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A GENERALISED KLEENE-ROSSER PARADOX FOR A SYSTEM CONTAINING THE COMBINATOR K

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The following is the statement of the Kleene-Rosser paradox as given in reference [1].[1]

"If in a given logic CI-CVI of the list which follows are rules of procedure or valid methods of proof and C1 and C2 are provable formulas, then in that logic every well formed formula with no free variables is provable.

CI-CIII. *Cf.* [2], Church's rules of procedure I-III.

CIV. $\wedge xy \vdash x$.

CV. If FA and $Fx \supset_x Gx$, where x is a variable not occurring in F or G, then GA.

CVI. If the variable x does not occur in F, G or M as a free variable, and if $M, Fx \vdash Gx$, then $M, FA \vdash Fx \supset_x Gx$.

C1. $\vdash x \supset_x . \; y \supset_y \wedge xy$.

C2. $\vdash x \supset_x . \wedge xy \supset_y y$."

In this article we show that in a system of combinatory logic [3], which contains the combinator K (which is such that $\mathsf{K}xy = x$), the paradox can be strengthened to apply even without the conditions CIV, C1 and C2.

We can thus obtain a contradiction from merely some properties of equality (CI-III), CV the basic rule for restricted generality and the given deduction theorem for restricted generality CVI.

Using a particular definition of $\wedge$, namely the one Curry uses in [4], we can prove CIV, viz. we show:

$$X, Y \vdash \wedge XY, \tag{a}$$

$$\wedge XY \vdash Y, \tag{b}$$

To prove these properties Curry uses a completely unrestricted

1. This is a direct quotation except in that $\wedge xy$ is used for the conjunction of x and y instead of Kleene and Rosser's xy. For details on combinatory logic see [3].

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deduction theorem, but it can easily be shown that with CVI his proof still goes through.

As we have $\vdash \mathsf{I}(\mathsf{QKK})$, CVI applied to (a) twice gives

$$\vdash x \supset_x . y \supset_y \wedge xy.$$

By (a) $x \vdash \wedge x(\mathsf{QKK})$, so by (b) and CVI, $x \vdash \wedge xy \supset_y y$ and as before

$$\vdash x \supset_x . \wedge xy \supset_y y.$$

Thus with only CI-CIII, CV and CVI we get the paradox. If we want to retain the rules for equality and Rule Ξ it is therefore necessary to take a weaker version of CVI such as the one proved in [5].

REFERENCES

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