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ON RELEVANTLY DERIVABLE DISJUNCTIONS

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Where A and B are negation-free formulas of one of the *relevant logics* E, R, or P,[1] we show that

(1) $\vdash A \vee B$ *if and only if* $\vdash A$ *or* $\vdash B$.

Results from [7] and [8] will be presupposed.

1. Our strategy in proving that (1) holds for the relevant logics will be as follows. First, we shall determine a set of conditions such that the negation-free formulas of any logic which simultaneously satisfies all of these conditions has property (1). Second, using results from [7] and [8], we show that the relevant logics satisfy all of these conditions. We close with observations related to the intuitionist logic J and the Lewis system S4, noting now that (1) is one of the more famous properties of J.[2]

2. For present purposes, a logic L is a triple $\langle F, O, T\rangle$, where $\{\rightarrow, \wedge, \vee, -\} = O$, F is a set of formulas built up from sentential variables and the operations of O, and T is the set of theorems of L, which we require to be closed under *modus ponens* for $\rightarrow$, adjunction, and substitution for sentential variables. Where L is $\langle F, O, T\rangle$, an L-theory is any triple $\langle F, O, T'\rangle$, where $T \subseteq T'$ and T' is closed under *modus ponens* for $\rightarrow$ and adjunction. Where no ambiguity results, we identify a theory with its set T' of theorems, and we write $\vdash_{T'} A$ if $A \in T'$.

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1. We assume the sentential logics E and R formulated as in [2] (taking disjunction as an additional primitive). The Anderson-Belnap system P results when Belnap's axioms (1) and (7) are dropped in favor of the weaker scheme $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$; the implicational part of this system is motivated in Anderson's [1].

2. This fact, and the corresponding fact about S4, may be proved by Gentzen techniques, as e.g. in [3]. (Such techniques have lately been applied by Dunn to secure some of the present results for R; hopefully such techniques will work also for the other relevant logics.) Because of the special character of intuitionist negation, (1) holds without restriction for J, of course; that it holds was first announced in Gödel's [4].

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Let T be an L-theory. T is *consistent* if for no A both $\vdash_T A$ and $\vdash_T \overline{A}$; *complete* if for all A, $\vdash_T A$ or $\vdash_T \overline{A}$. Finally, if both T and T' are L-theories and $T \subseteq T'$, we call T' an *extension* of T and T a *sub-theory* of T'.

3. We are interested in logics L which have the following properties:

(2) *Every complete L-theory has a consistent and complete sub-theory.*[3]
(3) $\vdash_L \overline{A} \wedge \overline{B} \rightarrow \overline{A \vee B}$.
(4) *There is a complete L-theory T such that, for all negation-free formulas A of L, $\vdash_T A$ iff $\vdash_L A$.*

Lemma 1. *Let L be a logic for which* (2)-(4) *hold, and let A and B be negation-free formulas of L, then if $\vdash_L A \vee B$, $\vdash_L A$ or $\vdash_L B$.*

Proof. Suppose A and B are negation-free non-theorems of L. By (4) there is a complete, and by (2) a consistent and complete, L-theory T such that $\vdash_T \overline{A}$ and $\vdash_T \overline{B}$. By adjunction, (3), and *modus ponens*, $\vdash_T (\overline{A \vee B})$. Hence not $\vdash_T A \vee B$, since T may be assumed consistent; *a fortiori*, not $\vdash_L A \vee B$. So if $\vdash_L A \vee B$, $\vdash_L A$ or $\vdash_L B$, which was to be proved.

4. We turn to the determination of sufficient conditions for (2), and to the proof that the relevant logics satisfy these conditions. This determination rests on a sharpening of the normalization technique of [8], to which the reader is referred for motivating remarks.

The reader is presumed familiar with the notion of a matrix $\mathfrak{M}$ for a logic L (henceforth, *L-matrix*), and with associated standard notions.[4] A *complete L-matrix* is here a triple $\mathfrak{M} = \langle M, O, D\rangle$, where (i) M is a non-empty set, (ii) O is the set of matrix operations (corresponding to the operations of L), (iii) D is a non-empty subset (of designated elements of) M closed under adjunction and *modus ponens* and containing $I(A)$ when I is an interpretation of L in $\mathfrak{M}$ and $A \in T$, and (iv) for all $a \in M$, $a \in D$ or $\overline{a} \in D$. A *normal L-matrix* is here a complete L-matrix which is also consistent, satisfying, in addition to (i)-(iv), (v) for all $a \in M$, $a \notin D$ or $\overline{a} \notin D$.

Let $\mathfrak{M} = \langle M, O, D\rangle$ be a complete L-matrix, where $O = \{\rightarrow, \wedge, \vee, -\}$. The *truth-partition of M* is the partition of M into disjoint sets T, N, F, where F is $M - D$, $N = \{a \in M\colon\ a \in D \,\&\, \overline{a} \in D\}$, and $T = D - N$; note that $\mathfrak{M}$ is normal iff N is empty. We define the *normalization* $\mathfrak{M}^*$ of $\mathfrak{M}$ in the manner of [8] as follows:

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3. (2) is an interesting property, worth some notice on its own. Although it holds trivially in the classical case (since the inconsistent classical theory is itself trivial), in general (2) is a non-trivial converse to Lindenbaum's lemma (i.e., every consistent theory has a consistent and complete extension); for relevant logics, both Lindenbaum's lemma and (2) hold; in particular, since the relevant logics admit nontrivial inconsistent theories, (2) holds non-trivially for them.

4. L-matrices were called L-algebras in [8], on the ground that 'matrix' is ambiguous; but Dunn has convinced me that 'matrix' is still more clear than 'algebra' in this context to logicians, whence the present usage.

Let N^* be a set disjoint from M and in 1-1 correspondence with N, and for each element a of N let the corresponding element of N^* be a^*. Then,

(i) Let $M^* = M \cup N^*$.
(ii) Let h: $M^* \to M$ be defined for $a \in M$ by $ha = a$ and for $a^* \in N^*$ by $ha^* = a$.
(iii) Let operations $\to^*$, $\wedge^*$, $\vee^*$, $\bar{\ }^*$ of O^* be defined on M^* with the aid of the corresponding operations $\to$, $\wedge$, $\vee$, $\bar{\ }$ of M, using the function h of (ii) as follows:
(a) If $a \in N$, $b \in N^*$, $ha \to hb \in N$, then $a \to^* b = (ha \to hb)^*$.
(b) Otherwise, $a \to^* b = ha \to hb$.
(c) If $a \notin D$ *or* $b \notin D$, $ha \wedge hb \in N$, then $a \wedge^* b = (ha \wedge hb)^*$.
(d) Otherwise $a \wedge^* b = ha \wedge hb$.
(e) If $a \notin D$ *and* $b \notin D$, $ha \vee hb \in N$, then $a \vee^* b = (ha \vee hb)^*$.
(f) Otherwise $a \vee^* b = ha \vee hb$.
(g) If $a \in N$, $\bar{a}^* = (\bar{a})^*$.
(h) Otherwise $\overline{a^*} = \overline{ha}$.

The matrix $\mathfrak{M}^* = \langle M^*, O^*, D\rangle$ is the *normalization* of $\mathfrak{M}$. We relate the notions just introduced to (2).

Lemma 2. *Let* $L = \langle F, O, T\rangle$ *be a logic such that, for every complete* L*-matrix* $\mathfrak{M}$, *its normalization* $\mathfrak{M}^*$ *is an* L*-matrix. Then every complete* L*-theory has a consistent and complete* L*-subtheory—i.e.*, (2) *holds for* L.

Proof. Let $T = \langle F, O, T'\rangle$ be a complete L-theory. We may consider T' itself a matrix (i.e., we identify T' with its so-called *Lindenbaum matrix*), and we note that T' is a complete L-matrix. Let $\langle F^*, O^*, T'\rangle$ be the normalization of $\langle F, O, T'\rangle$. Consider the interpretation I of L in $\langle F^*, O^*, T'\rangle$ such that $I(p) = p$ for each sentential variable p. Let T_I be the set of formulas A of L such that $I(A) \in T'$; it is readily verified that $\langle F, O, T_I\rangle$ is an L-theory, since on the hypothesis of the lemma the normalization of $\langle F, O, T'\rangle$ is an L-matrix; moreover by the definition of normalization exactly one of $I(B)$, $I(\bar{B})$ $(= \overline{I(B)}^*)$ is designated, so accordingly $\langle F, O, T_I\rangle$ is consistent and complete.

We complete the proof of lemma 2 by proving T_I a subtheory of T'. Let h: $F^* \to F$ be as in (ii) above, and consider the composite hI: $F \to F$. It is readily proved by induction that $hI(A) = A$ for all $A \in F$, using the definitions of (iii). But $A \in T_I \Longleftrightarrow I(A) \in T'$ (by definition of T_I) $\Rightarrow hI(A) \in T'$ (by definition of h) $\Rightarrow A \in T'$ (by the observation just made); accordingly $T_I \subseteq T'$, which was to be showed. We now prove the hypothesis of lemma 2 for the relevant logics.

Lemma 3. *Let* $\mathfrak{M} = \langle M, O, D\rangle$ *be a complete* P*-matrix. Then the normalization* $\mathfrak{M}^* = \langle M^*, O^*, D\rangle$ *of* $\mathfrak{M}$ *is a* P*-matrix. If in addition* $\mathfrak{M}$ *is an* E*-matrix or an* R*-matrix, so is* $\mathfrak{M}^*$.

Proof. It was proved in [1] that lemma 3 holds for E, for R, and for P under the slightly stronger hypothesis that $\mathfrak{M}$ is prime—i.e., whenever

$a \vee b \in D$, $a \in D$ or $b \in D$.[5] We employ a similar strategy here, showing that the axioms of P hold under all interpretations I of P in $\mathfrak{M}^*$ and that D is closed in $\mathfrak{M}^*$ under *modus ponens for* $\rightarrow^*$ and adjunction of $\wedge^*$, given that $\mathfrak{M}$ is known to be a P-matrix. Since no difficulties of principle arise the task of applying the technique of [8] in detail may be safely left to the reader; the generality of the present result requires special attention only in that, with respect to the truth-partition of the underlying complete P-matrix $\mathfrak{M}$, the possibilities $a \in F$, $b \in F$, $a \vee b \in N$, and $a \in T$, $b \in T$, $a \wedge b \in N$ arise. A similar proof establishes the result for E-matrices, and for R-matrices.

5. We can now prove (1) for the relevant logics. Half of (1) is trivial since all instances of $A \rightarrow A \vee B$ and $B \rightarrow A \vee B$ are provable. Furthermore,

Theorem 1. *Suppose A and B are negation-free* $\vdash_{\mathrm{R}} A \vee B$; *then* $\vdash_{\mathrm{R}} A$ *or* $\vdash_{\mathrm{R}} B$. *Furthermore if* $A \vee B$ *is a theorem of* E *or of* P, *so is at least one of* A, B.

Proof. By lemma 1, we need only show that (2)-(4) hold. But (2) holds by lemmas 2 and 3. Furthermore all forms of the DeMorgan laws are easily proved in the relevant logics, so the particular law (3) holds for E, R, and P. Moreover, it is observed in [7] that (4) holds for E, R, and P which completes the proof.

6. It was observed at the outset that (1) is a familiar property of the intuitionist logic J and of the negation-free fragment S4+ of S4 (formulated with strict implication primitive). Direct proof of these facts may be had from the present result, since it is shown in [6] that J is an exact subsystem of the positive part of R, and in [5] that S4+ is an exact subsystem of the positive part of E.[6]

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5. Alas, this hypothesis is not available to us in the present case, on pain of circularity in the proof of (1).

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