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RESOLUTION AND THE CONSISTENCY OF ANALYSIS

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§1. *Introduction.** In [2] we formulated a system $\mathcal{R}$, called a Resolution system, for refuting finite sets of sentences of type theory, and proved that $\mathcal{R}$ is complete in the (weak) sense that every set of sentences which can be refuted in the system $\mathcal{T}$ of type theory due to Church [5] can also be refuted in $\mathcal{R}$. The statement that $\mathcal{R}$ is in this sense complete is a purely syntactic one concerning finite sequences of wffs. However, it is clear that there can be no purely syntactic proof of the completeness of $\mathcal{R}$, since the completeness of $\mathcal{R}$ is closely related to Takeuti's conjecture [9] (since proved by Takahashi [8] and Pravitz [7]) concerning cut-elimination in type theory. As Takeuti pointed out in [9] and [10], cut-elimination in type theory implies the consistency of analysis. Indeed, Takeuti's conjecture implies the consistency of a formulation of type theory with an axiom of infinity; in such a system classical analysis and much more can be formalized. Hence, to avoid a conflict with Gödel's theorem, any proof of the completeness of resolution in type theory must involve arguments which cannot be formalized in type theory with an axiom of infinity. Indeed, the proof in [2] does involve a semantic argument. Nevertheless, it must be admitted that anyone who does not find the line of reasoning sketched above completely clear will have difficulty finding a unified and coherent exposition of the entire argument in the published literature. We propose to remedy this situation here.

We presuppose familiarity with §2 (The System $\mathcal{T}$) and Definitions 4.1 and 5.1 (The Resolution System $\mathcal{R}$) of [2], and follow the notation used there. In particular, $\Box$ stands for the contradictory sentence $\forall p_o p_o$. To distinguish between formulations of $\mathcal{T}$ with different sets of parameters, we henceforth assume $\mathcal{T}$ has no parameters, and denote by $\mathcal{T}(\mathbf{A}^1, \ldots, \mathbf{A}^n)$ a formulation of the system with parameters $\mathbf{A}^1, \ldots, \mathbf{A}^n$. If $\mathcal{H}$ is a set of sentences, $\mathcal{H} \vdash_{\mathcal{S}} \mathbf{B}$ shall mean that $\mathbf{B}$ is derivable from some finite subset of $\mathcal{H}$ in system $\mathcal{S}$. The deduction theorem is proved in §5 of [5]. We shall

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incorporate into our argument Gandy's results in §3 of [6] with some minor modifications. We also wish to thank Professor Gandy for the basic idea (attributed by him to Turing) used below in showing the relative consistency of the axiom of descriptions. (This idea is mentioned briefly at the top of page 48 of [6].) We shall have occasion to refer to the following wffs:

The set $\mathcal{E}$ of *axioms of extensionality*:

E^{o}: $\forall p_o \forall q_o \cdot p_o \equiv q_o \supset . p_o = q_o$.
$E^{(\alpha\beta)}$: $\forall f_{\alpha\beta} \forall g_{\alpha\beta} \cdot \forall x_\beta [f_{\alpha\beta} x_\beta = g_{\alpha\beta} x_\beta] \supset . f_{\alpha\beta} = g_{\alpha\beta}$.

The *axiom of descriptions* for type α:

D^{α}: $\forall f_{o\alpha} \cdot \exists_1 x_\alpha f_{o\alpha} x_\alpha \supset f_{o\alpha}[\iota_{\alpha(o\alpha)} f_{o\alpha}]$.

An *axiom of infinity* for type α:

J^{α}: $\exists r_{o\alpha\alpha} \forall x_\alpha \forall y_\alpha \forall z_\alpha \cdot \exists w_\alpha r_{o\alpha\alpha} x_\alpha w_\alpha \wedge$
$\sim r_{o\alpha\alpha} x_\alpha x_\alpha \wedge . \sim r_{o\alpha\alpha} x_\alpha y_\alpha \vee \sim r_{o\alpha\alpha} y_\alpha z_\alpha \vee r_{o\alpha\alpha} x_\alpha z_\alpha$.

We let $\mathcal{A}$ denote the system obtained when one adds to $\mathcal{T}(\iota_{\iota(o\iota)})$ the axioms $\mathcal{E}$, D^{ι}, and J^{ι}. (Description operators and axioms for higher types are not needed, since Church showed [5] that they can be introduced by definition. This matter is also discussed in [3]).

In §4 we shall show how the natural numbers can be defined, and Peano's Postulates can be proved, in $\mathcal{A}$. The basic ideas here go back to Russell and Whitehead [11], of course, but our simple axiom of infinity is not that of Principia Mathematica, but is due to Bernays and Schönfinkel [4]. The natural numbers can be treated in a variety of ways in type theory (e.g., as in [5]), but we believe that the treatment given here has certain advantages of simplicity and naturalness. The simplicity of the axiom of infinity J^{ι} is essential to our program in §3.

Once one has represented the natural numbers in $\mathcal{A}$, one can easily represent the primitive recursive functions. (With minor changes in type symbols, the details can be found in Chapter 3 of [1].) Syntactic statements about wffs can be represented in the usual way by wffs of $\mathcal{A}$ via the device of Gödel numbering. Thus there is a wff *Consis* of $\mathcal{A}$ whose interpretation is that $\mathcal{A}$ is consistent, and by Gödel's theorem it is not the case that $\vdash_{\mathcal{A}}$ *Consis*. Nevertheless, much of mathematics can be formalized in $\mathcal{A}$.

The completeness theorem for $\mathcal{R}$ (Theorem 5.3 of [2]) is also a purely syntactic statement, and hence can be represented by a wff R of $\mathcal{A}$. After preparing the ground in §2 with some preliminary results, in §3 we shall show that by using the completeness of $\mathcal{R}$ we can prove the consistency of $\mathcal{A}$. This argument will be purely syntactic, and could be formalized in $\mathcal{A}$, so $\vdash_{\mathcal{A}}[R \supset \textit{Consis}]$. Thus it is not the case that $\vdash_{\mathcal{A}} R$, so any proof of the completeness of resolution in type theory must transcend the rather considerable means of proof available in $\mathcal{A}$. Of course such a proof can be formalized in transfinite type theory or in Zermelo set theory.

§2. *Preliminary Definitions and Lemmas.* We first establish some preliminary results which will be useful in §3. The reader may wish to postpone the proofs of this section and proceed rapidly to §3. In presenting proofs of theorems of $\mathcal{T}$ (and extensions of $\mathcal{T}$), we shall make extensive use of proofs from hypotheses and the deduction theorem. Each line of a proof will have a number, which will appear at the left hand margin in parentheses. For the sake of brevity, this number will be used as an abbreviation for the wff which is asserted in that line. At the right hand margin we shall list the number(s) of the line(s) from which the given line is inferred (unless it is simply inferred from the preceding line). We use "hyp" to indicate that the wff is inferred with the aid of one or more of the hypotheses of the given line. Thus in

(.1)	$\vdash \mathbf{A}$	
(.2)	$\mathbf{B} \vdash \mathbf{B}$	hyp
(.3)	$\mathbf{B} \vdash \mathbf{C}$	.1, .2
(.4)	$\mathbf{D} \vdash \mathbf{C}$	.1, hyp

the hypothesis **B** is introduced in line .2, and **C** is inferred from **B** and the theorem **A** in line .3; **C** is also inferred from **A** and a different hypothesis **D** in line .4. However, if the wffs **B** and **C** are long, we may write this proof instead as follows:

(.1)	$\vdash \mathbf{A}$	
(.2)	$.2 \vdash \mathbf{B}$	hyp
(.3)	$.2 \vdash \mathbf{C}$	.1, .2
(.4)	$\mathbf{D} \vdash .3$	.1, hyp

A generally useful derived rule of inference is that if $\mathcal{H}$ is a set of hypotheses such that $\mathcal{H} \vdash \exists \mathbf{x}\mathbf{A}$ and $\mathcal{H}, \mathbf{A} \vdash \mathbf{B}$, where $\mathbf{x}$ does not occur free in **B** or any wff of $\mathcal{H}$, then $\mathcal{H} \vdash \mathbf{B}$. We shall indicate applications of this rule in the following fashion:

(.17)	$\mathcal{H} \vdash \exists \mathbf{x}\mathbf{A}$	...
(.20)	$\mathcal{H}, .20 \vdash \mathbf{A}$	choose $\mathbf{x}$ (.17)
(.23)	$\mathcal{H}, .20 \vdash \mathbf{B}$	...
(.24)	$\mathcal{H} \vdash \mathbf{B}$	.17, .23

If the wff **A** is long, we might write step (.17) as follows:

(.17) $\mathcal{H} \vdash \exists \mathbf{x}.20$

We shall present only abstracts of proofs, omitting many steps and using familiar laws of quantification theory, equality, and λ-conversion quite freely. We shall usually omit type symbols on occurrences of variables after the first.

Definition. For each wff **A** of $\mathcal{T}(\iota_{o\iota(o(o\iota))})$, let $\#\mathbf{A}$ be the wff of $\mathcal{T}$ which is the result of replacing the primitive constant $\iota_{o\iota(o(o\iota))}$ everywhere by the wff

$$[\lambda f_{o(o\iota)}\lambda z_\iota \cdot \exists x_{o\iota} \cdot f_{o(o\iota)} x_{o\iota} \wedge x_{o\iota} z_\iota].$$

Lemma 1. $\mathrm{E}^o, \mathrm{E}^{o\iota} \vdash_{\mathcal{T}} \#\mathrm{D}^{o\iota}$.

Proof: First note that $\#\mathrm{D}^{o\iota}$ conv $\forall f_{o(o\iota)} \cdot \exists_1 x_{o\iota} f x \supset f[\lambda z_\iota \cdot \exists x_{o\iota} \cdot f x \wedge x z]$

(.1)	$.1 \vdash \exists_1 x_{o\iota} f_{o(o\iota)} x_{o\iota}$	hyp
(.2)	$.1, .2 \vdash f_{o(o\iota)} x_{o\iota} \wedge \forall u_{o\iota} . fu \supset u = x$	choose x(.1)
(.3)	$.1, .2 \vdash x_{o\iota} z_\iota \equiv \exists x_{o\iota} . f_{o(o\iota)} x \wedge xz$	.2
(.4)	$\mathrm{E}^o, .1, .2 \vdash \forall z_\iota . x_{o\iota} z_\iota = \exists x_{o\iota} . f_{o(o\iota)} x \wedge xz$	.3, E^o
(.5)	$\mathrm{E}^o, \mathrm{E}^{o\iota}, .1, .2 \vdash x_{o\iota} = [\lambda z_\iota . \exists x_{o\iota} . f_{o(o\iota)} x \wedge xz]$	.4, $\mathrm{E}^{o\iota}$
(.6)	$\mathrm{E}^o, \mathrm{E}^{o\iota}, .1, .2 \vdash f_{o(o\iota)} [\lambda z_\iota . \exists x_{o\iota} . fx \wedge xz]$	.2, .5
(.7)	$\mathrm{E}^o, \mathrm{E}^{o\iota}, .1 \vdash .6$	.1, .6
(.8)	$\mathrm{E}^o, \mathrm{E}^{o\iota} \vdash \# \mathrm{D}^{o\iota}$	.7

Lemma 2. $\mathrm{J}^\iota \vdash \mathrm{J}^{o\iota}$

Proof: We assume J^ι.

(.1)	$.1 \vdash \forall x_\iota \forall y_\iota \forall z_\iota . \exists w_\iota r_{o\iota\iota} xw \wedge \sim rxx \wedge . \sim rxy \vee \sim ryz \vee rxz$	choose $r_{o\iota\iota}$

Let $\mathsf{K}_{o(o\iota)(o\iota)}$ be

$$[\lambda u_{o\iota} \lambda v_{o\iota} . \exists t_\iota v_{o\iota} t_\iota \wedge . \sim \exists s_\iota u_{o\iota} s_\iota \vee \exists s_\iota . u_{o\iota} s_\iota \wedge \forall t_\iota . v_{o\iota} t_\iota \supset r_{o\iota\iota} s_\iota t_\iota].$$

We shall establish in lines (.11), (.16) and (.31) that K has the properties necessary to establish $\mathrm{J}^{o\iota}$. To attack (.11) we consider two cases, (.2) and (.5).

(.2)	$.2 \vdash \sim \exists s_\iota x_{o\iota} s$	hyp (case 1)
(.3)	$.2 \vdash \mathsf{K} x_{o\iota} [\lambda t_\iota . t_\iota = t_\iota]$	.2, def. of K
(.4)	$.2 \vdash \exists w_{o\iota} \mathsf{K} x_{o\iota} w$	.3
(.5)	$.5 \vdash \exists s_\iota x_{o\iota} s_\iota$	hyp (case 2)
(.6)	$.5, .6 \vdash x_{o\iota} s_\iota$	choose s (.5)
(.7)	$.1, .5, .6, .7 \vdash r_{o\iota\iota} s_\iota w_\iota$	choose w_ι (.1)
(.8)	$.1, .5, .6, .7 \vdash \mathsf{K} x_{o\iota} [\lambda t_\iota . w_\iota = t_\iota]$	.6, .7, def. of K
(.9)	$.1, .5, .6, .7 \vdash \exists w_{o\iota} \mathsf{K} x_{o\iota} w$	.8
(.10)	$.1, .5 \vdash .9$	.9, .1, .5
(.11)	$.1 \vdash \exists w_{o\iota} \mathsf{K} x_{o\iota} w$	.4, .10

Next we attack (.16). The proof is by contradiction.

(.12)	$.12 \vdash \mathsf{K} x_{o\iota} x_{o\iota}$	hyp
(.13)	$.12 \vdash \exists s_\iota . x_{o\iota} s \wedge \forall t_\iota . xt \supset r_{o\iota\iota} st$	.12, def. of K
(.14)	$.12 \vdash \exists s_\iota r_{o\iota\iota} ss$	.13 (instantiate t with s)
(.15)	$.1 \vdash \forall s_\iota \sim r_{o\iota\iota} ss$	.1
(.16)	$.1 \vdash \sim \mathsf{K} x_{o\iota} x_{o\iota}$	.14, .15

Finally we attack (.31).

(.17)	$.17 \vdash \mathsf{K} x_{o\iota} y_{o\iota} \wedge \mathsf{K} y_{o\iota} z_{o\iota}$	hyp
(.18)	$.17 \vdash \exists t_\iota y_{o\iota} t \wedge \exists t_\iota z_{o\iota} t$	.17, def. of K
(.19)	$.17 \vdash \sim \exists s_\iota x_{o\iota} s \vee \exists s_\iota . xs \wedge \forall q_\iota . y_{o\iota} q \supset r_{o\iota\iota} sq$	.17, def. of K

In (.20) and (.21) we consider the two possibilities set forth in (.19).

(.20)	$.17, \sim \exists s_\iota x_{o\iota} s \vdash \mathsf{K} x_{o\iota} z_{o\iota}$	.18, hyp, def. of K
(.21)	$.17, .21 \vdash \exists s_\iota . x_{o\iota} s \wedge \forall q_\iota . y_{o\iota} q \supset r_{o\iota\iota} sq$	hyp
(.22)	$.17, .21, .22 \vdash x_{o\iota} s_\iota \wedge \forall q_\iota . y_{o\iota} q \supset r_{o\iota\iota} sq$	choose s (.21)

(.23)	$.17 \vdash \exists q_\iota .24$	.17, .18, def. of K
(.24)	$.17, .24 \vdash y_{o\iota} q_\iota \wedge \forall t_\iota .\ z_{o\iota} t \supset r_{o\iota\iota} qt$	choose q (.23)
(.25)	$.17, .21, .22, .24, z_{o\iota} t_\iota \vdash r_{o\iota\iota} s_\iota q_\iota \wedge rqt_\iota$	hyp, .22, .24
(.26)	$.1, .17, .21, .22, .24, z_{o\iota} t_\iota \vdash r_{o\iota\iota} s_\iota t_\iota$	.1, .25
(.27)	$.1, .17, .21, .22, .24 \vdash \forall t_\iota .\ z_{o\iota} t \supset r_{o\iota\iota} s_\iota t$	.26
(.28)	$.1, .17, .21, .22, .24 \vdash \mathsf{K} x_{o\iota} z_{o\iota}$	.18, .22, .27, def. of K
(.29)	$.1, .17, .21 \vdash .28$	.23, .21, .28
(.30)	$.1, .17 \vdash .28$	.19, .20, .29
(.31)	$.1 \vdash \sim \mathsf{K} x_{o\iota} y_{o\iota} \vee \sim \mathsf{K} y z_{o\iota} \vee \mathsf{K} xz$	.30
(.32)	$.1 \vdash \mathrm{J}^{o\iota}$	.11, .16, .31
(.33)	$\mathrm{J}^{\iota} \vdash \mathrm{J}^{o\iota}$	.32

We next repeat Gandy's definitions in [6] with some minor modifications.

Definition. By induction on γ, we define wffs $\mathsf{Mod}_{o\gamma}$ and $\mathsf{M}_{o\gamma\gamma}$ for each type symbol γ.

$\mathbf{A}_\gamma \overset{\mathsf{M}}{=} \mathbf{B}_\gamma$ stands for $\mathsf{M}_{o\gamma\gamma} \mathbf{A}_\gamma \mathbf{B}_\gamma$.
$\mathsf{Mod}_{o\kappa}$ stands for $[\lambda x_\kappa \exists p_o p_o]$ for $\kappa = o, \iota$.
M_{ooo} stands for $[\lambda p_o \lambda q_o .\ p_o \equiv q_o]$.
$\mathsf{M}_{o\iota\iota}$ stands for $[\lambda x_\iota \lambda y_\iota .\ x_\iota = y_\iota]$.
$\mathsf{Mod}_{o(\alpha\beta)}$ stands for $[\lambda f_{\alpha\beta} .\ \forall x_\beta \forall y_\beta .\ \mathsf{Mod}_{o\beta} x_\beta \wedge \mathsf{Mod}_{o\beta} y_\beta \wedge x_\beta \overset{\mathsf{M}}{=} y_\beta \supset .\ \mathsf{Mod}_{o\alpha}[f_{\alpha\beta} x_\beta] \wedge .\ f_{\alpha\beta} x_\beta \overset{\mathsf{M}}{=} f_{\alpha\beta} y_\beta]$.
$\mathsf{M}_{o(\alpha\beta)(\alpha\beta)}$ stands for $[\lambda f_{\alpha\beta} \lambda g_{\alpha\beta} .\ \forall x_\beta .\ \mathsf{Mod}_{o\beta} x_\beta \supset .\ f_{\alpha\beta} x_\beta \overset{\mathsf{M}}{=} g_{\alpha\beta} x_\beta]$.

Lemma 3. $\vdash_{\mathcal{T}} x_\alpha \overset{\mathsf{M}}{=} x_\alpha \wedge .\ x_\alpha \overset{\mathsf{M}}{=} y_\alpha \supset .\ z_\alpha \overset{\mathsf{M}}{=} x_\alpha \equiv .\ z_\alpha \overset{\mathsf{M}}{=} y_\alpha$.

Proof: By induction on α.

Definition. For each wff $\mathbf{A}$ of $\mathcal{T}$, $\mathbf{A}^{\mathsf{T}}$ is the result of replacing $\Pi_{o(o\alpha)}$ by $[\lambda f_{o\alpha} .\ \forall x_\alpha .\ \mathsf{Mod}_{o\alpha} x_\alpha \supset f_{o\alpha} x_\alpha]$ everywhere in $\mathbf{A}$.

Lemma 4. *If* $\mathbf{A}^1, \ldots, \mathbf{A}^n$, *and* $\mathbf{B}$ *are sentences of* $\mathcal{T}$ *such that* $\mathbf{A}_1, \ldots, \mathbf{A}^n \vdash_{\mathcal{T}} \mathbf{B}$, *then* $(\mathbf{A}^1)^{\mathsf{T}}, \ldots, (\mathbf{A}^n)^{\mathsf{T}} \vdash_{\mathcal{T}} \mathbf{B}^{\mathsf{T}}$.

Proof: This is an immediate consequence of Theorem 3.26 of [6], since Gandy's full translation $\mathbf{C}^{\mathsf{F}}$ of $\mathbf{C}$ is $\mathbf{C}^{\mathsf{T}}$ when $\mathbf{C}$ is a sentence. Our modifications of Gandy's definitions do not injure the proof.

Lemma 5. $\vdash_{\mathcal{T}} \mathsf{Mod}[\mathsf{M}_{o\alpha\alpha} z_\alpha]$.

Proof: $\mathsf{Mod}[\mathsf{M}_{o\alpha\alpha} z_\alpha]$ is equivalent to

$\forall x_\alpha \forall y_\alpha [\mathsf{Mod}\ x_\alpha \wedge \mathsf{Mod}\ y_\alpha \wedge x \overset{\mathsf{M}}{=} y \supset .\ \mathsf{Mod}[\mathsf{M}_{o\alpha\alpha} z_\alpha x_\alpha] \wedge .\mathsf{M}_{o\alpha\alpha} z_\alpha x_\alpha \equiv \mathsf{M}_{o\alpha\alpha} z_\alpha y_\alpha]$.

This is readily proved using the definition of Mod_{oo} and Lemma 3.

Lemma 6. $\vdash_{\mathcal{T}} (\mathrm{E}^\gamma)^{\mathsf{T}}$ *for each* E^γ *in* $\mathcal{E}$.

Proof: $(\mathrm{E}^o)^{\mathsf{T}}$ is equivalent to

$\forall p_o[\mathsf{Mod}\ p_o \supset \forall q_o .\ \mathsf{Mod}\ q_o \supset .\ [p_o \equiv q_o] \supset \forall f_{oo} .\ \mathsf{Mod}\ f_{oo} \supset .\ f_{oo} p_o \supset f_{oo} q_o]$,

which is easily proved using the definition of $\mathsf{Mod}\ f_{oo}$. $(\mathrm{E}^{\alpha\beta})^{\mathsf{T}}$ is equivalent to

$\forall f_{\alpha\beta}[\mathrm{Mod}\, f \supset \forall g_{\alpha\beta}.\ \mathrm{Mod}\, g \supset.\ \forall x_\beta[\mathrm{Mod}\, x \supset \forall h_{o\alpha}.\ \mathrm{Mod}\, h \supset.\ h[fx] \supset h.gx]$
$\supset \forall k_{o(\alpha\beta)}.\ \mathrm{Mod}\, k \supset.\ kf \supset kg]$,

which we prove as follows:

(.1)	$.1 \vdash \mathrm{Mod}\, f_{\alpha\beta} \wedge \mathrm{Mod}\, g_{\alpha\beta}$	hyp
(.2)	$.2 \vdash \forall x_\beta[\mathrm{Mod}\, x \supset \forall h_{o\alpha}.\ \mathrm{Mod}\, h \supset .h[fx] \supset h.gx]$	hyp
(.3)	$.3 \vdash \mathrm{Mod}\, k_{o(\alpha\beta)}$	hyp
(.4)	$\vdash \mathrm{Mod}_{(o\alpha)}\ .\ \mathrm{M}_{o\alpha\alpha} \cdot f_{\alpha\beta}x_\beta$	Lemma 5
(.5)	$.2,\ \mathrm{Mod}\, x_\beta \vdash [\mathrm{M}_{o\alpha\alpha}.\ f_{\alpha\beta}x_\beta]\ [f_{\alpha\beta}x_\beta] \supset.\ [\mathrm{M}_{o\alpha\alpha} \cdot f_{\alpha\beta}x_\beta]\ .g_{\alpha\beta}x_\beta$	.2, .4 (instantiate $h_{o\alpha}$ with $\mathrm{M}[fx]$)
(.6)	$\vdash \mathrm{M}_{o\alpha\alpha}[f_{\alpha\beta}x_\beta]\ [f_{\alpha\beta}x_\beta]$	Lemma 3
(.7)	$.2,\ \mathrm{Mod}\, x_\beta \vdash f_{\alpha\beta}x_\beta \overset{\mathrm{M}}{=} g_{\alpha\beta}x_\beta$	.5, .6
(.8)	$.2 \vdash f_{\alpha\beta} \overset{\mathrm{M}}{=} g_{\alpha\beta}$	.7, def. of $\mathrm{M}_{o(\alpha\beta)(\alpha\beta)}$
(.9)	$.1,\ .2,\ .3 \vdash k_{o(\alpha\beta)}f_{\alpha\beta} \equiv k_{o(\alpha\beta)}g_{\alpha\beta}$	.3, def. of $\mathrm{Mod}\, k_{o(\alpha\beta)}$, .1, .8
(.10)	$\vdash (\mathrm{E}^{\alpha\beta})^{\mathsf{T}}$	.9

Lemma 7. $\vdash_{\mathcal{T}} \mathrm{Mod}\, r_{o\iota\iota}$.

Proof: $\mathrm{Mod}\, z_{o\iota}$ is equivalent to

$\forall x_\iota\, \forall y_\iota[\mathrm{Mod}\, x_\iota \wedge \mathrm{Mod}\, y_\iota \wedge x_\iota = y_\iota \supset.\ \mathrm{Mod}[z_{o\iota}x_\iota] \wedge.\ z_{o\iota}x_\iota \equiv z_{o\iota}y_\iota]$

so $\vdash \forall z_{o\iota}\ \mathrm{Mod}\, z_{o\iota}$. $\mathrm{Mod}\, r_{o\iota\iota}$ is equivalent to

$\forall x_\iota\, \forall y_\iota[\mathrm{Mod}\, x_\iota \wedge \mathrm{Mod}\, y_\iota \wedge x_\iota = y_\iota \supset.\ \mathrm{Mod}[r_{o\iota\iota}x_\iota] \wedge \forall w_\iota.\ \mathrm{Mod}\, w_\iota \supset.$
$r_{o\iota\iota}x_\iota w_\iota \equiv r_{o\iota\iota}y_\iota w_\iota]$,

which is easily proved.

Lemma 8. $\mathrm{J}^\iota \vdash_{\mathcal{T}} (\mathrm{J}^\iota)^{\mathsf{T}}$.

Proof: $(\mathrm{J}^\iota)^{\mathsf{T}}$ is equivalent to

$\exists r_{o\iota\iota}[\mathrm{Mod}\, r_{o\iota\iota} \wedge \forall x_\iota.\ \mathrm{Mod}\, x_\iota \supset \forall y_\iota.\ \mathrm{Mod}\, y_\iota \supset \forall z_\iota.\ \mathrm{Mod}\, z_\iota \supset.$
$\exists w_\iota[\mathrm{Mod}\, w_\iota \wedge r_{o\iota\iota}x_\iota w_\iota] \wedge \sim r_{o\iota\iota}x_\iota x_\iota \wedge.\ \sim r_{o\iota\iota}x_\iota y_\iota \vee \sim r_{o\iota\iota}y_\iota z_\iota \vee r_{o\iota\iota}x_\iota z_\iota]$.

This is easily derived from J^ι with the aid of Lemma 7.

Definition: Let θ be the substitution $\mathop{\mathrm{S}}^{\mathbf{x}^1 \ldots \mathbf{x}^n}_{\mathbf{A}^1 \ldots \mathbf{A}^n}$, i.e., the simultaneous substitution of $\mathbf{A}^i$ for all free occurrences of $\mathbf{x}^i$ for $1 \leqslant i \leqslant n$, where $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are distinct variables and $\mathbf{A}^i$ has the same type as $\mathbf{x}^i$ for $1 \leqslant i \leqslant n$. If $\mathbf{B}$ is any wff, we let $\theta * \mathbf{B}$ denote $\eta[[\lambda \mathbf{x}^1 \ldots \lambda \mathbf{x}^n \mathbf{B}]\, \mathbf{A}^1 \ldots \mathbf{A}^n]$. If θ is the null substitution (i.e., $n = 0$), then $\theta * \mathbf{B}$ denotes $\eta \mathbf{B}$.

Note that if $\mathbf{x}_\alpha$ and $\mathbf{y}_\beta$ are distinct variables, $[[\lambda \mathbf{x}_\alpha \lambda \mathbf{y}_\beta \mathbf{B}]\, \mathbf{A}_\alpha \mathbf{C}_\beta]$ conv $[[\lambda \mathbf{y}_\beta \lambda \mathbf{x}_\alpha \mathbf{B}]\mathbf{C}_\beta \mathbf{A}_\alpha]$, so the definition above is unambiguous. Clearly, if there are no conflicts of bound variables, $\theta * \mathbf{B}$ is simply $\eta\, \theta \mathbf{B}$, the η-normal form of the result of applying the substitution θ to $\mathbf{B}$. From the definition it is evident that if $\mathbf{B}$ conv $\mathbf{C}$, then $\theta * \mathbf{B} = \theta * \mathbf{C}$.

§3. *The Consistency of $\mathcal{A}$.*

Theorem. *$\mathcal{A}$ is consistent.*

Proof: The proof is by contradiction, so we suppose $\mathcal{A}$ is inconsistent. Thus

(1) $\mathrm{J}^{\iota}, \mathcal{E}, \mathrm{D}^{\iota} \vdash_{\mathcal{T}(\iota_{\iota(o\iota)})} \Box$.

(2) $\mathrm{J}^{o\iota}, \mathcal{E}, \mathrm{D}^{o\iota} \vdash_{\mathcal{T}(\iota_{o\iota(o(o\iota))})} \Box$.

Proof: Replace the type symbol ι by the type symbol $(o\iota)$ everywhere in the sequence of wffs which constitutes a proof of $\Box$ whose existence is asserted in step 1. By checking the axioms and rules of inference of $\mathcal{T}$ one easily sees that a proof of $\Box$ satisfying the requirements of step 2 is obtained.

(3) $\mathrm{J}^{o\iota}, \mathcal{E}, \#\mathrm{D}^{o\iota} \vdash_{\mathcal{T}} \Box$.

Proof: The replacement of $\mathbf{A}$ by $\#\mathbf{A}$ everywhere in the proof whose existence is asserted in step 2 yields a proof satisfying step 3, possibly after the insertion of a few applications of the rule of alphabetic change of bound variables.

(4) $\mathrm{J}^{o\iota}, \mathcal{E} \vdash_{\mathcal{T}} \Box$ by Lemma 1.

(5) $\mathrm{J}^{\iota}, \mathcal{E} \vdash_{\mathcal{T}} \Box$ by Lemma 2.

(6) $(\mathrm{J}^{\iota})^{\mathsf{T}}, \{(\mathrm{E}^{\gamma})^{\mathsf{T}} \mid \mathrm{E}^{\gamma} \in \mathcal{E}\} \vdash_{\mathcal{T}} \Box$

Proof: By Lemma 4, since $\vdash_{\mathcal{T}} \Box^{\mathsf{T}} \supset \Box$.

(7) $(\mathrm{J}^{\iota})^{\mathsf{T}} \vdash_{\mathcal{T}} \Box$ by Lemma 6.

(8) $\mathrm{J}^{\iota} \vdash_{\mathcal{T}} \Box$ by Lemma 8.

We next introduce parameters $\bar{r}_{o\iota\iota}$ and $\bar{g}_{\iota\iota}$. Let:

$\mathcal{J} = \{\forall x_{\iota} \bar{r}_{o\iota\iota} x_{\iota}[\bar{g}_{\iota\iota} x_{\iota}],\ \forall x_{\iota} \sim \bar{r}_{o\iota\iota} x_{\iota} x_{\iota},\ \forall x_{\iota} \forall y_{\iota} \forall z_{\iota} . \sim \bar{r}_{o\iota\iota} x_{\iota} y_{\iota} \vee \sim \bar{r}_{o\iota\iota} y_{\iota} z_{\iota} \vee r_{o\iota\iota} x_{\iota} z_{\iota}\}$.

(9) $\mathcal{J} \vdash_{\mathcal{T}(\bar{r}_{o\iota\iota}, \bar{g}_{\iota\iota})} \Box$.

Proof: $\mathrm{J}^{\iota} \vdash_{\mathcal{T}(\bar{r}, \bar{g})} \Box$ by (8), and $\mathcal{J} \vdash_{\mathcal{T}(\bar{r}, \bar{g})} \mathrm{J}^{\iota}$.

(10) $\mathcal{J} \vdash_{\mathcal{R}} \Box$

Proof: This follows from (9) by the completeness of resolution in type theory, i.e., Theorem 5.3 of [2]. The proof of this theorem is the one non-syntactic step in our present proof of the consistency of $\mathcal{A}$.

(11) It is not the case that $\mathcal{J} \vdash_{\mathcal{R}} \Box$.

Proof: An η-wff of the form $\bar{r}_{o\iota\iota}\mathbf{A}_{\iota}\mathbf{B}_{\iota}$ will be called *positive* if the number of occurrences of $\bar{g}_{\iota\iota}$ in $\mathbf{A}_{\iota}$ is strictly less than the number of occurrences of $\bar{g}_{\iota\iota}$ in $\mathbf{B}_{\iota}$, and otherwise *negative*. An η-wff of the form $\sim r_{o\iota\iota}\mathbf{A}_{\iota}\mathbf{B}_{\iota}$ will be called positive iff $\bar{r}_{o\iota\iota}\mathbf{A}_{\iota}\mathbf{B}_{\iota}$ is negative, and negative iff $\bar{r}_{o\iota\iota}\mathbf{A}_{\iota}\mathbf{B}_{\iota}$ is positive. Let $\mathcal{G}$ be the set of wffs $\mathbf{G}$ having one of the following six forms:

(a) $\forall \mathbf{x}_{\iota}\ \bar{r}\,\mathbf{x}[\bar{g}\mathbf{x}]$

(b) $\forall \mathbf{x}_{\iota} \sim \bar{r}\mathbf{x}\mathbf{x}$

(c) $\forall \mathbf{x}_{\iota} \forall \mathbf{y}_{\iota} \forall \mathbf{z}_{\iota}\ [\sim \bar{r}\mathbf{x}\mathbf{y} \vee \sim \bar{r}\mathbf{y}\mathbf{z} \vee \bar{r}\mathbf{x}\mathbf{z}]$ where $\mathbf{x}_{\iota}$, $\mathbf{y}_{\iota}$, and $\mathbf{z}_{\iota}$ are distinct variables.

(d) $\forall \mathbf{y}_\iota \forall \mathbf{z}_\iota\, [\sim\bar{r}\,\mathbf{A}_\iota\mathbf{y} \vee \sim\bar{r}\mathbf{y}\mathbf{z} \vee \bar{r}\,\mathbf{A}_\iota\mathbf{z}]$ where $\mathbf{y}_\iota$ and $\mathbf{z}_\iota$ are distinct from one another and from the free variables of $\mathbf{A}_\iota$.
(e) $\forall \mathbf{z}_\iota\, [\sim\bar{r}\,\mathbf{A}_\iota\mathbf{B}_\iota \vee \sim\bar{r}\,\mathbf{B}_\iota\mathbf{z} \vee \bar{r}\,\mathbf{A}_\iota\mathbf{z}]$ where $\mathbf{z}_\iota$ is distinct from the free variables of $\mathbf{A}_\iota$ and of $\mathbf{B}_\iota$.
(f) $\mathbf{G}$ is a disjunction of wffs, each of the form $\bar{r}\,\mathbf{A}_\iota\mathbf{B}_\iota$ or $\sim\bar{r}\,\mathbf{A}\mathbf{B}$, at least one of which is positive.

Let $\mathcal{C}$ be the set of wffs $\mathbf{C}$ such that for each substitution θ, $\theta * \mathbf{C}$ is in $\mathcal{I}$. We assert that if $\mathcal{J} \vdash_{\mathcal{R}} \mathbf{C}$, then $\mathbf{C} \in \mathcal{C}$. Clearly $\mathcal{J} \subseteq \mathcal{C}$, so it suffices to show that $\mathcal{C}$ is closed under the rules of inference of $\mathcal{R}$. For each rule of inference of $\mathcal{R}$ and any substitution θ, we show that $\theta * \mathbf{E} \in \mathcal{I}$ for any wff $\mathbf{E}$ derived from wff(s) of $\mathcal{C}$ by that rule.

Suppose $\mathbf{M} \vee \mathbf{A}$ and $\mathbf{N} \vee \sim\mathbf{A}$ are in $\mathcal{C}$, and $\mathbf{M} \vee \mathbf{N}$ is obtained from them by cut. Then $\theta * [\mathbf{M} \vee \mathbf{A}]$ and $\theta * [\mathbf{N} \vee \sim\mathbf{A}]$ must each have form (f). (For $\theta * [\mathbf{N} \vee \sim\mathbf{A}] = [(\theta * \mathbf{N}) \vee \sim(\theta * \mathbf{A})]$; even if $\mathbf{N}$ is null, this cannot have any of the forms (a)-(e), so $\theta * \mathbf{A}$ must have the form $\bar{r}\,\mathbf{B}_\iota\mathbf{C}_\iota$.) $\theta * [\mathbf{M} \vee \mathbf{A}] = [(\theta * \mathbf{M}) \vee \theta * \mathbf{A}]$; if $\theta * \mathbf{A}$ is negative, $\theta * \mathbf{M}$ must contain a positive wff (so $\mathbf{M}$ cannot be null), so $\theta * [\mathbf{M} \vee \mathbf{N}]$ does also. If $\theta * \mathbf{A}$ is positive, then $\theta * [\sim\mathbf{A}]$ is negative, so $\theta * \mathbf{N}$ must contain a positive wff, so $\theta * [\mathbf{M} \vee \mathbf{N}]$ does also, and hence has form (f).

Suppose $\mathbf{D}$ is in $\mathcal{C}$, and $[\lambda \mathbf{x}_\alpha \mathbf{D}]\mathbf{B}_\alpha$ is obtained from $\mathbf{D}$ by substitution. Let ρ be the substitution $\mathop{\mathrm{S}}\limits^{\mathbf{x}_\alpha}_{\mathbf{B}_\alpha}$, and let $\theta \circ \rho$ be the substitution which is the composition of θ with ρ (i.e., $(\theta \circ \rho) * \mathbf{C} = \theta * (\rho * \mathbf{C})$ for each wff $\mathbf{C}$). Then $\theta * [[\lambda \mathbf{x}_\alpha \mathbf{D}]\mathbf{B}_\alpha] = \theta * \eta[[\lambda \mathbf{x}_\alpha \mathbf{D}]\mathbf{B}_\alpha] = \theta * (\rho * \mathbf{D}) = (\theta \circ \rho) * \mathbf{D} \in \mathcal{I}$ since $\mathbf{D} \in \mathcal{C}$, so $[[\lambda \mathbf{x}_\alpha \mathbf{D}]\mathbf{B}_\alpha] \in \mathcal{C}$.

Suppose $\mathbf{D} \in \mathcal{C}$ and $\mathbf{E}$ is derived from $\mathbf{D}$ by universal instantiation. Thus $\mathbf{D}$ has the form $\mathbf{M} \vee \Pi_{o(o\alpha)}\mathbf{A}_{o\alpha}$, where $\mathbf{M}$ may be null. By considering the null substitution we see that $\eta\,\mathbf{D} \in \mathcal{I}$, so $\mathbf{D}$ has the form $\Pi_{o(o\iota)}\mathbf{A}_{o\iota}$ and $\mathbf{E}$ has the form $\mathbf{A}_{o\iota}\mathbf{x}_\iota$. It is easily checked by examining forms (a)-(e) that if $\mathbf{H}$ is any wff obtained from a wff of $\mathcal{I}$ by universal instantiation, then $(\theta * \mathbf{H}) \in \mathcal{I}$. But $(\eta\,\mathbf{A}_{o\iota})\mathbf{x}_\iota$ is obtained from $\eta\mathbf{D}$ by universal instantiation, so $\theta * \mathbf{E} = \theta * [(\eta\,\mathbf{A}_{o\iota})\mathbf{x}_\iota]$ is in $\mathcal{I}$.

The verification that $\mathcal{C}$ is closed under the remaining rules of inference of $\mathcal{R}$ is trivial, so our assertion is proved. Now $\Box$ is not in $\mathcal{C}$, so it is not the case that $\mathcal{J} \vdash_{\mathcal{R}} \Box$.

(12) The contradiction between (10) and (11) proves our theorem.

§4. *The Natural Numbers in $\mathcal{A}$.* We shall define the natural numbers to be equivalence classes of sets of individuals having the same finite cardinality. We let σ denote the type symbol $(o(o\,\iota))$. σ is the type of natural numbers.

Definitions:

O_σ stands for $[\lambda p_{o\iota}\, \forall x_\iota\, \sim p_{o\iota}x_\iota]$.
$\mathsf{S}_{\sigma\sigma}$ stands for $[\lambda n_{o(o\iota)}\lambda p_{o\iota}\,.\; \exists x_\iota\,.\; p_{o\iota}x_\iota \wedge n_{o(o\iota)}[\lambda t_\iota\,.\; t_\iota \neq x_\iota \wedge p_{o\iota}t_\iota]]$.
$\mathsf{N}_{o\sigma}$ stands for $[\lambda n_\sigma \forall p_{o\sigma}.\; [p_{o\sigma}\mathsf{O}_\sigma \wedge \forall x_\sigma.\; p_{o\sigma}x_\sigma \supset p_{o\sigma}\mathsf{S}_{\sigma\sigma}x_\sigma] \supset p_{o\sigma}n_\sigma]$.

$\dot{\forall} x_\sigma A$ stands for $\forall x_\sigma[\mathsf{N}_{o\sigma} x_\sigma \supset A]$.
$\dot{\exists} x_\sigma A$ stands for $\exists x_\sigma[\mathsf{N}_{o\sigma} x_\sigma \wedge A]$.

Thus zero is the collection of all sets with zero members, i.e., the collection containing just the empty set $[\lambda x_\iota \Box]$. S represents the successor function. If $n_{(o(o\iota))}$ is a finite cardinal (say 2), then a set $p_{o\iota}$ (say $\{a, b, c\}$) is in $\mathsf{S}n$ iff there is an individual (say c) which is in $p_{o\iota}$ and whose deletion from $p_{o\iota}$ leaves a set ($\{a, b\}$) which is in n. $\mathsf{N}_{o\sigma}$ represents the set of natural numbers, i.e., the intersection of all sets which contain O and are closed under S.

We now prove Peano's Postulates (Theorems 1, 2, 3, 4, and 7 below.) In this section $\vdash B$ means B is a theorem of $\mathcal{A}$.

1 $\vdash \mathsf{N}_{o\sigma}\mathsf{O}_\sigma$ — by the def. of N

2 $\vdash \forall x_\sigma.\ \mathsf{N}_{o\sigma} x_\sigma \supset \mathsf{N}_{o\sigma}.\ \mathsf{S}_{\sigma\sigma} x_\sigma$

Proof:

(.1)	$\mathsf{N}x_\sigma,\ .1 \vdash p_{o\sigma}\mathsf{O} \wedge \forall x_\sigma.\ px \supset p.\ \mathsf{S}x$	hyp
(.2)	$\mathsf{N}x_\sigma,\ .1 \vdash p_{o\sigma} x_\sigma$	.1, hyp, def. of N
(.3)	$\mathsf{N}x_\sigma,\ .1 \vdash p_{o\sigma}.\ \mathsf{S}x_\sigma$	.1, .2
(.4)	$\mathsf{N}x_\sigma \vdash \mathsf{N}.\ \mathsf{S}x_\sigma$	.3, def. of N

3 The Induction Theorem:

$\vdash \forall p_{o\sigma}.\ [p_{o\sigma}\mathsf{O}_\sigma \wedge \dot{\forall} x_\sigma.\ p_{o\sigma} x_\sigma \supset p_{o\sigma}.\ \mathsf{S}_{\sigma\sigma} x_\sigma] \supset \dot{\forall} x_\sigma p_{o\sigma} x_\sigma$

Proof: Let $\mathrm{P}_{o\sigma}$ be $[\lambda t_\sigma.\ \mathsf{N}t \wedge p_{o\sigma} t]$.

(.1)	$.1 \vdash p_{o\sigma}\mathsf{O} \wedge \forall x_\sigma.\ \mathsf{N}x \supset.\ px \supset p.\ \mathsf{S}x$	hyp
(.2)	$\mathsf{N}y_\sigma \vdash [\mathrm{P}\mathsf{O} \wedge \forall x_\sigma.\ \mathrm{P}x \supset \mathrm{P}.\ \mathsf{S}x] \supset \mathrm{P}y_\sigma$	hyp, def. of N
(.3)	$.1 \vdash \mathrm{P}\mathsf{O}$	def. of P, .1, Theorem 1
(.4)	$.1 \vdash \forall x_\sigma.\ \mathrm{P}x \supset \mathrm{P}.\ \mathsf{S}x$	def. of P, .1, Theorem 2
(.5)	$.1, \mathsf{N}y_\sigma \vdash \mathrm{P}y_\sigma$	.2, .3, .4
(.6)	$.1 \vdash \dot{\forall} y_\sigma p y_\sigma$	.5, def. of $\dot{\forall}$, P

4 $\vdash \dot{\forall} n_\sigma.\ \mathsf{S}_{\sigma\sigma} n_\sigma \neq \mathsf{O}_\sigma$

Proof by contradiction:

(.1)	$.1 \vdash \mathsf{S}n_\sigma = \mathsf{O}$	hyp
(.2)	$\vdash \mathsf{O}_\sigma[\lambda x_\iota \Box]$	def. of O
(.3)	$.1 \vdash \mathsf{S}n_\sigma[\lambda x_\iota \Box]$	.1, .2
(.4)	$.1 \vdash \exists x_\iota \Box$	.3, def. of S
(.5)	$\vdash \mathsf{S}n_\sigma \neq \mathsf{O}$	.4
(.6)	$\vdash \dot{\forall} n_\sigma.\ \mathsf{S}n \neq \mathsf{O}$	.5, def. of $\dot{\forall}$

Our first step in proving Theorem 7 is to show that if we remove any element from a set of cardinality $\mathsf{S}n$ we obtain a set of cardinality n.

5 $\vdash \dot{\forall} n_\sigma \forall p_{o\iota}.\ \sim p_{o\iota} w_\iota \wedge \mathsf{S}_{\sigma\sigma} n_\sigma[\lambda t_\iota.\ t_\iota = w_\iota \vee p_{o\iota} t_\iota] \supset n_\sigma p_{o\iota}$

The proof is by induction on n. First we treat the case $n = \mathsf{O}$.

(.1)	$.1 \vdash \sim p_{o\iota} w_\iota \wedge \mathsf{S}\mathsf{O}[\lambda t_\iota \,.\, t = w \vee pt]$	hyp
(.2)	$.1 \vdash \exists x_\iota . 3$	.1, def. of S
(.3)	$.1, .3 \vdash [x_\iota = w_\iota \vee p_{o\iota} x] \wedge \mathsf{O}[\lambda t_\iota .\ t \neq x \wedge . t = w \vee pt]$	choose x (.2)
(.4)	$.1, .3 \vdash \sim . w_\iota \neq x_\iota \wedge . w = w \vee p_{o\iota} w_\iota$	.3, def. of O
(.5)	$.1, .3 \vdash w_\iota = x_\iota$	.4
(.6)	$.1, .3 \vdash \forall t_\iota .\ p_{o\iota} t \equiv . t \neq x_\iota \wedge . t = w_\iota \vee pt$	.1, .5
(.7)	$.1, .3 \vdash p_{o\iota} = [\lambda t_\iota .\ t \neq x_\iota \wedge . t = w_\iota \vee pt]$	.6, E^{o}, $\mathrm{E}^{o\iota}$
(.8)	$.1, .3 \vdash \mathsf{O}\, p_{o\iota}$	.3, .7
(.9)	$\vdash \forall p_{o\iota} .\ \sim p_{o\iota} w_\iota \wedge \mathsf{S}\mathsf{O}[\lambda t_\iota .\ t = w \vee pt] \supset \mathsf{O} p$	.2, .8

Next we treat the induction step

(.10)	$.10 \vdash \mathsf{N} n_\sigma \wedge \forall p_{o\iota} .\ \sim p w_\iota \wedge \mathsf{S} n[\lambda t_\iota .\ t = w \vee pt] \supset np$	(inductive) hyp
(.11)	$.11 \vdash \sim p_{o\iota} w_\iota \wedge [\mathsf{S}\mathsf{S} n_\sigma][\lambda t_\iota .\ t = w \vee pt]$	hyp
(.12)	$.11 \vdash \exists x_\iota . 13$	.11, def. of S
(.13)	$.11, .13 \vdash [x_\iota = w_\iota \vee p_{o\iota} x] \wedge . \mathsf{S} n_\sigma[\lambda t_\iota .\ t \neq x \wedge . t = w \vee pt]$	choose x (.12)

From (.11) we must prove $[\mathsf{S} n] p$. We consider two cases in (.14) and (.17).

(.14)	$.14 \vdash x_\iota = w_\iota$	hyp (case 1)
(.15)	$.11, .13, .14 \vdash p_{o\iota} = [\lambda t_\iota .\ t \neq x_\iota \wedge . t = w_\iota \vee pt]$	.11, .14
(.16)	$.11, .13, .14 \vdash [\mathsf{S} n_\sigma] p_{o\iota}$	.13, .15

In case 2 we shall use the inductive hypothesis.

(.17)	$.17 \vdash x_\iota \neq w_\iota$	hyp (case 2)
(.18)	$.17 \vdash [\lambda t_\iota .\ t \neq x_\iota \wedge . t = w_\iota \vee p_{o\iota} t] = [\lambda t_\iota .\ t = w_\iota \vee . t \neq x_\iota \wedge p_{o\iota} t]$	.17
(.19)	$.11, .13, .17 \vdash \mathsf{S} n_\sigma[\lambda t_\iota . t = w_\iota \vee . t \neq x_\iota \wedge p_{o\iota} t]$	.13, .18
(.20)	$.10, .11, .13, .17 \vdash n_\sigma[\lambda t_\iota .\ t \neq x_\iota \wedge p_{o\iota} t]$	.10, .11, .19
(.21)	$.11, .13, .17 \vdash p_{o\iota} x_\iota$	.13, .17
(.22)	$.10, .11, .13, .17 \vdash [\mathsf{S} n_\sigma] p_{o\iota}$	def. of S, .20, .21
(.23)	$.10, .11 \vdash [\mathsf{S} n_\sigma] p_{o\iota}$	.16, .22, .12
(.24)	$.10 \vdash \forall p_{o\iota} .\ \sim p w_\iota \wedge [\mathsf{S}\mathsf{S} n_\sigma][\lambda t_\iota \,.\, t = w \vee pt] \supset [\mathsf{S} n_\sigma] p$	.23

This completes the induction step. The theorem now follows from .9 and .24 by the Induction Theorem.

It will be observed that so far in this section we have not used the axiom of infinity J^{ι}. We shall use it in proving the next theorem, which will also be used to prove Theorem 7.

6 $\vdash \dot\forall n_\sigma .\ n_\sigma p_{o\iota} \supset \exists w_\iota \sim p_{o\iota} w_\iota$

(.1)	$.1 \vdash \forall x_\iota \forall y_\iota \forall z_\iota .\ \exists w_\iota r_{o\iota\iota} x w \wedge \sim rxx \wedge . \sim rxy \vee \sim ryz \vee rxz$	choose r (J^{ι})

Let $\mathsf{P}_{o\sigma}$ be $[\lambda n_\sigma \forall p_{o\iota} .\ np \supset \exists z_\iota \forall w_\iota .\ r_{o\iota\iota} z w \supset \sim pw]$.

We may informally interpret rzw as meaning that z is below w. Thus $\mathsf{P} n$ means that if p is in n, then there is an element z which is below no member of p. We shall prove $\dot\forall n_\sigma \mathsf{P} n$ by induction on n.

(.2)	$\mathsf{O} p_{o\iota} \vdash \sim p_{o\iota} w_\iota$	def. of O
(.3)	$\vdash \mathsf{P}\mathsf{O}$	.2, def. of P

Next we treat the induction step.

(.4)	$.4 \vdash \mathsf{N} n_\sigma \wedge \mathsf{P} n$	(inductive) hyp
(.5)	$.5 \vdash \mathsf{S} n_\sigma p_{o\iota}$	hyp
(.6)	$.5 \vdash \exists x_\iota\ .7$	.5, def. of S
(.7)	$.5, .7 \vdash p_{o\iota} x_\iota \wedge n_\sigma[\lambda t_\iota\,.\, t \neq x \wedge pt]$	choose x (.6)
(.8)	$.4, .5, .7 \vdash \exists z_\iota\ .9$	.4, def. of P, .7
(.9)	$.4, .5, .7, .9 \vdash \forall w_\iota .\ r_{o\iota\iota} z_\iota w \supset .\ w = x_\iota \vee \sim p_{o\iota} w$	choose z (.8)

Thus from the inductive hypothesis we see that there is an element z which is under nothing in $p - \{x\}$. We must show that there is an element which is under nothing in p. We consider two cases, (.10) and (.14).

(.10)	$.10 \vdash \sim r_{o\iota\iota} z_\iota x_\iota$	hyp (case 1)
(.11)	$.4, .5, .7, .9, .10 \vdash r_{o\iota\iota} z_\iota w_\iota \supset w \neq x_\iota$	.10
(.12)	$.4, .5, .7, .9, .10 \vdash \forall w_\iota .\ r_{o\iota\iota} z_\iota w \supset \sim p_{o\iota} w$	.9, .11
(.13)	$.4, .5, .7, .9, .10 \vdash \exists z_\iota\ .12$	.12

Next we consider case 2, and show that x is under nothing in p.

(.14)	$.14 \vdash r_{o\iota\iota} z_\iota x_\iota$	hyp (case 2)
(.15)	$.1, .14, r_{o\iota\iota} x_\iota w_\iota \vdash r_{o\iota\iota} z_\iota w_\iota$	.14, hyp, .1
(.16)	$.1, .4, .5, .7, .9, .14, r_{o\iota\iota} x_\iota w_\iota \vdash w_\iota = x_\iota \vee \sim p_{o\iota} w$	.9, .15
(.17)	$\vdash w_\iota = x_\iota \supset .\ r_{o\iota\iota} x w \supset r x x$	
(.18)	$.1 \vdash \sim r_{o\iota\iota} x_\iota x$	.1
(.19)	$.1, .4, .5, .7, .9, .14 \vdash \forall w_\iota .\ r_{o\iota\iota} x_\iota w \supset \sim p_{o\iota} w$	.16, .17, .18
(.20)	$.1, .4, .5, .7, .9, .14 \vdash \exists z_\iota \forall w_\iota .\ r_{o\iota\iota} z w \supset \sim p_{o\iota} w$	.19
(.21)	$.1, .4, .5 \vdash .20$	.13, .20, .8, .6
(.22)	$.1 \vdash \mathsf{N} n_\sigma \wedge \mathsf{P} n \supset \mathsf{PS} n$	.21, def. of P
(.23)	$.1 \vdash \dot\forall n_\sigma \mathsf{P} n_\sigma$	.3, .22, Theorem 3

Having finished the inductive proof, we proceed to prove the main theorem.

(.24)	$.24 \vdash \mathsf{N} n_\sigma \wedge n p_{o\iota}$	hyp
(.25)	$.1, .24 \vdash \exists z_\iota \forall w_\iota .\ r_{o\iota\iota} z w \supset \sim p_{o\iota} w$	.23, .24, def. of P
(.26)	$.1 \vdash \forall z_\iota \exists w_\iota r_{o\iota\iota} z w$	.1
(.27)	$.1, .24 \vdash \exists w_\iota \sim p_{o\iota} w_\iota$	.25, .26
(.28)	$.1 \vdash \dot\forall n_\sigma .\ n p_{o\iota} \supset \exists w_\iota \sim p_{o\iota} w_\iota$	.27
(.29)	$\vdash .28$	J^ι

7 $\vdash \dot\forall n_\sigma \dot\forall m_\sigma .\ \mathsf{S}_{\sigma\sigma} n_\sigma = \mathsf{S}_{\sigma\sigma} m_\sigma \supset n_\sigma = m_\sigma$

Proof:

(.1)	$.1 \vdash \mathsf{N} n_\sigma \wedge \mathsf{N} m_\sigma \wedge \mathsf{S} n = \mathsf{S} m$	hyp
(.2)	$.2 \vdash n_\sigma p_{o\iota}$	hyp
(.3)	$.1, .2, \vdash \exists w_\iota \sim p_{o\iota} w_\iota$	.1, .2, Theorem 6
(.4)	$.1, .2, .4 \vdash \sim p_{o\iota} w_\iota$	choose w (.3)
(.5)	$.1, .2, .4 \vdash p_{o\iota} = [\lambda t_\iota\,.\, t \neq w_\iota \wedge .\ t = w \vee pt]$	.4, E^{o}, $\mathrm{E}^{o\iota}$
(.6)	$.1, .2, .4 \vdash n_\sigma[\lambda t_\iota\,.\, t \neq w_\iota \wedge .\ t = w \vee p_{o\iota} t]$	.2, .5
(.7)	$.1, .2, .4 \vdash \mathsf{S} n_\sigma[\lambda t_\iota\,.\, t = w_\iota \vee p_{o\iota} t]$	.6, def. of S

(.8)	$.1, .2, .4 \vdash \mathcal{S}\, m_\sigma[\lambda t_\iota .\ t = w_\iota \vee p_{o\iota} t]$	.1, .7
(.9)	$.1, .2, .4 \vdash m_\sigma p_{o\iota}$	.1, .4, .8, Theorem 5
(.10)	$.1 \vdash n_\sigma p_{o\iota} \supset m_\sigma p$	.3, .9
(.11)	$.1 \vdash m_\sigma p_{o\iota} \supset n_\sigma p$	proof as for .10
(.12)	$.1 \vdash \forall p_{o\iota} .\ n_\sigma p \equiv m_\sigma p$	.10, .11
(.13)	$.1 \vdash n_\sigma = m_\sigma$	.12, E^o, $E^{o\iota}$

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