

## THE FINITE MODEL PROPERTY FOR **MIPQ** AND SOME CONSEQUENCES

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The proof given in [1] that **MIPQ** has the finite model property is shown to be erroneous. Through the use of slightly different techniques we prove that this property nonetheless does hold. We then prove a representation theorem for finite Monadic Heyting algebras which, together with the finite model property for **MIPQ**, yields an algebraic proof of a theorem concerning a translation map from **MIPQ** to a bimodal calculus (S4, S5)-**C** (see [3]).

**1 MIPQ has the finite model property** Following a suggestion of Prior, R. A. Bull considers in [1] a modal calculus (**MIPQ**) which contains as its base the intuitionist propositional calculus (**IC**) instead of the usual classical logic. For reasons that will become clear in section 2, from here on this calculus will be called S5-**IC**. Hence S5-**IC** contains **IC** and the following rules:

- R1  $\frac{\alpha \rightarrow \beta}{L\alpha \rightarrow \beta}$
- R2  $\frac{\alpha \rightarrow \beta}{\alpha \rightarrow M\beta}$
- R3  $\frac{\alpha \rightarrow \beta}{\alpha \rightarrow L\beta}$ , if  $\alpha$  is fully modalized
- R4  $\frac{\alpha \rightarrow \beta}{M\alpha \rightarrow \beta}$ , if  $\beta$  is fully modalized.

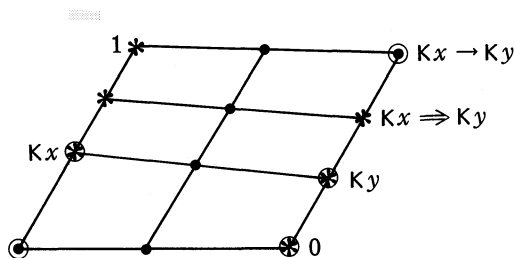
For this modal extension of **IC**, R. A. Bull shows completeness with respect to canonical models. We recall that in a canonical model  $\langle A, B, \cup, \cap, \rightarrow, 0, 1, K, I \rangle$ ,  $A$  is a Heyting algebra,  $B$  is a relatively complete subalgebra of  $A$  and  $K, I$  are two operators on  $A$  such that  $Kx = \text{Min}\{y \in B: x \leq y\}$  and  $Ix = \text{Max}\{y \in B: y \leq x\}$ . In [1] it is proven that S5-**IC** is characterized by finite models, but it appears that the proof in question is wrong. For, starting from a canonical model, the author uses a construction which, opposed to what is maintained, does not yield a finite canonical model. Let us consider the argument in [1]: given a canonical model  $\langle A, B, \cup, \cap, \rightarrow, 0, 1, K, I \rangle$

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and a sequence  $a_1, \dots, a_n \in A$ , take the finite sublattice generated by  $0, 1, a_1, \dots, a_n$ . Call it  $A'$  and define

$$a \Rightarrow b = \text{Max} \{c \in A' : c \leq a \rightarrow b\}, \quad a, b \in A'$$

It is known that  $\langle A', \cup, \cap, \Rightarrow, 1 \rangle$  is a Heyting algebra. Now if the finite sublattice  $B' = B \cap A'$ , could be shown to be closed with respect to  $\Rightarrow$ , then  $B'$  would be a relatively complete subalgebra of  $A'$  determining  $K'$  and  $I'$  and thus concluding the construction of a finite canonical model from an arbitrary one. Unfortunately the proof that  $B'$  is a subalgebra of  $A'$  is erroneous. To be more precise, given the above notations, it can be shown that in general  $B'$  is not a Heyting subalgebra of  $A'$ . Consider, for instance, the following example:



The twelve elements lattice represented in this figure is a Heyting algebra  $A$ ; a sublattice  $A'$  is given by the starred elements and a relatively complete subalgebra  $B$  of  $A$  is represented by the circled elements. Setting  $B' = A' \cap B$ , it is immediate that  $B'$  is not a subalgebra of  $A'$  for it does not contain  $Kx \Rightarrow Ky$ . Note that the proof in [1] could not be corrected by simply taking the subalgebra  $B_0$  generated by  $B'$  in  $A'$ . For in order to prove the finite model property we want a subalgebra  $B_0$  of  $A'$  such that if  $K_0, I_0$  are the operators with range  $B_0$ , then  $K_0 a = Ka$  and  $I_0 a = Ia$  for each  $a$  such that  $a, Ka, Ia \in A'$ .  $B_0$  does not in general satisfy to this condition as can readily be seen, if we consider the example above and take  $B_0 = A'$  and  $a = Kx \Rightarrow Ky$ .

Yet as we shall see below, S5-IC does have in fact the finite model property. To show this, we first recall the concept of Monadic Heyting Algebras. We say that  $\langle A; \cup, \cap, \rightarrow, K, I \rangle$  is a *Monadic Heyting algebra* (HM) if  $\langle A, \cup, \cap, \rightarrow \rangle$  is a Heyting algebra and  $K, I: A \rightarrow A$  (called associated quantifiers on  $A$ ) are such that

- (i)  $K[A]$ , the range of  $K$  in  $A$ , is a subalgebra of  $A$
- (ii)  $K[A] = I[A]$
- (iii) (a)  $x \leq y$  iff  $Kx \leq y$ ,  $x \in A, y \in K[A]$   
 (b)  $x \leq y$  iff  $x \leq Iy$ ,  $x \in K[A], y \in A$ .

In [4] Monadic Heyting algebras were shown to be equivalent to canonical models and since the former structures prove to be more convenient for the problems we want to deal with, we shall from here on think of the HM's as the structures which give completeness for S5-IC. We then prove

**Theorem 1** Let  $\mathfrak{G} = \langle A; \cup, \cap, \rightarrow, K, I \rangle$  be a Monadic Heyting algebra and let  $A_0 \subseteq A$  be a finite subset of  $A$ . Then there exists a finite **HM**,  $\mathfrak{G}' = \langle A'; \cup, \cap, \Rightarrow, K, I' \rangle$  such that

- (i)  $A_0 \cup \{1_A\} \cup \{0_A\} \subseteq A'$ ,
- (ii)  $\langle A', \cup, \cap, K \rangle$  is a (finite) existential sublattice of  $\mathfrak{G}$ ,<sup>1</sup>
- (iii) if  $x, y, x \rightarrow y \in A'$ , then  $x \rightarrow y = x \Rightarrow y$ ,
- (iv) if  $x, Ix \in A'$ , then  $Ix = I'x$ .

*Proof:* Let  $\langle A', \cup, \cap, K \rangle$  be the existential sublattice generated by  $A_0 \cup \{1, 0\}$ . It is known that the structure thus obtained is finite ([8]). Letting  $I_0$  be the interior operator determined by  $A'$ , (i.e.,  $I_0[A] = A'$ ), and defining

$$\begin{aligned} x \Rightarrow y &= I_0(x \rightarrow y), & x, y &\in A' \\ I'x &= I_0Ix, & x &\in A' \end{aligned}$$

we show that  $\mathfrak{G}' = \langle A'; \cup, \cap, \Rightarrow, K, I' \rangle$  is an **HM**. It is straightforward to check that  $\langle A', \cup, \cap, \Rightarrow \rangle$  is a Heyting algebra for which condition (iii) holds. We prove first that the sublattice  $K[A']$  is a subalgebra of  $\langle A', \cup, \cap, \Rightarrow \rangle$ , i.e., that it is closed with respect to  $\Rightarrow$ . Note that  $K[A]$  is a Heyting subalgebra of  $\mathfrak{G}$ , hence for  $x, y \in A'$ ,  $Kx \rightarrow Ky = Kz$  for some  $z \in A$ . Thus  $Kx \Rightarrow Ky \leq Kx \rightarrow Ky = Kz$  which by condition (iii,a) on **HM**'s, yields  $K(Kx \Rightarrow Ky) \leq Kz$ . But  $A'$  is closed with respect to  $K$ , so  $K(Kx \Rightarrow Ky) = I_0K(Kx \Rightarrow Ky) \leq I_0Kz = Kx \Rightarrow Ky$ . Hence  $Kx \Rightarrow Ky = K(Kx \Rightarrow Ky) \in K[A']$ . To prove that  $K[A'] = I'[A']$ , it is sufficient to show that for all  $x \in A'$ ,  $I'Kx = Kx$  and  $KI'x \leq I'x$ . Since  $Kx = IKx$  (see [4]) we have  $Kx = I_0Kx = I_0IKx = I'Kx$ . Second from  $I'x \leq Ix$ , it follows  $KI'x \leq KIx = Ix$  and therefore  $KI'x = I_0KI'x \leq I_0Ix = I'x$ . That  $K$  satisfies (iii,a) of the definition of **HM**'s is immediate since it is the original operator restricted to  $A'$  and  $K[A'] \subseteq K[A]$ . In order to prove that  $I'$  satisfies (iii,b) we suppose that  $x \in K[A']$  and  $x \leq y$ . Then for some  $z \in A'$ ,  $Kz = x \leq y$  and thus  $Kz \leq Iy$ . But  $Kz = I_0Kz \leq I_0Iy = I'y$ . The second half of (iii,b) is straightforward. Last note that if  $x, Ix \in A'$ , then  $Ix = I_0Ix = I'x$ . Consequently  $\mathfrak{G}'$  is an **HM** having the required properties.

**Theorem 2** S5-IC has the finite model property, i.e., for each non-theorem  $\alpha$  of S5-IC, there is a finite **HM** which falsifies  $\alpha$ .<sup>2</sup>

*Proof:* Suppose that  $\alpha$  is a non-theorem of S5-IC with  $n$  subformulas. Then there is an **HM**  $\mathfrak{G}$  and a homomorphism  $v: \mathfrak{F} \rightarrow \mathfrak{G}$  such that  $v(\alpha) \neq 1_A$ . Let  $A_0 = \{a_1, \dots, a_n\}$  be the set of elements that  $v$  assigns to the subformulas of  $\alpha$ . Then by Theorem 1, there is a finite **HM**  $\mathfrak{G}'$  satisfying conditions (i) through (iv). Clearly these conditions ensure the existence of a homomorphism  $v': \mathfrak{F} \rightarrow \mathfrak{G}'$  such that  $v'(\alpha) = v(\alpha)$ . For by (i) it is possible to

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1. We recall that an existential lattice is a pair  $\langle A, K \rangle$  where  $A$  is a distributive lattice with 0, 1 and  $K$  is an operator on  $A$  such that  $x \leq Kx$ ,  $K0 = 0$ ,  $K(x \cup y) = Kx \cup Ky$ ,  $K(x \cap Ky) = Kx \cap Ky$ .
  2. A formula  $\alpha \in \mathfrak{F}$  (where  $\mathfrak{F}$  is the algebra of formulas of S5-IC) is true in an **HM**  $\mathfrak{G}$  if for every homomorphism  $v: \mathfrak{F} \rightarrow \mathfrak{G}$ ,  $v(\alpha) = 1_{\mathfrak{G}}$ .

choose  $v'$  in such a way that  $v'$  coincides with  $v$  on the propositional variables of  $\alpha$ . Hence by (i) through (iv),  $v(\beta) = v'(\beta)$  for every subformula  $\beta$  of  $\alpha$  and hence  $v'(\alpha) = v(\alpha) \neq 1_A = 1_{A'}$ , i.e.,  $\alpha$  is falsified in  $\mathfrak{G}'$ .

**2 A consequence of the finite model property** In a sense which was explained in [1] and [2], S5-IC turns out to be an intuitionistic analogue of Lewis' S5. In [3] we proposed a general criterion by means of which it is possible to find the intuitionist counterpart of many other classical modal systems. This criterion is based upon the existence of a theorem-preserving translation  $\mathsf{T}$ , from S5-IC to a bimodal calculus (S4, S5)-C. While the theorem concerning the existence of translation  $\mathsf{T}$  in [3] makes use of both algebraic semantics and syntactic methods, we want to show in this paper that once the finite model property for S5-IC has been established, this same theorem can be proven using algebraic techniques only.<sup>3</sup>

In order to make this paper more readable, I recall a few concepts that can be found in [3]. Let (S4, S5)-C be a "bimodal calculus" having as primitive connectives ( $\vee$ ,  $\neg$ ,  $L_1$ ,  $L_2$ ) with the following axioms and rules:

(b<sub>1</sub>) all classical propositional tautologies with the usual definitions for  $\wedge$ ,  $\rightarrow$ .

(b<sub>2</sub>) S4 axioms and rules on  $L_1$ :  $L_1\alpha \rightarrow \alpha$   
 $L_1(\alpha \rightarrow \beta) \rightarrow L_1\alpha \rightarrow L_1\beta$   
 $L_1\alpha \rightarrow L_1L_1\alpha$   
 $\frac{\vdash \alpha}{\vdash L_1\alpha}$

(b<sub>3</sub>) S5 axioms and rules on  $L_2$ :  $L_2\alpha \rightarrow \alpha$   
 $L_2(\alpha \rightarrow \beta) \rightarrow L_2\alpha \rightarrow L_2\beta$   
 $M_2\alpha \rightarrow L_2M_2\alpha$ , where  $M_2 = \neg L_2\neg$   
 $\frac{\vdash \alpha}{\vdash L_2\alpha}$

(b<sub>4</sub>)  $M_2L_1\alpha \rightarrow L_1M_2\alpha$ .

Now let  $\mathsf{T}$  be the function from S5-IC formulas to (S4, S5)-C formulas such that for each propositional variable  $p$ ,

$$\mathsf{T}(p) = L_1p$$

and if  $\alpha, \beta$  are formulas of S5-IC, then

$$\begin{aligned}\mathsf{T}(\alpha \vee \beta) &= \mathsf{T}\alpha \vee \mathsf{T}\beta \\ \mathsf{T}(\alpha \wedge \beta) &= \mathsf{T}\alpha \wedge \mathsf{T}\beta \\ \mathsf{T}(\alpha \rightarrow \beta) &= L_1(\mathsf{T}\alpha \rightarrow \mathsf{T}\beta) \\ \mathsf{T}(\neg \alpha) &= L_1\neg \mathsf{T}\alpha \\ \mathsf{T}(M\alpha) &= M_2\mathsf{T}\alpha \\ \mathsf{T}(L\alpha) &= L_1L_2\mathsf{T}\alpha\end{aligned}$$

In [3], we proved

3. Then, this paper settles a question mentioned in [3].

**Theorem 3** If  $\vdash_{S5-IC} \alpha$ , then  $\vdash_{(S4, S5)-C} \top \alpha$ ,

using first, completeness results for S5-IC and (S4, S5)-C in the class of all HM's and all bimodal algebras<sup>4</sup> respectively, and second the following

**Lemma 1** Let  $\mathfrak{B} = \langle B, I_1, K_2 \rangle$  be a bimodal algebra. If we define  $G(B) = \{x \in B : x = I_1 x\}$  and

$$K = K_2 \upharpoonright G(B) \\ I = I_1 \upharpoonright_2 \upharpoonright G(B), \text{ where } I_2 = \sim K_2 \sim$$

then  $G(\mathfrak{B}) = \langle G(B), K, I \rangle$  is an HM to be called the HM associated with the bimodal algebra  $\mathfrak{B}$ .

Our aim is to give an algebraic proof of the converse of Theorem 3. We shall be able to do so, once we establish that every finite HM is, up to isomorphism, associated with a bimodal algebra and such will be the topic of section 3.

**3 A representation theorem for finite HM's** From here on  $\Rightarrow, \rightarrow$  will stand for the Heyting and Boolean relative pseudo complement respectively. Similarly  $-$  will denote the Heyting pseudo complement while  $\sim$  will represent Boolean complementation. Now let  $\langle A; K, I \rangle$  be a finite HM. It is well known (see [6]) that there is a topological Boolean algebra  $\langle B; I_1 \rangle$  such that  $A = G(B) = I_1[B]$  with  $x \Rightarrow y = I_1(x \rightarrow y)$  and  $-x = I_1(\sim x)$  for all  $x, y \in G(B)$ . Then  $K[A] = I[A] \subseteq A \subseteq B$ . Let  $B'$  be the Boolean subalgebra of  $B$  generated by  $K[A]$ . Bearing these facts in mind we can prove the following two lemmas

**Lemma 2** There is a (Boolean) quantifier  $K_2$  on  $B$  such that  $K_2[B] = B'$ .

*Proof:*  $B'$  is a finite, hence relatively complete, subalgebra of  $B$ .

**Lemma 3** For every  $x \in B'$ ,  $I_1 x \in K[A] = I[A]$ .

*Proof:* If  $x \in B'$ , then  $x = \bigcap_{i=1}^m (a_i \rightarrow b_i)$  with  $a_i, b_i \in K[A]$  (see [6]). So:

$$I_1 x = I_1 \bigcap_{i=1}^m (a_i \rightarrow b_i) = \bigcap_{i=1}^m I_1(a_i \rightarrow b_i) = \bigcap_{i=1}^m (a_i \Rightarrow b_i) \in K[A].$$

And now for the representation theorem for finite HM's:

**Theorem 4** For every finite HM,  $\mathfrak{G} = \langle A; K, I \rangle$ , there exists a finite bimodal algebra  $\mathfrak{B} = \langle B, I_1, K_2 \rangle$  such that  $\mathfrak{G}$  is (up to isomorphism) the HM associated with  $\mathfrak{B}$ .

*Proof:* Let  $\langle B; I_1 \rangle$  be as in the comments preceding Lemma 2. Let  $K_2$  be as in Lemma 2 and put  $K_1 = \sim I_1 \sim$ ,  $I_2 = \sim K_2 \sim$ . We prove that  $\mathfrak{B} = \langle B, I_1, K_2 \rangle$  is

4. A bimodal algebra (see [3]) is a triple  $\langle B, I_1, K_2 \rangle$  where  $B$  is a boolean algebra,  $K_2$  a boolean quantifier and  $I_1$  an interior operator having the following property: for all  $x \in B$ ,  $K_2 I_1 x \leq I_1 K_2 x$  or equivalently  $K_2 I_1 x = I_1 K_2 I_1 x$ .

a bimodal algebra with  $\mathfrak{G} = G(\mathfrak{B})$ . We show first that  $K = K_2 \upharpoonright A$ . For every  $x \in A$ ,  $x \leq Kx \in B' = K_2[B]$  (Lemma 2) thus  $K_2x \leq Kx$ . On the other hand  $K_2x \in B'$  so by Lemma 3,  $I_1K_2x \in K[A]$ . But  $x = I_1x \leq I_1K_2x$  and hence  $Kx \leq I_1K_2x \leq K_2x$ . We can now prove that  $\mathfrak{B}$  is a bimodal algebra, i.e., that  $K_2I_1x \leq I_1K_2x$  for every  $x \in B$ . Since  $A = I_1[B]$  and  $K$  and  $K_2$  coincide on elements of  $A$ , we have  $K_2I_1x = KI_1x = I_1KI_1x = I_1K_2I_1x \leq I_1K_2x$ . Last we prove  $I = I_1I_2 \upharpoonright A$ . Let  $x \in A$ ; from  $x \geq Ix \in B'$  we get  $I_2x \geq Ix$  and hence  $I_1I_2x \geq I_1Ix = Ix$ . Vice versa, from Lemma 3 we have  $I_1x \geq I_1I_2x \in K[A]$ , thus  $I_1x \geq I_1I_2x$ . It follows  $Ix \geq I_1I_2x$ .

**4 A theorem on translation  $\uparrow$**  We can now give an algebraic proof of the converse of Theorem 3. First we recall that in [3], we proved

**Lemma 4** *Let  $\mathfrak{B} = \langle B; I_1, K_2 \rangle$  be a bimodal algebra. For every formula  $\alpha$  of S5-IC,  $\uparrow\alpha$  is true in  $\mathfrak{B}$  iff  $\alpha$  is true in  $G(\mathfrak{B})$ , the HM associated with  $\mathfrak{B}$ .*

We then prove

**Theorem 5** *If  $\vdash_{(S4, S5)-C} \uparrow\alpha$ , then  $\vdash_{S5-IC} \alpha$ .*

*Proof:* Suppose that  $\alpha$  is a non-theorem of S5-IC, then by Theorem 2, there exists a finite HM,  $\mathfrak{G}$  which falsifies  $\alpha$ . By Theorem 4, there is a bimodal algebra  $\mathfrak{B}$  such that  $\mathfrak{G} = G(\mathfrak{B})$  is the HM associated with  $\mathfrak{B}$ . Then by Lemma 4,  $\uparrow\alpha$  is not true in  $\mathfrak{B}$ , hence by completeness  $\uparrow\alpha$  is not a theorem of (S4, S5)-C.

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