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# GENERALIZATIONS OF THE DISTRIBUTIVE AND ASSOCIATIVE LAWS

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**1 Introduction** Let $x \triangle y$ and $x \bigcirc y$ denote two truth-value functions: $\{0, 1\} \times \{0, 1\} \to \{0, 1\}$, where 1 and 0 denote "true" and "false" respectively. The two functions "and" and "or" satisfy the law

$$(*) \qquad x \triangle (y \bigcirc z) = (x \triangle y) \bigcirc (x \triangle z)$$

in either order. We would like to weaken (*) so that more functions satisfy the relationship. To do so, we use

$$(**) \qquad x \triangle (y \bigcirc z) = (x \triangle y) \bigcirc (x \triangle z) \bigcirc (x \triangle \mathrm{I})$$

where I is the identity of $x \bigcirc y$. (**) is a generalization of (*) for the reason that all functions $x \bigcirc y$ that have identities and all $x \triangle y$ that together satisfy (*) also satisfy (**), but not conversely. This is shown in Theorem 1.

"And" and "or" satisfy the associative law

$$x \triangle (y \triangle z) = (x \triangle y) \triangle z,$$

and so does "equivalence" and "exclusive or." However, we shall demonstrate that for all truth-functions $x \triangle y$, the truth-values of $x \triangle (x \triangle z) \equiv (x \triangle y) \triangle z$ and $x \triangle (y \triangle z) \underline{\vee} (x \triangle y) \triangle z$ are independent of $y$.

**2 The Generalized Distributive Law** We wish to prove the following:

**Theorem 1** (**) *holds*

(a) *for all* $x \triangle y$ *if* $x \bigcirc y$ *is either* $x \equiv y$ *or* $x \underline{\vee} y$;

*and*

(b) *for all* $x \triangle y$ *such that* $y \leqslant z$ *implies* $x \triangle y \leqslant x \triangle z$ *if* $x \bigcirc y$ *is* $x \vee y$ *or* $x \wedge y$.

*Proof:* Note that $x \wedge y$, $x \vee y$, $x \equiv y$, and $x \underline{\vee} y$ are the only functions that have identities, so Theorem 1 has all the possible combinations. All four of them happen to be commutative and associative. For part (a), let us show

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$$x \triangle (y \equiv z) = (x \triangle y) \equiv (x \triangle z) \equiv (x \triangle 1).$$

If $z = 1$, this becomes $x \triangle y = (x \triangle y) \equiv (x \triangle 1) \equiv (x \triangle 1)$ which is clearly true. Thus it holds for $z = 1$ along with $y = 0$ and $y = 1$; by symmetry, the equation holds for $y = 1$ and $z = 0$. Finally, if $y = z = 0$, then we get

$$x \triangle 1 = (x \triangle 0) \equiv (x \triangle 0) \equiv (x \triangle 1),$$

which also simplifies. Since $x \equiv y$ and $x \underline{\vee} y$ are De Morgan complements, the other case of part (a) follows. As for part (b), let us take $x \bigcirc y$ to be $x \wedge y$ and assume $x \triangle (y \wedge z) = (x \triangle y) \wedge (x \triangle z) \wedge (x \triangle 1)$. If $y \leqslant z$, this reduces to $x \triangle y = (x \triangle y) \wedge (x \triangle z) \wedge (x \triangle 1)$. Since $y \leqslant z$ if and only if $(y \wedge z) = y$, it follows that $x \triangle y \leqslant x \triangle z$. Assume now that $y \leqslant z$ implies $x \triangle y \leqslant x \triangle z$. Then it is true that $y \leqslant z$ implies $x \triangle (y \wedge z) \leqslant (x \triangle y) \wedge (x \triangle z) \wedge (x \triangle 1)$. By substituting truth-values, we can show that only equality holds. If $z = 1$, then we have

$$x \triangle y = (x \triangle y) \wedge (x \triangle 1) \wedge (x \triangle 1) = x \triangle y.$$

Thus it holds for $z = 1$ along with $y = 0$ and $y = 1$. Substituting $y = z = 0$ results in

$$x \triangle 0 = (x \triangle 0) \wedge (x \triangle 0) \wedge (x \triangle 1) = x \triangle 0.$$

The rest of part (b) follows by a similar argument.

In (**), $x \triangle \mathrm{I}$ is an "error term" independent of $y$ and $z$, which means that the distributive law "almost" holds. If $x \triangle \mathrm{I} = \mathrm{I}$ for all $x$, then (**) is (*); whereas if $x \triangle y$ is either $x \equiv y$ or $x \underline{\vee} y$ and if $x \triangle \mathrm{I} = \bar{\mathrm{I}}$ for all $x$, then (**) is

$$x \triangle (y \bigcirc z) = -[(x \triangle y) \bigcirc (x \triangle z)].$$

Schröder expressed all truth-functions in the form

$$x \triangle y = \varphi_{11}xy + \varphi_{10}x\bar{y} + \varphi_{01}\bar{x}y + \varphi_{00}\overline{xy}$$

where "$x + y$" is "or," "$xy$" is "and," "$\bar{x}$" is "not $x$," and $\varphi_{ij}$ is either 1 or 0 depending on which truth-function we are considering. Note that the error term is either

$$x \triangle 1 = \varphi_{11}x + \varphi_{01}\bar{x} \text{ or } x \triangle 0 = \varphi_{10}x + \varphi_{00}\bar{x}.$$

**3 The Generalized Associative Law** As a generalization of the associative law, we have:

**Theorem 2** *For all truth-functions $x \triangle y$, the truth-values of the expressions $x \triangle (y \triangle z) \equiv (x \triangle y) \triangle z$ and $x \triangle (y \triangle z) \underline{\vee} (x \triangle y) \triangle z$ are independent of $y$.*

*Proof:* One method of proof is to substitute functions into the first expression for $x \triangle y$ in the form: $xy$, $x\bar{y}$, $\bar{x}y$, $\overline{xy}$, $x + y$, $x + \bar{y}$, $\bar{x} + y$, $\bar{x} + \bar{y}$, $x \equiv y$, $x \underline{\vee} y$, $x$, $y$, $\bar{x}$, $\bar{y}$, 1, and 0, and test each case with truth-values. Since $x \underline{\vee} y = -(x \equiv y)$, the other expression is also independent of $y$.

**4** Conclusion The two laws demonstrate relationships between more truth-functions than "and" and "or." However, they are derived from and do not replace the basic axioms of propositional logic.

## REFERENCE

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