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NATURAL DEDUCTION RULES FOR S1°-S4°

THOMAS W. SATRE

In [3], Zeman has provided bases for modal systems S1°-S4° in the style of Lemmon's formalizations of S1-S4 in [1]. In [2] we have formulated natural deduction rules for these latter systems together with the D- and E-systems of [1]. In this paper similar rules are provided for S1°-S4° and T°. It is assumed throughout that the reader is acquainted with the axioms and rules of [1], [2], and [3], and it is assumed that S1°-S4° and T° are formulated in the style of Lemmon as given in [3].

In [2] it is shown that the usual form of the deduction theorem will hold for any system of modal logic which has the symbols and formation rules of classical propositional calculus and is obtained by adding to propositional calculus proper axioms and rules, together with formation rules for any new symbols introduced in the proper axioms and rules, provided that each proper rule applies only to theorems of the system, i.e., to formulas which are provable from zero hypotheses. Since the proper rules of [3] are all of this type, we are assured that the deduction theorem holds for each of the systems to be studied in this paper.

As a stock of natural deduction rules we take the rules of [2], adding the following:

$\Box I_{12}$: Given a proof of B from Δ alone as hypotheses, then (i) given as premises $\Box\Delta$, derive $\Box B$ depending upon the hypotheses upon which $\Box\Delta$ depend; and (ii) if each hypothesis of Δ is of the form of $\Box C$, for some wff C, then we may derive $\Box B$ depending upon Δ.

$\Box E_4$: Given a proof of $\Box A$ as a theorem, we may derive A as a theorem.

$\Box S_3$: Given proofs of $\Box(A \supset B)$ and of $\Box(B \supset A)$ as theorems, we may derive $\Box(\Box A \supset \Box B)$ as a theorem.

$\Box S_4$: Given as a premise $\Box(A \supset (B \supset C))$, we may derive $\Box(\Box A \supset (\Box B \supset \Box C))$ depending upon the same hypotheses, if any, of the premise.

Rules $\Box E_4$ and $\Box S_3$ are just (Gc) and (Gb′) of [3]. $\Box I_{12}$ is $\Box I_{10}$ of [2] liberalized to allow the case where Δ is empty, and $\Box S_4$ is a new rule.

The natural deduction systems obtained are:

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S1°: **PC**; (Ga′); (Gb′); (Gc); G3. NS1°: NPC; $\Box E_4$; $\Box I_8$, where

Λ is the set of all substitution instances of tautologies and of G3; $\Box S_3$.

S2°: **PC**; (Ga′); (Gb); (Gc); G1′. NS2°: NPC; $\Box E_4$; $\Box I_8$, where

Λ is the set of all substitution instances of tautologies and of G1′; $\Box S_2$.

S3°: **PC**; (Ga′); (Gc); G4. NS3°: NPC; $\Box E_4$; $\Box I_8$, where

Λ is the set of all substitution instances of tautologies and of G4; $\Box S_4$.

S4°: **PC**; (Ga); (Gc); G4. NS4°: NPC; $\Box E_4$; $\Box I_{12}$.
T°: **PC**; (Ga); (Gc); G1′. NT°: NPC; $\Box E_4$; $\Box I_1$.

We first show the appropriate deductive equivalences for S1°-S3°. It will be noted that **PC** and NPC are deductively equivalent, (Gc) is $\Box E_4$, (Gb) is $\Box S_2$, (Gb′) is $\Box S_3$, and (Ga′) for each system is just $\Box I_8$ for the corresponding natural deduction system in the case where Δ is empty. Thus, for the equivalence of S1° and NS1° it suffices to derive $\Box I_8$ for NS1° in S1° and to derive G3 in NS1°. To derive $\Box I_8$ for NS1° in S1°, suppose that we have a proof of $A_1, \ldots, A_n \vdash B$ in NPC. Thus, $A_1, \ldots, A_n \vdash B$ in **PC**. By the deduction theorem we have

1. $\vdash A_1 \supset (\ldots \supset (A_n \supset B) \ldots)$ in **PC**.

By Ga′ we infer

2. $\vdash \Box(A_1 \supset (\ldots \supset (A_n \supset B) \ldots))$ in S1°.

Since G1′ is provable in S1° by application of (Gc) to line (14) of [3], we use this theorem together with (2) to infer

3. $\vdash \Box A_1 \supset (\ldots \supset (\Box A_n \supset \Box B) \ldots)$ in S1°.

Hence, if we are given $\Box A_1, \ldots, \Box A_n$ as premises, the corresponding theorem at (3) yields a proof of $\Box B$ from the same hypotheses, if any. Further, if $\vdash B$ in NS1° and B is in Λ, then either $\vdash B$ in **PC** or B is an axiom of S1°. In either case we have $\vdash \Box B$ in S1° by (Ga′). Finally, G3 is proven in NS1° by the same proof given in [2] of G3 in NS1.

The deductive equivalence of S2° and NS2° is obtained by deriving $\Box I_8$ for NS2° in S2° and deriving G1′ in NS2°. The proof of the former is that given above for S1°, except that where Δ is empty we are dealing with Λ for NS2° and rule (Ga′) for S2°. The proof of the latter is that given in [2] for NS2. Similarly, we obtain a derivation of $\Box I_8$ for NS3° in S3° by noting that in this case Λ is specified for NS3°. It is easily verified that in the presence of **PC**, or NPC, $\Box$S4 is deductively equivalent to G4. Thus, the equivalences of S1°-S3° to the natural deduction systems NS1°-NS3° are established.

For the deductive equivalence of S4° and NS4° we note that (Ga) is just $\Box I_{12}$ where Δ is empty. Thus (Ga) is derivable in NS4°. G4 is proven in NS4° as follows:

1 (1) $\Box(p \supset (q \supset r))$ Hyp

2	(2)	$\Box p$	Hyp
3	(3)	$\Box q$	Hyp
4	(4)	$p \supset (q \supset r)$	Hyp
5	(5)	p	Hyp
6	(6)	q	Hyp
4, 5, 6	(7)	r	4-6 NPC
1, 2, 3	(8)	$\Box r$	7 $\Box I_{12}$ (i), Δ = 4-6
1, 2	(9)	$\Box q \supset \Box r$	3, 8 CP
1	(10)	$\Box p \supset (\Box q \supset \Box r)$	2, 9 CP
1	(11)	$\Box(\Box p \supset (\Box q \supset \Box r))$	10 $\Box I_{12}$ (ii)
	(12)	$\Box(p \supset (q \supset r)) \supset \Box(\Box p \supset (\Box q \supset \Box r))$	1, 11 CP

We derive the two parts of $\Box I_{12}$ in S4° as follows: (i) Let $A_1, \ldots, A_n \, r B$ in S4° be given. By the deduction theorem and (Ga)

4. $\vdash \Box(A_1 \supset (\ldots \supset (A_n \supset B) \ldots))$ in S4°.

By repeated substitution on G1′, which is a theorem of S4°, we have

5. $\vdash \Box A_1 \supset (\ldots \supset (\Box A_n \supset \Box B) \ldots)$ in S4°.

So given $\Box A_1, \ldots, \Box A_n$ as premises, the theorem at (5) permits us to derive $\Box B$ from the same hypotheses, if any. (ii) Further, if each A_i $(1 \leq i \leq n)$ is of the form $\Box C_i$, then by the deduction theorem we have

6. $\vdash \Box C_1 \supset (\ldots \supset (\Box C_n \supset B) \ldots)$ in S4°.

So, (Ga) and G1′ yield

7. $\vdash \Box\Box C_1 \supset (\ldots \supset (\Box\Box C_n \supset \Box B) \ldots)$ in S4°.

However, by (Gc) we infer from line (35) of [3]

8. $\vdash \Box p \supset \Box\Box p$ in S4°.

By substitution for variables and **PC** we infer

9. $\vdash \Box C_1 \supset (\ldots \supset (\Box C_n \supset \Box B) \ldots)$ in S4°.

So, if we are given as premises $A_1, \ldots, A_n$, then the theorem at (9) yields a proof of $\Box B$ from the same hypotheses, if any.

For T°, the proof in [2] that (a) and 1′ are deductively equivalent to $\Box I_1$ in the presence of **PC** or NPC also proves the deductive equivalence of (Ga) and G1′ with $\Box I_1$. Thus we have natural deduction rules for the systems of [3].

Zeman notes ([3], p. 459) that the systems S3° and S4° cannot be formalized by just weakening axiom G2 of S3 and S4 to the rule (Gc). Stated in terms of our rules, the problem is that an unrestricted elimination rule such as $\Box E_1$ of [2] which permits the derivation of G1′ from G1 also permits the proof of G2. But a restricted rule as $\Box E_4$ which does not permit this proof does not permit the derivation of G1′ from G1 either. We can easily prove G1′ from G4. However, the $\Box I$ rules which I have formulated which permit easy proofs of G4 also permit the proof of $\Box p \supset \Box\Box p$, and so these rules fail to distinguish between the systems S3° and S4°. Further, the $\square$I rule for NS3° must provide the restriction of axiom scheme Ga′. For this reason G4 is provided for in NS3° by the special rule $\square S_4$ for the distribution of $\square$. The rule $\square I_8$ for NS3° will yield G1′ in the absence of G2 or $\square E_1$, which is deductively equivalent to G2.

The composite rule $\square I_{12}$ for NS4° is motivated by the fact that while in S4 we can prove both

10. $\vdash \square A \supset B$ if and only if $\vdash \square A \supset \square B$

and

11. If $\vdash A \supset B$, then $\vdash \square A \supset \square B$,

for all formulas A and B, in NS4° we cannot. We can still prove (11) in NS4°, but without G2 we obtain in place of (10) only

12. If $\vdash \square A \supset B$, then $\vdash \square A \supset \square B$

for formulas A and B. The reader may easily verify that the parts (i) and (ii) of $\square I_{12}$ yield proofs of (11) and (12) respectively. I have not found a proof of (12) using $\square I_{12}$ (i), in the absence of G2, except by the use of the theorem $\square p \supset \square\square p$ whose proof requires $\square I_{12}$ (ii), and I have not found a proof of (11) using $\square I_{12}$ (ii) without using G1′ whose proof requires $\square I_{12}$ (i). Thus, I conjecture that neither part of $\square I_{12}$ is redundant.

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Bradley University
Peoria, Illinois