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INDEXINGS OF SETS

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In [1] we gave two new definitions of an ordered pair of sets in terms of more basic sets, the simplest one being

$$[x, y] = \{\{\emptyset, x\}, \{y\}\} \tag{1}$$

Actually of course the concept of an ordered pair of sets is independent of the concept of a set. To see this, consider any common axiomatization of set theory, drop the definition of an ordered pair—e.g., (1)—add a new primitive symbol "[,]" to the language and the new proper axiom

$$[x_1, x_2] = [y_1, y_2] \Rightarrow x_1 = y_1 \ \& \ x_2 = y_2. \tag{2}$$

Then apply Padoa's principle (cf. [2; pp. 169-172]). In fact, conceptually speaking, an ordered pair of sets isn't even a set. A definition such as (1) is merely a representation within set theory of the concept (2) of an ordered pair.

In the present paper we wish to consider the concept of an indexing and some possible representations of this concept within class-set theory. Using informal language we state some of the properties we might want indexings to have.

(I1) An indexing on index class Λ is a function with domain Λ.

(I2) Any two indexings I and J are equal if and only if they are defined on the same Λ and $I_\lambda = J_\lambda$ for all $\lambda \in \Lambda$, where I_λ denotes the λ-th component of I.

(I3) Every class can be indexed; i.e., given any class X, we can find an index class Λ and a 1:1 indexing I on Λ such that $X = \{I_\lambda | \lambda \in \Lambda\}$. (We don't want to require that all indexings be 1:1 because in mathematics one often speaks of an indexed class $\langle x_\lambda | \lambda \in \Lambda \rangle$ of sets, allowing $x_\lambda = x_\mu$ for some $\lambda \neq \mu$.)

(I4) The representations or definitions of n-tuples and sequences can be defined as special cases of indexings.

(I5) One can define the concept of subindexings and hence of subsequences in terms of indexings.

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Another possible condition has to do with what is called the *generalized Cartesian product* of an indexed class $\langle x_\lambda | \lambda \epsilon \Lambda \rangle$ of (not necessarily distinct) sets: in particular $\times_{\lambda \epsilon \Lambda} x_\lambda$ is defined as the class of all indexings s on Λ such that $s_\lambda \epsilon x_\lambda$ for all $\lambda \epsilon \Lambda$.

(I6) The Cartesian product $x_0 \times x_1$ is a special case of the generalized Cartesian product $\times_{\lambda \epsilon \Lambda} x_\lambda$ in the case when Λ is some "standard" two element ordered index set having elements called "0" and "1".

The standard approach to indexings in set theory is to define them as (or represent them by) functions, where a function is defined as usual by the following property.

(F) *A function F is a class of ordered pairs such that* $[x, y] \epsilon F$ & $[x, z] \epsilon F \Rightarrow y = z$.

Then it is easy to show that indexings satisfy (I1-I5). However, for the usual definitions of an ordered pair (I6) fails. This is no accident as we will soon show.

Convention: Assume that we haven't developed the integers yet so that we are free to choose which sets will represent the various integers. Let 0 and 1 denote the choices for the first two integers and let $2 = \{0, 1\}$. Let *2-tuples* denote indexings on index set 2. Denote ordered pairs by $[u, v]$ and 2-tuples by $\langle u, v \rangle$.

Lemma. (I6) *holds if and only if ordered pairs are 2-tuples: in symbols,*

$\times_{\lambda \epsilon 2} x_\lambda = x_0 \times x_1$ *for all sets* $x_0, x_1 \Leftrightarrow \langle u, v \rangle = [u, v]$ *for all sets* u, v.

Proof. The proof of ($\Leftarrow$) is trivial once we notice that what we are asked to prove is

$$\{\langle u, v \rangle | u \epsilon x_0, v \epsilon x_1\} = \{[u, v] | u \epsilon x_0, v \epsilon x_1\}.$$

To prove ($\Rightarrow$), consider any two sets u_0, u_1 and let $x_i = \{u_i\}$, $i = 0, 1$.

Theorem 1. *Conditions* (I1) *and* (I6) *are mutually exculsive if a function is defined via* (F).

Comment: This theorem is valid no matter what definition we use for ordered pairs; it is even valid if we don't define an ordered pair of sets in terms of more basic set operations but instead work in a set theory with "[,]" as well as "ϵ" for primitives with "[,]" satisfying (2).

Proof. Assume indexings and functions satisfy properties (I1) and (F) respectively. Then for any set theory satisfying $x \notin x$ for all sets x we have

$$\langle 1, y \rangle = \{[0, 1], [1, y]\} \neq [1, y],$$

hence (I6) fails to hold by our lemma.

An alternative approach to indexings would be to retain the definition (F) of a function, reject (I1), and choose definitions of ordered pairs and indexings such that (I2-I6) are satisfied. We now give such a presentation in which indexings, though formally not functions, will act like functions. Our treatment here is a natural generalization of [1; section 3] but is entirely self-contained.

By the *pre-ordered pair* of any sets x and y we mean

$$(x, y) = \{x, \{y\}\}.$$

Define B as the class of all sets which aren't singletons, with the one exception $\{\emptyset\}$: in symbols,

$$B = V - \{\{x\} \mid x \neq \emptyset\}.$$

Define an *index class* as any subclass of B.

Theorem 2. $(x_1, x_2) = (y_1, y_2) \Rightarrow x_1 = y_1$ & $x_2 = y_2$ *if* $x_1, y_1 \in B$.

Thus pre-ordered pairs satisfy the ordered pair property on B but not in general. In fact B is the largest such class since for any $x = \{x_1\} \notin B$ we have $(\{x_1\}, x) = (\{x\}, x_1)$.

We define the *pre-product* of any two classes X, Y by

$$X * Y = \{(x, y) \mid x \in X, y \in Y\}.$$

Then we say that class I is an *indexing* if and only if

(i) $I \subseteq B * V = \{(x, y) \mid x \in B\}$, and
(ii) $(x, y) \in I$ & $(x, z) \in I \Rightarrow y = z$.

I is an *indexing on* Λ if and only if $\Lambda = \{x \mid (x, y) \in I$ for some $y\} \subseteq B$. In keeping with standard indexing notation we define the λ-th component I_λ of an indexing I by $I_\lambda = y \Leftrightarrow (\lambda, y) \in I$.

That our indexings satisfy (I2) follows easily from theorem 2. They also satisfy (I3) since given any X, the class $I = \{((\emptyset, u), u) \mid u \in X\}$ is an indexing of X on the index class $\Lambda = \{\emptyset\} * X$. It is also clear how one would define the pre-composition of two indexings to get subindexings.

We now show how sequences and n-tuples can be formally developed as special cases of our formal indexings.

Theorem 3. $\beta \in B$ *and* $\beta \subseteq B$ *for any von-Neumann ordinal* β.

Proof. If $\beta \notin B$ for some β, then $\beta = \{x\}$ for some $x \neq \emptyset$. But every von-Neumann ordinal is equal to the set of all smaller ordinals: in symbols,

$$\beta = \{\alpha \mid \alpha < \beta\} = \{\alpha \mid \alpha \in \beta\}. \tag{3}$$

Thus the only ordinal which contains only one element is $1 = \{0\} = \{\emptyset\} \in B$. To show $\beta \subseteq B$ we just use (3) and the result that all ordinals are in B.

Since $\alpha \subseteq B$ for any ordinal α, we define an α-sequence as an indexing on the set α. If $\alpha = \omega$, then we have the ordinary sequence used in analysis. Since the von-Neumann integer $n = \{0, 1, \ldots, n-1\}$ is just a von-Neumann ordinal, we have $n \subseteq B$; hence we define an n-*tuple* as an indexing on a set n. The obvious choice of definition for an *ordered pair* is now

$$[x, y] = \langle x, y\rangle = \{(0, x), (1, y)\}$$

with the desired property that ordered pairs are 2-tuples, hence indexings satisfy (I6). In summary we have

Theorem 4. *Our indexings satisfy* (I2-I6) *but not* (I1).

Comment: As long as a function is defined by (**F**) and ordered pairs are defined by or satisfy (2), function-like objects which aren't functions such as our indexings do exist, whether or not we single them out as we have. The reason of course is that the usual intuitive concept of a function is much broader than what is embodied in (**F**). In particular one usually conceives of a function from a set A into a set B as a rule which associates a unique element of B to each element of A, where the notion of a rule is still rather vague. However, according to this conception our indexings obviously are functions, even though they don't satisfy (**F**). Definition (**F**) is merely a representation (and in some senses, an incomplete representation) within set theory of this concept.

REFERENCES

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