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AN ADDITIONAL REMARK ON SELF-CONJUGATE FUNCTIONS ON BOOLEAN ALGEBRAS

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In this note I add several remarks to my paper [1]. Let f be a self-conjugate function on a Boolean algebra, in [1] it was shown that if $f^n = \mathrm{id}$ and $f \neq \mathrm{id}$, then n is even and $f^2 = \mathrm{id}$. Necessary and sufficient conditions for a self-conjugate function f to have the property $f^2 = f$ were given. Here we extend results of this type. The references in the proofs are from [1]. Let

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1\rangle$$

be a **BA**.

Lemma *If $f: A \to A$ is self-conjugate, then*

$$x \leqslant f^2(x) \leqslant f^4(x) \leqslant \ldots \leqslant f^{2n}(x) \leqslant \ldots$$

and

$$f(x) \leqslant f^3(x) \leqslant f^5(x) \leqslant \ldots \leqslant f^{2n+1}(x) \leqslant \ldots$$

for any $x \in A$, and all $n > 0$.

Proof: By replacing x by $f(1)$ and y by $f^n(x)$ in 1.2(iii) we obtain

$$f^n(x) = f(1) \cdot f^n(x) \leqslant f(1 \cdot f^{n+1}(x)) = f^{n+2}(x)$$

for any $n > 0$. Now if we show that $x \leqslant f^2(x)$ we are done. Let $c = x - f^2(x)$. By 1.1, 1.3, and 1.5,

$$f^2(x) \cdot c = 0 \leftrightarrow f(x) \cdot f(c) = 0 \leftrightarrow f(c) = 0 \leftrightarrow c = 0.$$

Hence $x \leqslant f^2(x)$ as desired.

Theorem *If $f: A \to A$ is self-conjugate, $f^n = f^m$, $n < m$ and for $i \leqslant n$, $j < m$, $f^i \neq f^j$, then*

(i) $m = n + 2$, *if n, m have the same parity,*
(ii) $m = n + 1$, *if n, m have different parities.*

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Proof: (i) By lemma

$$f^n(x) \leqslant f^{n+2}(x) \leqslant \ldots \leqslant f^m(x)$$

Hence, $f^n(x) = f^{n+2}(x)$, for all $x \in A$, and therefore $m = n + 2$.
(ii) Again by the lemma we get $f^n(x) \leqslant f^{m-1}(x)$. So $f^{n+1}(x) \leqslant f^m(x) = f^n(x)$. Hence, by 1.4, $f^n(x) = f^{n+1}(x)$ and $m = n + 1$.

We have now established some conditions for which occur when powers of self-conjugate functions start repeating. We show that these results are the best possible. Let

$$\mathfrak{A} = \langle A, \cup, \cap, -, 0, 1, c_{ij} \rangle_{i,j<\omega, i<j}$$

where A is the power of ${}^{\omega}U$, where U is a countably infinite set, $\cup$, $\cap$, and $-$ are the standard set-theoretic operations and, for each $i < j$, c_{ij} is a unary function on A defined as follows:

$c_{ij}(X) = \{y \in A\text{: for some } x \in X \text{ we have } x_i = y_j,\ y_i = x_j \text{ and } x_k = y_k \text{ for } k \neq i, j\}$

for any $X \in A$.

By 1.25, c_{ij} is self-conjugate for each i, j. Let

$$f_n = \mathrm{id} + \sum_{i,j<n} c_{ij}.$$

f_n has the property that $f_n^i \neq f_n^j$ for $i \leqslant n$, $j < n$, and $f_n^n = f_n^{n+1}$.

For a self-conjugate function f such that $f^i \neq f^j$, for $i \leqslant n$, $j < n + 2$, and $f^n = f^{n+2}$, the function $f = \langle f_n, c_{01} \rangle$ on $\mathfrak{A} \times \mathfrak{A}$ suffices.

REFERENCE

[1] Sudkamp, T. A., "Self-conjugate functions on Boolean algebras," *Notre Dame Journal of Formal Logic*, vol. XIX (1978), pp. 504-512.

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