Notre Dame Journal of Formal Logic
Volume XIV, Number 4, October 1973
NDJFAM

A SIMPLIFIED DECISION PROCEDURE FOR CATEGORICAL SYLLOGISMS

HARRY J. GENSLER, S.J.

In this article I would like to (1) present a simplified decision procedure (the "star-test") for categorical syllogisms, (2) prove the equivalence of this procedure with a more traditional set of rules, (3) present an extended form of the star-test in a simple syllogistic calculus, (4) show how the procedure may be used to derive syllogistic conclusions, (5) present a generalized version of the extended star-test, and (6) sketch a parallel inferential proof-procedure.

1 To test a categorical syllogism (on the modern interpretation): if you asterisk just the distributed terms in the premises and the undistributed terms in the conclusion, then the syllogism is valid if and only if every term is asterisked exactly once and there is exactly one right hand asterisk. Let me give a couple of examples:

no P^* is F^*
some C is F
$\therefore$ some C^* is not P

is valid; after asterisking every distributed term in the premises and every undistributed term in the conclusion, it is found that every term is asterisked exactly once and there is exactly one right hand asterisk. But:

no P^* is F^*
some C is not F^*
$\therefore$ some C^* is P^*

is invalid; after doing the asterisking it is found that "P" and "F" are asterisked twice while there are three right hand asterisks. This "star-test" is very easy to learn and remember and takes only five seconds to apply.

A parallel star-test for the Aristotelian interpretation is as follows (the difference between the two tests is italicized): if you asterisk just the distributed terms in the premises and the undistributed terms in the

Received March 3, 1971

conclusion, then the syllogism is valid if and only if every term is asterisked *at least* once and there is exactly one right hand asterisk. To take another example:

all A^* is B
all A^* is C
∴ some B^* is C^*

is valid on the Aristotelian interpretation (every term is asterisked *at least* once) while it is invalid on the modern interpretation ("A" is not asterisked *exactly* once).

2 Now let me prove the equivalence of the star-test with a more traditional set of rules. The set of rules I will use is taken from Salmon's little book [1]; he in turn adapts them from Culbertson [2]. According to these rules, a categorical syllogism is valid on the modern interpretation if and only if:

S1. The middle term is distributed exactly once.
S2. No end term is distributed exactly once.
S3. The number of negative premises equals the number of negative conclusions.

Three equivalent conditions for the modern star-test would be:

*1. The middle term is asterisked exactly once.
*2. Each end term is asterisked exactly once.
*3. There is exactly one right hand asterisk.

Now the equivalence of the star-test with Salmon's rules will be proven by showing the equivalence of *1 with S1, *2 with S2, and *3 with S3.

An occurrence of a middle term will be asterisked if and only if it is distributed; thus *1 is equivalent to S1.

Recall that an end term may be distributed or undistributed in the premises and distributed or undistributed in the conclusion; thus we have four possible cases as follows ("**D**" is for "distributed" while "**U**" is for "undistributed"):

	(1)	(2)	(3)	(4)
premises:	**D***	**U**	**D***	**U**
conclusion:	**D**	**D**	**U***	**U***

(1) and (4) satisfy S2; (2) and (3) violate S2. Asterisking the distributed (**D**) terms in the premises and the undistributed (**U**) terms in the conclusion, we find that (1) and (4) have the end term asterisked exactly once (and thus satisfy *2) while (2) and (3) do not have the end term asterisked exactly once (and thus violate *2). Thus *2 and S2 are satisfied or violated in the same cases and so are equivalent.

Recall that each of the three statements in the syllogism may be either positive or negative; thus we have eight possible cases as follows ("**N**" is for "negative" while "**P**" is for "positive"):

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
major premise:	N*	P	N*	P	N*	P	N*	P
minor premise:	N*	N*	P	P	N*	N*	P	P
conclusion:	N	N	N	N	P*	P*	P*	P*

(2), (3), and (8) satisfy S3; (1), (4), (5), (6), and (7) violate S3. Now a categorical statement has the right hand term distributed if and only if it is a negative statement; this is easy to see by checking the four types of categorical statements (the distributed terms are underlined):

all <u>*A*</u> is *B*	(positive & right hand term undistributed)
some *A* is *B*	(positive & right hand term undistributed)
some *A* is not <u>*B*</u>	(negative & right hand term distributed)
no <u>*A*</u> is <u>*B*</u>	(negative & right hand term distributed)

Now a premise has a right hand asterisk if and only if its right hand term is distributed (and hence if and only if it is a negative statement); and a conclusion has a right hand asterisk if and only if its right hand term is undistributed (and hence if and only if it is a positive statement). The eight cases are asterisked above in accord with whether the respective statement would have a right hand asterisk; thus the "**N**" premises and the "**P**" conclusions are asterisked. Now it is easy to see that (2), (3), and (8) would have exactly one right hand asterisk (and thus satisfy *3) while (1), (4), (5), (6), and (7) would not have exactly one right hand asterisk (and thus violate *3). Thus *3 and S3 are satisfied or violated in the same cases and so are equivalent. Thus the star-test is equivalent to Salmon's rules.

The equivalence of the Aristotelian star-test with Salmon's Aristotelian rules is similarly shown. According to Salmon, a categorical syllogism is valid on the Aristotelian interpretation if and only if:

S′1. The middle term is distributed *at least* once.
S′2. Each end term which is distributed in the conclusion is also distributed in the premises.
S′3. = S3.

Three equivalent conditions for the Aristotelian star-test would be:

*′1. The middle term is asterisked *at least* once.
*′2. Each end term is asterisked *at least* once.
*′3. = *3.

The equivalence of *′1 with S′1 is obvious. The equivalence of *′2 with S′2 is easily seen; cases (1), (3), and (4) satisfy both of them while case (2) violates both of them. The third condition is unchanged. Thus the Aristotelian star-test is equivalent to Salmon's Aristotelian rules.

3 Now let me show how an extended form of the star-test (extended to cover arguments with any number of premises and containing singular and identity statements as well as categorical statements) may be embodied within a simple calculus. I will first give the calculus and then comment upon it.

Syllogistic Calculus (SC)

VOCABULARY: “all,” “no,” “some,” “is,” “not,” capital letters, and small letters (for purposes of this calculus, corresponding capital letters and small letters will be regarded as different letters).

FORMATION RULES: Sequences of the form “all *A* is *B*,” “some *A* is *B*,” “no *A* is *B*,” “some *A* is not *B*,” “*x* is *A*,” “*x* is not *A*,” “*x* is *y*,” or “*x* is not *y*” are *wffs*—where any capital letters may substitute for “*A*” and “*B*” and any small letters may substitute for “*x*” and “*y*.”

Vertical sequences of one or more wffs satisfying these three conditions are *syls* (syllogisms):

(1) Each letter in the sequence occurs exactly twice.
(2) Each wff has at least one letter in common with the wff just below it (if there is one).
(3) The first wff has at least one letter in common with the last wff.

DEFINITION: A letter occurring anywhere after a “no” or “not” in a wff, or else just after an “all,” is *distributed* in that wff.

VALIDITY TEST: If you asterisk just the distributed letters in all but the last wff and the undistributed letters in the last wff, then the syllogism is *valid* if and only if every capital letter is asterisked exactly once and there is exactly one asterisk on the right hand side. (For Aristotle's interpretation, substitute “at least” for the first “exactly”.)

The SC is a calculus in the sense of an artificial language with notational rules for determining wff-hood and validity-or-theoremhood. The capital letters substitute for general terms (and correspond with the predicate letters of quantificational logic) while the small letters substitute for singular terms (and correspond with the individual constants of quantificational logic). “*a* is *F*” is like the “*Fa*” of quantificational logic (and “*a* is not *F*” like “$\sim Fa$”), while “*a* is *b*” is like the “$a = b$” of identity logic (and “*a* is not *b*” like “$a \neq b$”). A syllogism is basically an argument composed of wffs in which the terms form a chain. The definition of “distributed” is notational and somewhat unconventional. According to this definition, the underlined terms below would be distributed:

all $\underline{F}$ is G	a is F
some F is G	a is not $\underline{F}$
some F is not $\underline{G}$	a is b
no $\underline{F}$ is $\underline{G}$	a is not $\underline{b}$

The star-test here is slightly different from that given previously; one difference concerns the distinction between the capital letters and the small letters. Basically, each capital letter must be asterisked exactly once while the number of times a small letter is asterisked is irrelevant. So:

a is not b^*	$a \neq b$
b is F	Fb
$\therefore$ a^* is not F	$\therefore \sim Fa$

is invalid (the capital "F" has to be asterisked exactly once) while the corresponding syllogism with all singular terms:

a is not b^*	$a \neq b$
b is c	$b = c$
$\therefore$ a^* is not c	$\therefore a \neq c$

is valid (the number of times the small "c" is asterisked is irrelevant). The only way a pure-identity syllogism (one with no capital letters) can come out invalid is by not having exactly one right hand asterisk.

A syllogism may contain any number of premises; the following are examples of valid syllogisms containing five, one, and zero premises:

a is B	some A is B	$\therefore$ all A is A^*	$\therefore$ x^* is x^*
a is c	$\therefore$ some B^* is A^*		
c is D			
all D^* is E			
no F^* is E^*			
$\therefore$ some B^* is not F			

The star-test works the same regardless of the number of premises. The following are some simple arguments that come out as valid on the Aristotelian interpretation (where the capital letters substitute for non-empty general terms) but not on the modern interpretation:

all A^* is B	no A^* is B^*	$\therefore$ some A^* is A^*
$\therefore$ some A^* is B^*	$\therefore$ some A^* is not B	

A variation on the star-test may be used to determine whether or not the wffs of a syllogism form an inconsistent set. If you asterisk just the distributed letters in the syllogism, then the wffs of the syllogism form an inconsistent set if and only if every capital letter is asterisked exactly once and there is exactly one asterisk on the right hand side. For example, the following four syllogisms contain inconsistent wffs:

all A^* is B	some A is not B^*	x is not x^*	some A is not A^*
no B^* is C^*	all A^* is B		
d is C			
d is e			
e is F			
all F^* is A			

For Aristotle's interpretation, substitute "at least" for the first "exactly."

I like to start off a freshman logic course with some variation on **SC**. The system is very compact, very easy to master, and covers many arguments besides simply two-premise categorical syllogisms. Also the calculus-form, the use of any number of premises, and the singular term/ general term distinction help prepare the students for propositional and quantificational logic.

4 Conclusion-drawing with the star-test on the modern interpretation is relatively simple. Suppose you have a sequence of one or more wffs, satisfying the three below conditions, from which you wish to derive a syllogistically valid conclusion:

(1) No letter in the sequence occurs more than twice.
(2) Each wff has exactly one letter in common with the wff just below it (if there is one).
(3) Exactly two letters in the sequence occur just once.

Asterisk the distributed letters in these wffs. If some capital letter occurs twice without being asterisked just once, or if there are two or more right hand asterisks, then no valid conclusion is possible. If there is no right hand asterisk, then put one in the conclusion line. Determine which two letters will occur in the conclusion (these will be the two letters that have occurred just once already). Then:

(1) If you have two small letters, put one on the right hand side and one on the left; put "is" (if there is a right hand conclusion asterisk) or "is not" (if there is not) between the two.
(2) If you have a capital letter and a small letter, put the capital letter on the right hand side. If this capital letter is not then asterisked exactly once in the sequence, then no valid conclusion is possible; if it is, then put the small letter on the left hand side. Put "is" (if there is a right hand conclusion asterisk) or "is not" (if there is not) between the two.
(3) If you have two capital letters: if there is a right hand conclusion asterisk, then put on the right hand side one of these capitals that is not already asterisked; but if there is no right hand conclusion asterisk, then put on the right hand side one that is already asterisked. (If this cannot be done, then no conclusion is possible; this will happen if both capitals are asterisked above while there is a right hand conclusion asterisk or, alternatively, if neither capital is asterisked above while there is no right hand conclusion asterisk.) Put the other letter on the left hand side; asterisk it if and only if it is not asterisked above. Then fill in the rest of the conclusion in accord with the resulting asterisk-pattern as follows:

some . . .* is . . .*
all . . . is . . .*
some . . .* is not . . .
no . . . is . . .

In practice this is much easier to do than to explain.

5 Let us define a *zyl* (*zyllogism*) as a vertical sequence of one or more **SC** wffs. Then a syl (syllogism) would be a special case of a zyl in which the terms form a chain. The **SC** star-test correctly decides the validity of syls, but not the validity of zyls. For example, the following valid zyl would come out as invalid on the **SC** star-test:

no X* is Y*
∴ all A is A*

while the following invalid zyl comes out as valid on the test:

∴ some A* is B*

The **SC** star-test fails for zyls because zyls can include unnecessary wffs (as "no X is Y" in the first example) and the letters in a zyl need not form a chain (as happens in the second example); there are other reasons as well. Can the star-test be generalized to cover zyls? I suggest the following generalizations. I give the Aristotelian star-test first because it is less complex.

To test a zyl (on the Aristotelian interpretation): First asterisk just the distributed letters in all but the last wff and the undistributed letters in the last wff. Then the zyl is valid if and only if some group of wffs from the zyl can be arranged to form a syl in which every capital letter is asterisked at least once and there is exactly one right hand asterisk. This works for the above examples. It also works for this complex example:

all A* is B
no D* is E*
all B* is C
no F* is G*
some A is not C*
∴ x* is not z

This is valid because the following group of wffs from the zyl (keeping the above asterisking) can be arranged to form a syl satisfying the above conditions:

all A* is B
all B* is C
some A is not C*

In this case the original conclusion need not be used (because the premises are inconsistent).

The modern star-test is more complex. It would not do to just substitute "every capital letter is asterisked *exactly once*" in the Aristotelian star-test; for then the following zyls which are valid on the modern interpretation would come out as invalid:

x is A	no A* is A*
∴ some A* is A*	∴ all A is B*

The point is that if you extract from a zyl some syl passing the "*at least* *once*'' condition but not satisfying the ''*exactly once*'' condition, then the premises plus the denial of the conclusion of the zyl entail that any twice asterisked capital letter of the syl is empty; but if the premises plus the denial of the conclusion of the zyl also entail that this term is not empty then the zyl should come out as valid. In the first example ''*x* is *A*'' insures that ''*A*'' is not empty while the denial of ''some *A* is *A*'' insures that ''*A*'' is empty; so the premise cannot be true while the conclusion is false. In the second example the denial of ''all *A* is *B*'' insures that ''*A*'' is not empty while ''no *A* is *A*'' insures that ''*A*'' is empty; so the premise cannot be true while the conclusion is false. Such thinking may render the following modern star-test more intelligible.

To test a zyl (on the modern interpretation): First asterisk just the distributed letters in all but the last wff and the undistributed letters in the last wff. Then the zyl is valid if and only if some group of wffs from the zyl can be arranged to form a *syl* in which every capital letter is asterisked *at least once* and there is exactly one right hand asterisk, and for each twice asterisked capital letter in that syl there is some group of wffs from the zyl which can be arranged to form an auxiliary sequence of wffs such that (1) that capital letter occurs unasterisked in the bottom wff of the sequence and (2) every initial asterisked capital letter in the sequence is just below a wff in which this capital letter occurs unasterisked.

The above examples pass the test. In the first example ''some *A** is *A**'' is the syl and ''*x* is *A*'' is the auxiliary sequence; in the second example ''no *A** is *A**'' is the syl and ''all *A* is *B**'' is the auxiliary sequence. The role of the auxiliary sequence is to assure that, in the state of affairs which would obtain if the premises were true while the conclusion was false, the twice asterisked letter is not empty.

Another example may help one to grasp the generalized modern star-test a bit better. The following complex argument:

some *G* is not *H**
all *A** is *B*
all *G** is *F*
no *C** is *B**
all *F** is *D*
all *A** is *C*
∴ some *D** is not *A*

is valid on the modern interpretation. It divides into the following syl and auxiliary sequence:

all *A** is *B*	some *G* is not *H**
no *C** is *B**	all *G** is *F*
all *A** is *C*	all *F** is *D*
	some *D** is not *A*

This satisfies the test. The role of the auxiliary sequence here may be better understood if it is remembered that ''some *D** is not *A*'' is from the conclusion of the zyl and is asterisked just like its contradictory "all D^* is A" would be if the latter occurred in the premises (the asterisking of just the undistributed letters in the conclusion is analogous to the RAA method of adding the contradictory of the conclusion to the premises); so "some D^* is not A" here works just like "all D^* is A" would. So if the premises of the zyl are true while its conclusion is false, then (since by the first member of the auxiliary sequence "G" cannot be empty) "F," "D," and finally "A" cannot be empty.

I am not at all sure that these generalized star-tests work universally.

6 Now let me sketch an inferential proof-procedure for **SC** zyls. Pairs of wffs of the following forms are *contradictories*: "all A is B"/"some A is not B"; "no A is B"/"some A is B"; "x is A"/"x is not A"; "x is y"/"x is not y." A *formal proof* of a zyl is a list of wffs consisting of that zyl with its last wff underlined with another zyl below such that: 1) the first member of the second zyl is the contradictory of the underlined wff, 2) all other wffs of the second zyl follow from previous not-underlined wffs by means of one of the inference rules below, and 3) the list contains some pair of not-underlined contradictory wffs. The following forms of inference are recognized as valid:

R1 all A is B, x is A ∴ x is B.
R2 no A is B, x is A ∴ x is not B.
R3 some A is B ∴ x is A, x is B—provided that "x" has not occurred previously.
R4 some A is not B ∴ x is A, x is not B—provided that "x" has not occurred previously.
R5 x is y ∴ y is x.
R6 x is y, x is A ∴ y is A.
R7 x is y, x is z ∴ y is z.
R8 ∴ x is x.

Basically R1-R2 are an interpretation of the *dictum de omni et nullo*, R3-R4 are a form of existential instantiation, and R5-R8 are adaptations of the usual identity rules; R5 is redundant (it follows from R7-R8) but is included to simplify proofs. Formal **SC** proofs are *reductio ad absurdum* arguments. For the Aristotelian interpretation one might add an R9 to read:

R9 ∴ x is A—provided that "x" has not occurred previously.

R9 would insure that each general term is non-empty.

An example might help. Consider the strict formal proof of Barbara on the left and the looser proof-with-explanation on the right. The proof on the right (minus wff-numbers and justifications) shows how I like to do *reductio* proofs; the explanations should be self-explanatory.

all M is P	1 all M is P
all S is M	2 all S is M
<u>all S is P</u>	[∴] all S is P

some S is not P	3	asm: some S is not P
a is S	4	∴ a is S (from 3, R4)
a is not P	5	∴ a is not P (from 3, R4)
a is M	6	∴ a is M (from 2, 4, R1)
a is P	7	∴ a is P (from 1, 6, R1)
	8	∴ all S is P (from assumption 3 and contradictories 5 and 7)

Formal proofs of valid two-premise categorical syls take exactly five steps each and do not require the use of R5-R8.

Formal proofs are less convenient than the star-test but far more intuitive. Proof strategy is simple: underline the last wff of the zyl, then ignore this wff for the rest of the proof and add its contradictory below the line, apply R3-R4 once on each wff starting with "some" (using a different new letter with each such wff), derive all you can using R1-R2 and R5-R7, and if you get something of the form "x is not x" then use R8; you will get an inconsistency if and only if the original zyl was valid.

REFERENCES

[1] Salmon, Wesley C., *Logic,* Englewood Cliffs, N.J., Prentice-Hall, Inc. (1963), pp. 40-41

[2] Culbertson, James T., *Mathematics and Logic for Digital Devices,* Princeton, N.J., D. Van Nostrand Company, Inc. (1958), p. 99.

John Carroll University
Cleveland, Ohio