is of the form $\wedge\pi\chi$, then (3) can be shown by an argument similar to that above, except that when the theorem

$$\wedge\pi\chi \rightarrow \vee\pi\left(\wedge\alpha_0 \ldots \wedge\alpha_{n-1}[\pi(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1})] \wedge \chi\right)$$

is used in place of its analogue above, from which we conclude that $\wedge\alpha_0 \ldots \wedge\alpha_{n-1}[\rho(\alpha_0, \ldots, \alpha_{n-1}) \longleftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1})] \in K$, for some n-place predicate variable ρ which can be properly substituted for π in χ, we must subsequently use (U.I.*) in order to infer that $g(\rho) = g(\sigma)$ and therefore that $\mathfrak{a}(\rho) = \alpha(\sigma)$.

Now since K is maximally consistent, every universally closed instance of our comprehension axiom, (A4), is in K and is therefore—by (3) above and the semantic fact that a sentence is true in $\mathfrak{A}$ if at least one assignment satisfies it—true in $\mathfrak{A}$, from which it follows that $\mathfrak{A}$ is normal. But then ϕ is true in $\mathfrak{A}$ since by hypothesis it is valid. Therefore, by (3) above, $\phi \in K$, which completes what was to be shown.

Before concluding this paper it is noteworthy for reasons already cited that our present semantical notions provide a completeness theorem for $\mathbf{T}^{**}$ + (Ext*) + (I*) as well. Let us call a quasi-Fregean model $\langle D, \langle \mathcal{F}_n\rangle_{n\in\omega}, f\rangle$

a *restricted* model if for each $n \epsilon \omega$, there is an $x \epsilon D$ such that for all $X \epsilon \mathscr{F}_n, f(X) = x$. Clearly, essentially the same proofs as provided above for $\mathbf{T}^{**} + (\mathrm{Ext}^*)$ show that $\vdash_{\mathbf{T}^{**} + (\mathrm{Ext}^*) + (\mathrm{I}^*)} \phi$ iff ϕ is true in every normal restricted quasi-Fregean model. That the model constructed in our proof of completeness is in this case a restricted model follows from the fact that now $[\pi] = [\sigma]$, for all n-place predicate variables, π, σ, since $\wedge\pi \wedge\sigma\pi \equiv \sigma \epsilon K$ as K is now assumed to be maximally consistent in $\mathbf{T}^{**} + (\mathrm{Ext}^*) + (\mathrm{I}^*)$.

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