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FOR SO MANY INDIVIDUALS

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In [2], Tarski introduces the numerical quantifiers. These are expressions $(\exists_k x)$ which mean "there are at least k individuals x such that", where k is any nonnegative integer. Thus $(\exists_1 x)$ is the ordinary quantifier $(\exists x)$. The numerical quantifiers may be defined in terms of the ordinary quantifier and identity as follows:

$$(\exists_0 x)\, A \text{ for } A \rightarrow A$$
$$(\exists_{k+1} x)\, A \text{ for } (\exists_k x)\, (\exists y)\, (-(x = y) \,\&\, A \,\&\, A(y/x)),$$

where y is the first variable which does not occur in A and $A(t/x)$ is the result of substituting a term t for all free occurrences of x in A.

Because of their definability, the numerical quantifiers have rarely been considered on their own account. However, in this paper I consider a predicate logic without identity which is enriched with numerical quantifiers as primitive. In section 1, I present the syntax and semantics for this logic; and in sections 2 and 3, I establish its completeness.

1. *The Logic* L.

Syntax

Formulas These are constructed in the usual way from relation letters of given degree, (individual) constants, (individual) variables, the truth-functional connectives $\vee$ and $-$, the quantifier (x) and the quantifiers $(\exists_k x)$, $k = 2, 3, \ldots$. We use $(\exists_0 x)\, A$ to abbreviate $A \rightarrow A$ and $(\exists_1 x)\, A$ to abbreviate $(\exists x)\, A$, i.e. $-(x)\, -A$. Also we suppose that there are a denumerable number of individual variables and at least one predicate letter.

Axioms (where $k = 2, 3 \ldots$, and $l = 1, 2, \ldots$)

1. *All tautologous formulas*
2. $(x)\, A \rightarrow A(t/x)$, *t free for x in A*
3. $(x)\, (A \rightarrow B) \rightarrow ((x)\, A \rightarrow (x) B)$
4. $A \rightarrow (x)\, A$, *x not free in A*
5. $(\exists_k x)\, A \rightarrow (\exists_l x)\, A$, $l < k$
6. $(\exists_k x)\, A \leftrightarrow \bigvee_{i=0}^{k} (\exists_i x)\, (A \,\&\, B) \,\&\, (\exists_{k-i} x)\, (A \,\&\, -B)$

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7. $(x)(A \to B) \to ((\exists_k x) A \to (\exists_k x) B)$
8. $(\exists_k x) A \to (\exists_k y) A(y/x)$, *y free for x in A and not free in A*

Rules of Inference

Modus Ponens. *From $A, A \to B$ infer B.*
Generalisation. *From A infer $(x)A$.*

Semantics

A *structure* $\mathfrak{A}$ for a language $\mathfrak{L}$ consists of:
(a) a non-empty domain $|\mathfrak{A}|$
(b) an assignment of an n-adic relation $\mathfrak{A}(R)$ on $|\mathfrak{A}|$ to each n-th place relation letter in $\mathfrak{L}$
(c) an assignment of an element $\mathfrak{A}(a)$ of $|\mathfrak{A}|$ to each constant in $\mathfrak{L}$.

We may extend our language $\mathfrak{L}$ to a language $\mathfrak{L}'$ by adding each element of $|\mathfrak{A}|$ as a constant to $\mathfrak{L}$. We may then define the truth of a sentence (i.e. closed formula) of $\mathfrak{L}'$ in the usual manner. The clause for $(\exists_k x)$, $k = 2, 3, \ldots$, is:

$$(\exists_k x) A \text{ is true in } \mathfrak{A} \text{ if and only if } \mathrm{card}\ \{a \in |\mathfrak{A}| : A(a/x)\} \geq k.$$

Validity and modelhood etc. can then be defined in the usual manner.

2 *A Preliminary Result.* We say that a theory T has the *Henkin property* if whenever $(\exists x) A(x) \in T$ then $A(a) \in T$ for some constant a. (I use $A(x)$ for a formula with at most one free variable x. $A(t)$ is then $A(x)(t/x)$).

Fix on a consistent and complete theory T with the Henkin property and in a language $\mathfrak{L}$. As in the standard Henkin completeness proof for the predicate calculus it suffices to construct a canonical model $\mathfrak{A}$ for T. However, we cannot simply let the domain of $\mathfrak{A}$ be the set C of constants in $\mathfrak{L}$. Firstly because several constants may correspond to one individual; and secondly because one constant may correspond to several individuals.

We say constants a and b are indistinguishable, $a \sim b$, if for each formula $A(x)$, $A(a) \leftrightarrow A(b) \in T$. Clearly, $\sim$ is an equivalence relation. So to overcome the first difficulty we can let the elements of $\mathfrak{A}$ be equivalence classes $[a]$ with respect to $\sim$.

We say $A(x)$ *defines* $[a]$ if $[a]$ is the one and only member of $C/\sim$ such that $A(a) \in T$. Now $[a]$ corresponds to several individuals if some formula $A(x)$ defines $[a]$ and $(\exists_k x) A(x) \in T$ for some $k > 1$. So put

$$d([a]) = \begin{cases} 1 \text{ if no } A(x) \text{ defines } [a] \\ k \text{ if } k \text{ is the greatest number such that for some } A(x), \\ \qquad\qquad A(x) \text{ defines } [a] \text{ and } -(\exists_{k+1} x) A(x) \in T \\ \omega \text{ otherwise.} \end{cases}$$

Then $d([a])$ gives the number of individuals corresponding to $[a]$. Then we may overcome the second difficulty by letting $d([a])$ individuals correspond to each $[a]$. Put

$$N(A(x)) = \sum d([a]), \text{ for } [a] \text{ such that } A(a) \in T.$$

Then $N(A(x))$ gives the number of individuals "satisfying" $A(x)$. Therefore we require the following lemma:

Lemma. *For* $k > 0$, $N(A(x)) \geq k$ *if and only if* $(\exists_k x)\, A(x) \in T$.

Proof. By induction on k.
$k = 1 \Rightarrow$. Suppose $N(A(x)) \geq 1$. Then clearly for some a, $A(a) \in T$. But then by axiom-scheme 2, $(\exists x)\, A(x) \in T$.

$\Leftarrow$. Suppose $(\exists x)\, A(x) \in T$. Since T has the Henkin property, $A(a) \in T$ for some constant a. So $N(A(x)) \geq 1$. $k > 1 \Rightarrow$. Suppose $N(A(x)) \geq k$ i.e. $\sum d([a])$, for $[a]$ such that $A(a) \in T \geq k$. We distinguish two cases:

Case 1 $A(x)$ defines some $[a]$. So $d([a]) \geq k$. But then by the definition of d some $B(y)$ defines $[a]$ and $(\exists_k y)\, B(y) \in T$. For by axiom-scheme 5, if $-(\exists_k y)\, B(y) \in T$ then $-(\exists_l y)\, B(y) \in T$ for all $l > k$. Let z be a variable which does not occur in $A(x)$ or $B(y)$. Then $(z)\,(B(z) \rightarrow A(z)) \in T$. For otherwise $(\exists z)\,(B(z)\ \&\ -A(z)) \in T$ and so by the Henkin property $B(b)\ \&\ -A(b) \in T$ for some b. But then not $a \sim b$ and $B(y)$ does not define $[a]$, contrary to assumption. Now $(\exists_k z)\, B(z) \in T$ by axiom-scheme 8. So $(\exists_k z)\, A(z) \in T$ by axiom-scheme 7. Hence $(\exists_k x)\, A(x) \in T$, by axiom-scheme 8 again.

Case 2 $A(x)$ defines no $[a]$. Then there are distinct $[a]$ and $[b]$ such that $A(a), A(b) \in T$. So there is a formula $B(y)$ such that $B(a) \in T$ and $B(b) \notin T$. Let $X = \{[a] : A(a) \in T\}$, $Y = \{[a] : A(a)\ \&\ B(a) \in T\}$ and $Z = \{[a] : A(a)\ \&\ -B(a) \in T\}$. Then it is easy to see that $\{Y, Z\}$ is a partition of X. So $\text{card } X = \text{card } Y + \text{card } Z$ and $\text{card } Y$, $\text{card } Z > 0$. Hence there are integers $l, m > 0$ such that $l, m < k$, $l + m = k$, $N(A(z)\ \&\ B(z)) \geq l$ and $N(A(z)\ \&\ -B(z)) \geq m$, where z is a variable not in $A(x)$ or $B(y)$.

By the induction hypothesis, $(\exists_l z)\,(A(z)\ \&\ B(z))$, $(\exists_m z)\,(A(z)\ \&\ -B(z)) \in T$. So by axiom-scheme 7, $(\exists_k z)\, A(z) \in T$. Therefore $(\exists_k x)\, A(x) \in T$ by axiom-scheme 8.

$\Leftarrow$. Suppose $(\exists_k x)\, A(x) \in T$. Again we distinguish two cases:

Case 1 $A(x)$ defines some $[a]$. Then by axiom-scheme 5, it should be clear that $d([a]) \geq k$. So $N(A(x)) \geq k$.

Case 2 $A(x)$ defines no $[a]$. $A(a) \in T$ for some a by the Henkin property. So there are distinct $[a]$ and $[b]$ such that $A(a), A(b) \in T$. So $A(a)\ \&\ B(a)$, $A(b)\ \&\ -B(b) \in T$ for some formula $B(y)$. By axiom-scheme 8, $(\exists_k z)\, A(z) \in T$, z not in $A(x)$ or $B(y)$, and by axiom-scheme 6, $(\exists_i z)\,(A(z)\ \&\ B(z))$, $(\exists_{k-i} z)\,(A(z)\ \&\ -B(z)) \in T$ for some $i = 0, 1, \ldots, k$. Now $(\exists_1 z)\,(A(z)\ \&\ B(z))$, $(\exists_1 z)\,(A(z)\ \&\ -B(z)) \in T$. So by axiom-scheme 2 we may suppose $0 < i < k$. But then by the induction hypothesis $N(A(z)\ \&\ B(z)) \geq i$ and $N(A(z)\ \&\ -B(z)) \geq k - i$. Hence $N(A(x)) = N(A(z)) \geq k$.

3 *Completeness*. Let $D = \{\langle [a], l\rangle : 0 \leq l < d([a])\}$. Extend the language $\mathfrak{L}$ of T to $\mathfrak{L}'$ by adding the elements of D as constants. The *canonical structure* $\mathfrak{A}$ for $\mathfrak{L}'$ is then defined as follows:

(a) $|\mathfrak{A}| = D$

(b) $\mathfrak{A}(R) = \{\langle\langle [a_1], l\rangle, \ldots, \langle [a_n], l\rangle\rangle \in D^n : Ra_1 \ldots a_n \in T\}$ for each relation R in $\mathfrak{L}$ of degree n

(c) $\mathfrak{A}(e) = e$ for $e \in D$ and $\mathfrak{A}(a) = \langle [a], 0\rangle$ for a in $\mathfrak{L}$.

If A is a sentence of $\mathfrak{L}'$, let A' be any sentence of $\mathfrak{L}$ obtained by replacing each constant $\langle [a], l\rangle$ in A by a.

Theorem (On the Canonical Model). *For any sentence A of $\mathfrak{L}'$, A is true in $\mathfrak{A}$ if and only if $A' \epsilon T$.*

Proof. By induction on the length of A. We consider only the main case when $A = (\exists_k x)\, A(x)$. Now $(\exists_k x)\, A(x)$ is true in $\mathfrak{A}$

iff $\text{card}\ \{\langle [a], l\rangle \epsilon D : A(\langle [a], l\rangle)$ is true in $\mathfrak{A}\} \geq k$ (by semantical clause for $(\exists_k x)$)

iff $\text{card}\ \{\langle [a], l\rangle \epsilon D : A'(a) \epsilon T\} \geq k$ (by the induction hypothesis)

iff $N(A'(x)) \geq k$ (by the definitions of d and D)

iff $(\exists_k x)\ A'(x) \epsilon T$ (by the lemma).

Since our logic contains the ordinary predicate logic we know that every consistent set of sentences is contained in a consistent and complete theory with the Henkin property. So by standard methods we can obtain such results as the following.

Corollary 1 (Completeness) *A is a theorem of $\mathfrak{L}$ if and only if A is valid.*
Corollary 2 *Every consistent set of sentences has a model.*

Finally, it is worth noting the connection between this and [1]. The uniform monadic predicate logic with numerical quantifiers is isomorphic, syntactically and semantically, to S5n. $P(x)$ corresponds to p and $(\exists_k x)$ to M_k. On the other hand, the predicate logics with several variables introduce something new, as do the modal logics weaker than S5n.

REFERENCES

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