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BINOMIAL PAIRS, SEMI-BROUWERIAN AND BROUWERIAN SEMILATTICES

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This paper may be considered as a contribution to the axiomatization of intuitionistic logic, i.e., Brouwerian semilattices, within the wider realm of semi-Brouwerian semilattices. The latter occurred first within a purely algebraic context, as congruence lattices of semilattices (whose characterization as abstract lattices, *cf.* Grätzer [14], Problem 21, is, in a sense, still an open problem). As observed here for the first time, semi-Brouwerian semilattices form an equational class, an additional equation making them Brouwerian (Proposition 2.1). These equations have indeed a structural meaning that is fully investigated in section 1. In section 3, further conditions are given that make a semi-Brouwerian semilattice Brouwerian, some distributivity condition (Theorem 3.3) and the classical deduction theorem (Theorem 3.7) among them. Between Brouwerian and semi-Brouwerian semilattices, there is also a relationship similar to that between abelian groups and all groups (Theorem 3.6).

1 *Binomial pairs* Let S be a partially ordered set. A *closure operator* in S is a mapping β: $S \to S$ such that

$$(1.1) \qquad \begin{cases} \beta(\beta(z)) = \beta(z), \\ \quad z \leq \beta(z) \end{cases}$$

for each $z \in S$, moreover, for each $y, z \in S$,

$$(1.2) \qquad \textit{whenever } y \leq z, \textit{ then } \beta(y) \leq \beta(z).$$

There is a well-known one-to-one correspondence between closure operators β and certain subsets $B \subset S$, established by

$$(1.3) \qquad B = \beta(S),\ \beta(z) = \min\{b \in B \mid z \leq b\}\ (z \in S).$$

The subsets B occurring here are exactly those for which all those minima in (1.3) exist. The dual of a closure operator is a *kernel operator.*

A *weak closure operator* will be a mapping β: $S \to S$ just satisfying (1.1), (1.2) being no longer required. It is no longer determined by its image $\beta(S)$. E.g., in the semilattice

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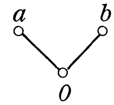


there are two weak closure operators with image $\{a, b\}$. However, a weak closure operator β is still determined by the induced equivalence relation $\rho = \beta^{-1} \circ \beta$. We will call an equivalence relation a *monomial equivalence* (*relation*) once every equivalence class has a maximum (for congruences of a join-semilattice, *cf.* E. T. Schmidt [27], Definition 4.7).

Theorem 1.1 *In a partially ordered set S, there is a one-to-one correspondence between weak closure operators β and monomial equivalences ρ, established by*

$$\rho = \beta^{-1} \circ \beta,\ \beta(z) = \max \{y \in S \mid (y, z) \in \rho\}\ (z \in S). \tag{1.4}$$

Dually, ρ is a *comonomial equivalence* once every equivalence class has a minimum. (Katriňák [18], Definition 3.4, used "comonomial" for certain monomial congruences of a meet-semilattice.) We get the dual one-to-one correspondence between weak kernel operators and comonomial equivalences. For a nice example of a weak kernel operator, consider Tarski's function cf. in the class of ordinals (or in any well-ordered set).

We now call ρ a *binomial equivalence* once ρ is both monomial and comonomial. A *binomial pair* is an ordered pair (α, β) of mappings $\alpha, \beta\colon S \to S$ so that

$$\begin{cases} \alpha^{-1} \circ \alpha = \beta^{-1} \circ \beta, \\ \alpha \circ \alpha = \alpha,\ \beta \circ \beta = \beta \end{cases} \tag{1.5}$$

and

$$\alpha(z) \leqslant z \leqslant \beta(z) \tag{1.6}$$

for each $z \in S$. I.e., α is a weak kernel operator, β a weak closure operator, and both induce the same equivalence relation.

Corollary 1.2 *In a partially ordered set S, there is a one-to-one correspondence between binomial pairs (α, β) and binomial equivalences ρ, established by*

$$\begin{cases} \rho = \alpha^{-1} \circ \alpha = \beta^{-1} \circ \beta, \\ \alpha(z) = \min \{y \in S \mid (y, z) \in \rho\}, \\ \beta(z) = \max \{y \in S \mid (y, z) \in \rho\}\ \ (z \in S). \end{cases} \tag{1.7}$$

Clearly, each member of a binomial pair determines its partner:

Corollary 1.3 *Let S be a partially ordered set, α and β mappings of S into itself. Then (α, β) is a binomial pair iff*

$$\begin{cases} \alpha(z) = \min \{y \in S \mid \beta(y) = \beta(z)\}, \\ \beta(z) = \max \{y \in S \mid \alpha(y) = \alpha(z)\} \end{cases} \tag{1.8}$$

for each $z \in S$.

For if (α, β) is a binomial pair, (1.8) holds as an immediate consequence of (1.7). Conversely, (1.8) makes α the weak kernel operator corresponding to the comonomial equivalence $\beta^{-1} \circ \beta$, so that $\alpha^{-1} \circ \alpha = \beta^{-1} \circ \beta$. (1.8) also makes β the weak closure operator corresponding to the monomial equivalence $\alpha^{-1} \circ \alpha$, so that again $\beta^{-1} \circ \beta = \alpha^{-1} \circ \alpha$. So this equivalence is binomial and (α, β) the corresponding binomial pair.

There is also an axiomatic, in fact, equational description of the partner. To that end, note that the purely set-theoretic conditions (1.5) are equivalent with the equations

$$\alpha \circ \beta = \alpha, \ \beta \circ \alpha = \beta. \tag{1.9}$$

For (1.5) makes z, $\alpha(z)$, and $\beta(z)$ equivalent under $\alpha^{-1} \circ \alpha = \beta^{-1} \circ \beta$, where $\alpha(\beta(z)) = \alpha(z)$ and $\beta(\alpha(z)) = \beta(z)$. Conversely, (1.9) implies $\alpha^{-1} \circ \alpha = \beta^{-1} \circ \beta$, also $\alpha \circ \alpha = \alpha \circ \beta \circ \alpha = \alpha \circ \beta = \alpha$. With that, we have

Corollary 1.4 *Let S be a partially ordered set and α: $S \to S$ a unary operation such that $\alpha(z) \leqslant z$ for each $z \in S$. Then the unary operation (if it exists) β: $S \to S$ that makes (α, β) a binomial pair is characterized by these three equations, holding for every $z \in S$:*

$$\begin{cases} \alpha(\beta(z)) = \alpha(z), \\ \beta(\alpha(z)) = \beta(z), \\ \beta(z) \wedge z = z. \end{cases} \tag{1.10}$$

An equivalence ρ is a *convex equivalence* once each equivalence class is a convex subset of S. Note that each order-preserving mapping of S into any partially ordered set T induces a convex equivalence in S since, more generally, the preimage of any convex subset of T will be a convex subset of S. For a convex binomial equivalence ρ, the equivalence class of $z \in S$ will be the closed interval $[\alpha(z), \beta(z)]$, where (α, β) is the corresponding *convex binomial pair.* The convex binomial equivalences actually correspond one-to-one to the decompositions of S into closed intervals.

Proposition 1.5 *Let α, β be mappings of S into itself. Then (α, β) is a convex binomial pair iff one of the following equivalent conditions holds, for each y, $z \in S$:*

$$\begin{cases} \alpha(y) = \alpha(z) \ \textit{iff} \ \alpha(z) \leqslant y \leqslant \beta(z), \\ \beta(y) = \beta(z) \ \textit{iff} \ \alpha(z) \leqslant y \leqslant \beta(z). \end{cases} \tag{1.11}$$

For the first equivalence of (1.11) makes $\alpha(z)$ the minimum, $\beta(z)$ the maximum of all elements y such that $\alpha(y) = \alpha(z)$. So $\rho = \alpha^{-1} \circ \alpha$ becomes a binomial equivalence and (α, β) the corresponding binomial pair. (1.11) also makes each equivalence class convex, so that (α, β) is a convex binomial pair.

For a neat example from universal algebra, let $\langle A, F\rangle$ be an algebra with finitary operations, S its congruence lattice, $S = \theta(A, F)$. We call the congruences y, $z \in S$ *equivalent with respect* to $a \in A$ provided that their congruence classes of a coincide, (1.12) $$(y, z) \in \rho_a \text{ iff } a/y = a/z.$$

ρ_a is a convex binomial equivalence.

The better known pairings of functions $\alpha, \beta: S \to S$ are the *adjoint situations* (α, β), alias *Galois connections of mixed type*. They are characterized by the condition:

(1.13) $$\alpha(y) \leq z \text{ iff } y \leq \beta(z)$$

for each $y, z \in S$. (*Cf.* Benado [4]; J. Schmidt [29]; Nöbeling [23]; also Blythe and Janowitz [7].) This is the case iff both α and β are order-preserving, $\beta \circ \alpha$ a closure operator, $\alpha \circ \beta$ a kernel operator. Another equivalent description:

(1.14) $$\begin{cases} \alpha(y) = \min \{z \in S \mid y \leq \beta(z)\}, \\ \beta(z) = \max \{y \in S \mid \alpha(y) \leq z\}, \end{cases}$$

for each $y, z \in S$.

Proposition 1.6 *Let S be a partially ordered set and α and β mappings of S into itself. The following are equivalent:*

(i) *(α, β) is a binomial pair α and β are order-preserving;*
(ii) *(α, β) is a Galois connection of mixed type, α is a kernel operator (or β a closure operator).*

Proof: (i) $\Rightarrow$ (ii): It suffices to show only one half of (1.13). Let $\alpha(y) \leq z$. We get $y \leq \beta(y) = \beta(\alpha(y)) \leq \beta(z)$. (ii) $\Rightarrow$ (i): $\alpha \circ \beta$ is a kernel operator and $\alpha(\beta(S)) = \alpha(S)$. If α is a kernel operator too, then $\alpha \circ \beta = \alpha$, hence $\beta = \beta \circ \alpha \circ \beta = \beta \circ \alpha$: (1.9) holds. Also $z \leq \beta(\alpha(z)) = \beta(z)$: (1.6) holds, making (α, β) a binomial pair.

We may call such a pair (α, β) consisting of a (strong) kernel operator α and a (strong) closure operator β inducing the same equivalence ρ a *strong binomial pair* and ρ a *strong binomial equivalence.* Note that (α, β) and ρ are then convex.

Corollary 1.7 *Let S be a partially ordered set, α a kernel operator. Then the unary operation (if it exists) $\beta: S \to S$ that makes (α, β) a strong binomial pair is characterized by the condition that β be order-preserving and*

(1.15) $$\alpha(\beta(z)) \leq z \leq \beta(\alpha(z))$$

for each $z \in S$.

For these conditions simply make (α, β) a Galois connection of mixed type.

In the sequel, we will always work in a (meet-) semilattice S. Congruences will be meet-congruences (they are convex!), homomorphisms will be meet-homomorphisms.

Proposition 1.8 *Let S be a semilattice $\alpha: S \to S$ a mapping. Then the following are equivalent:*

(i) *α is idempotent and $\alpha^{-1} \circ \alpha$ a congruence;*
(ii) $\alpha(y \wedge z) = \alpha(\alpha(y) \wedge \alpha(z))$ *for each* $y, z \in S$;
(iii) $\alpha(y \wedge z) = \alpha(y \wedge \alpha(z))$ *for each* $y, z \in S$.

Proof: (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i): evident.

Corollary 1.9 *Let S be a semilattice, α a weak kernel operator. Then α is a kernel operator iff α satisfies one of the conditions of* Proposition 1.8.

It is well-known that a kernel operator satisfies (ii). If α is a weak kernel operator satisfying (ii), we get $\alpha(y \wedge z) \leqslant \alpha(y) \wedge \alpha(z)$ which makes α order-preserving.

So a weak kernel operator α is order-preserving iff $\alpha^{-1} \circ \alpha$ is a subsemilattice of $S \times S$. It is well-known that a kernel operator α is meet-preserving iff $\alpha(S)$ is a subsemilattice of S. For a closure operator β, however, $\beta(S)$ is a subsemilattice of S anyway.

Corollary 1.10 *Let S be a semilattice, β a closure operator. Then β is meet-preserving iff $\beta^{-1} \circ \beta$ is a congruence.*

If β is an endomorphism, $\beta^{-1} \circ \beta$ is certainly a congruence (a subsemilattice of $S \times S$). Since β is order-preserving, we have $\beta(y \wedge z) \leqslant \beta(y) \wedge \beta(z)$ anyway. If $\beta^{-1} \circ \beta$ is a congruence, (ii) of Proposition 1.8 yields $\beta(y \wedge z) \geqslant \beta(y) \wedge \beta(z)$.

Ward [33] claims that every closure operator has the property of Proposition 1.8. For a study of meet-preserving closure operators, *cf.* Bergmann [5], Cignoli [8], R. E. Johnson [15], Varlet [32], also J. Schmidt [30]. In addition to Proposition 1.6, we now have

Corollary 1.11 *Let S be a semilattice (α, β) a binomial pair. Then (α, β) is a strong binomial pair iff β is meet-preserving.*

Proof: Corollaries 1.9 and 1.10.

Actually, one can say much more about α, β, and ρ in the strong case. For as a well-known property of arbitrary Galois connections of mixed type, α preserves arbitrary joins (as far as they exist), β arbitrary meets. Correspondingly, a strong binomial equivalence ρ is compatible, in an obvious sense, with aribtrary joins and meets. For a convex (weak) binomial pair (α, β), these statements still hold in a somewhat restricted form, expressing the fact that each closed interval is closed under joins and meets of non-empty subsets.

2 *Weak relative pseudo-complements* We apply the general observations of section 1 to the meet-preserving kernel operator α: $S \rightarrow S$ defined by $\alpha(z) = a \wedge z$ $(z \in S)$, where a is any element of S. Of course, a is uniquely determined by the image of α, the principal ideal $(a]$. The element a is also uniquely determined by the comonomial congruence $\rho = \alpha^{-1} \circ \alpha$ simply because the kernel operator α is determined by ρ (Theorem 1.1). Note also that the principal filter $[a)$, the congruence class of a, is the greatest

element of the factor semilattice S/ρ and the only element of S/ρ that is a filter. The congruence class of $z \in S$ is convex, with least element $a \wedge z$. Suppose now its greatest element

(2.1) $$a \rightharpoonup z = \max\{y \in S \mid a \wedge y = a \wedge z\}$$

exists too. I.e., the congruence class z/ρ is the closed interval $[a \wedge z, a \rightharpoonup z]$,

(2.2) $$a \wedge y = a \wedge z \textit{ iff } a \wedge z \leqslant y \leqslant a \rightharpoonup z$$

for each $y \in S$ (*cf.* (1.11)). We call $a \rightharpoonup z$ the *weak* (*relative*) *pseudo-complement of a with respect to z* (*in S*). Compare this with the definition of the usual (*strong*) *relative pseudo-complement*:

(2.3) $$a \rightharpoonup z = \max\{y \in S \mid a \wedge y \leqslant z\}.$$

It is implicitly defined by the equivalence

(2.4) $$a \wedge y \leqslant z \textit{ iff } y \leqslant a \rightharpoonup z,$$

to hold for each $y \in S$ (*cf.* (1.13)). I.e., the principal ideal $(a \rightharpoonup z]$ is the class of all elements $y \in S$ such that $y/\rho \leqslant z/\rho$ in the factor semilattice S/ρ. Clearly, a strong relative pseudo-complement is a weak one, the converse not being true in general. E.g., in the 3-semilattice of section 1, the weak $a \rightharpoonup b$ exists, which fails to be strong. In the sequel, $a \rightharpoonup z$ will always stand for the wider notion; any time it happens to be strong, we will say so. If S has a least element, 0, then, of course, the weak and the strong notion coincide for $z = 0$, and one calls the element $a \rightharpoonup 0$ simply the *pseudo-complement of a*,

(2.5) $$\neg a = a \rightharpoonup 0 = \max\{y \in S \mid a \wedge y = 0\}.$$

Note that the weak pseudo-complement $a \rightharpoonup z$ is nothing but the pseudo-complement of a in the principal filter $[a \wedge z)$:

(2.6) $$a \rightharpoonup z = {}_{\overline{a \wedge z}}\neg\, a.$$

In particular,

(2.7) $$a \rightharpoonup z = {}_{\overline{z}}\neg\, a, \text{ (if } z \leqslant a).$$

In [30], we called an element $a \in S$ *left Brouwerian* provided that the strong form of $a \rightharpoonup z$ exists for each $z \in S$. So we may call a *weakly left Brouwerian* provided that the weak $a \rightharpoonup z$ exists for each $z \in S$, i.e., if $\alpha = a\wedge$ is a partner of a (convex) binomial pair (α, β), where $\beta = a \rightharpoonup$. (Again, β determines the element a uniquely.) Note that a is (strongly) left Brouwerian iff our kernel operator α is a partner of a Galois connection of mixed type (α, β), i.e., of a strong binomial pair (α, β) (Proposition 1.6, (ii)).

Proposition 2.1 *Let S be a semilattice, $a \in S$. Then a is weakly left Brouwerian iff there is a unary operation—necessarily unique—$a \rightharpoonup$: $S \to S$ satisfying the following equations:*

L1. $a \wedge (a \rightharpoonup z) = a \wedge z$,

L2. $a \rightarrow (a \wedge z) = a \rightarrow z$,
L3. $(a \rightarrow z) \wedge z = z$.

The element a is (strongly) left Brouwerian iff, in addition, the equation

L4. $a \rightarrow (y \wedge z) = (a \rightarrow y) \wedge (a \rightarrow z)$

holds. (It suffices to assume $a \rightarrow$ order-preserving.)

Proof: Corollaries 1.4 and 1.11.

For the weak case, we still have, as a substitute for L4, the equation

$$a \rightarrow (y \wedge z) = a \rightarrow ((a \rightarrow y) \wedge (a \rightarrow z)), \tag{2.8}$$

due to Proposition 1.8.

A semilattice S is called *Brouwerian* if each element $a \in S$ is left Brouwerian. L1-L4 are the defining equations for Brouwerian semilattices given for lattices by Monteiro in [21]; *cf.* also Rasiowa-Sikorski [25]. McKinsey and Tarski [20] had characterized Brouwerian lattices by the conditions that $\beta = a \rightarrow$ be order-preserving and

$$a \wedge (a \rightarrow z) \leqslant z \leqslant a \rightarrow (a \wedge z) \tag{2.9}$$

for each $z \in S$. This is just a short way of stating that $(a \wedge, a \rightarrow)$ is a Galois connection of mixed type (*cf.* Corollary 1.7) and so characterizes (strongly) left Brouwerian elements a of a semilattice. For other axiomatizations of Brouwerian (semi)lattices, *cf.* Ribenboim [26] and Katriňák-Mitschke [19].

A semilattice S will be called *weakly Brouwerian* or *semi-Brouwerian* if each element $a \in S$ is weakly left Brouwerian. L1-L3 are the defining equations for these semilattices. Papert [24] was the first to consider weak relative pseudo-complementation, in the case of congruence lattices $S = \theta(J)$ of join-semilattices J. She restricted the operation $a \rightarrow z$ to the case $z \leqslant a$ and wrote $a * z$. Varlet [31] was the first to consider abstract semi-Brouwerian lattices (the congruence lattices $\theta(J)$ among them).

Let now S be an arbitrary semilattice and $a \in S$. The congruence class of a itself is the principal filter $[a)$. If a is weakly left Brouwerian, $[a) = [a, a \rightarrow a]$, which makes $a \rightarrow a$ a maximal element of S (and each maximal element b can be so represented, e.g., as $b \rightarrow b$). With that, we have the first part of

Proposition 2.2 *Let a be a weakly left Brouwerian element of the semilattice S. Then S has a greatest element, e, iff S is directed. In this case,*

(2.10) $a \leqslant z$ *iff* $a \rightarrow z = e$,

for each $z \in S$. In particular,

(2.11) $a \rightarrow a = a \rightarrow e = e$.

If S is semi-Brouwerian, S has definitely a greatest element.

For suppose S semi-Brouwerian and $a, b \in S$. Since

$$(a \wedge b) \wedge a = (a \wedge b) \wedge b = (a \wedge b) \wedge (a \wedge b) \quad \text{and} \quad a,\, b \leqslant (a \wedge b) \rightarrow (a \wedge b),$$

showing that S is directed.

Note also that e exists once the strong $a \rightarrow a$ exists for some a, which is the simplest reason why a Brouwerian semilattice has an identity. Let us also observe that e is always strongly left Brouwerian and

$$e \rightarrow z = z, \tag{2.12}$$

for each $z \in S$.

Proposition 2.3 *Let S be a semilattice. Then the following are equivalent:*

(i) *S is semi-Brouwerian;*
(ii) *each principal filter $[z)$ is pseudo-complemented;*
(iii) *S has an identity, e, and each closed interval is pseudo-complemented.*

Proof: (i)$\Longleftrightarrow$(ii) is trivial, *cf.* (2.6) and (2.7). (iii)$\Longrightarrow$(ii) is trivial. (i), (ii) $\Longrightarrow$ (iii): By Proposition 2.2, S has an identity. Clearly, a principal ideal $[z, y]$ of a pseudo-complemented semilattice $[z, e]$ is pseudo-complemented, the pseudo-complement of $a \in [z, y]$ being $y \wedge \urcorner_z a$, i.e., $y \wedge (a \rightarrow z)$.

Katriňák [16], [17] considered the case that all principal ideals $(y] = [0, y]$ are pseudo-complemented. In [18], Katriňák considered *semilattices with pseudo-complemented intervals* $[z, y]$ as "segment-pseudo-complemented" (abschnittspseudokomplementär). Such semilattices may, of course, fail to have a largest element, hence fail to be semi-Brouwerian. Varlet [31], Théorème 2, and E. T. Schmidt [28], section **14**, observed that a lattice with pseudo-complemented intervals is distributive iff it is modular. In fact, this is an immediate consequence of the existence, in a non-distributive modular lattice S, of a non-distributive (modular) sublattice of five elements. Dean and Oehmke [9], Theorems 6 and 8, proved this for the special case $S = \theta(J)$ (J a join-semilattice). Simultaneously with Papert [24], Theorem 7, they found that $\theta(J)$ is distributive iff J is a dual tree. This provides us with many examples of semi-Brouwerian semilattices which are not Brouwerian. For an intensive study of the semi-Brouwerian lattices $\theta(J)$, *cf.* Evans [10].

3 *Conditions making a semi-Brouwerian semilattice Brouwerian* Such a condition has already been given in Proposition 2.1, where we axiomatized semi-Brouwerian semilattices by equations keeping the first argument of the binary operation $\rightarrow$ fixed. Katriňák and Mitschke [19], 5.1, for the special case $z = 0$ also Balbes and Horn [3], Theorem 1.1, have given an equational characterization of Brouwerian or pseudo-complemented lattices respectively in which the second argument is kept fixed. We can say somewhat more. In [30], we called an element $z \in S$ *right Brouwerian* provided that the strong form of $a \rightarrow z$ exists for each $a \in S$. So we may again call z *weakly right Brouwerian* provided that the weak $a \rightarrow z$ exists for each $a \in S$. We state without proof:

Proposition 3.1 *Let S be a semilattice with identity e, let $z \in S$. Then z is right Brouwerian iff there is a unary operation $\to z$: $S \to S$ satisfying the conditions:*

R1. $a \wedge (a \to z) = a \wedge z$,
R2. $(a \wedge z) \to z = e$,
R3. $b \wedge ((a \wedge b) \to z) = b \wedge (a \to z)$.

Note that this operation determines z uniquely by virtue of (2.12). Assuming the above equations to hold for each $z \in S$, we get another equational characterization of Brouwerian semilattices indeed. Note that R1 (= L1) and R2 (*cf.* (2.10)) hold in every semi-Brouwerian semilattice. As observed by Katriňák, at least the following modification of R3 holds in a semi-Brouwerian semilattice:

R3$_S$. $b \wedge ((a \wedge b) \to z) = b \wedge (a \to (b \wedge z))$.

No longer is that an equation for $\to z$, where z is fixed. Recall also that the left equations L1-L3, L4 were inherited from section 1. Nothing of that sort seems to apply to the right equations.

We still may, for a fixed z, collect information about the weak $\to z$. As opposed to $a \to$, the operation $\to z$ respects the order, in fact, reverses it:

(3.1) $$\textit{if } a \leq b, \textit{ then } b \to z \leq a \to z.$$

For if $a \leq b$, then $a \wedge (b \to z) = a \wedge b \wedge (b \to z) = a \wedge b \wedge z = a \wedge z$, where $b \to z \leq a \to z$. In the strong case, $\to z$ reverses all existing joins into meets. Nothing of that sort can be stated here. (3.1) makes the iterated function

(3.2) $$\gamma_z(a) = (a \to z) \to z$$

order-preserving:

(3.3) $$\textit{if } a \leq b, \textit{ then } \gamma_z(a) \leq \gamma_z(b).$$

Now, if z is weakly right Brouwerian, then the principal filter $[z)$ is certainly pseudo-complemented, with the restriction of $\to z$ to $[z)$ as pseudo-complementation. Hence the famous results of Glivenko [13] (extended to meet-semilattices by Frink [12]) hold: The restriction of γ_z to $[z)$ is a closure operator in $[z)$, whose closed elements form a Boolean lattice, B_z, which, as far as meets are concerned, is a subsemilattice of $[z)$, hence of S, with the restriction of $\to z$ to B_z as complementation and z as least element. Also, the restrictions to $[z)$ of $\to z$ and its threefold iteration coincide and their image is again B_z. Note that $a \to z \in [z)$ anyway. With these observations, we have at least

(3.4) $$a \to z \leq ((a \to z) \to z) \to z = \gamma_z(a) \to z$$

for each $a \in S$, equality being guaranteed in case $a \geq z$, and

(3.5) $$\gamma_z(a) = \gamma_z(\gamma_z(a))$$

without restriction whatsoever. Hence in all of S, γ_z is still an order-preserving idempotent operator, with image B_z. However, γ_z fails to be a closure operator in S since $a \leqslant \gamma_z(a)$ is only guaranteed in $[z)$. Note that B_z is contained in the image of $\rightarrow z$, but this inclusion may be proper. We summarize these statements in

Theorem 3.2 *Let S be a semilattice, $z \in S$ be a weakly right Brouwerian element. Then γ_z: $S \rightarrow S$ is an order-preserving idempotent operator onto a Boolean lattice B_z which is a subsemilattice of S, with the restriction of $\rightarrow z$ as complementation and z as least element.*

Note that the operator γ_z is not necessarily meet-preserving. All we know from (3.3) is

$$\gamma_z(a \wedge b) \leqslant \gamma_z(a) \wedge \gamma_z(b). \tag{3.6}$$

Equality is only guaranteed here in case a, $b \geqslant z$, not globally.

An element z of a meet-semilattice, even a partially ordered set, S, is called *meet-distributive* provided that the following holds: whenever $a \wedge b \leqslant z$, one has $a' \wedge b' = z$ for some $a' \geqslant a$, $b' \geqslant b$. (If $a \vee z$ and $b \vee z$ exist, one may take these joins as a' and b' respectively.) S is *meet-distributive* if each element $z \in S$ is. A meet-distributive lattice is distributive in the usual sense. (Balbes [1], Theorem 4.1, has shown that a meet-distributive meet-semilattice is join-distributive!)

Theorem 3.3 *Let S be a semilattice, z a weakly right Brouwerian element. Then the following are equivalent:*

(i) *z is (strongly) right Brouwerian;*
(ii) *z is meet-distributive (in S);*
(iii) *γ_z: $S \rightarrow S$ is a meet-preserving closure operator.*

Proof: (i) $\Longrightarrow$ (iii) is well-known. Beyond (3.3) and (3.5), one shows

$$a \leqslant \gamma_z(a) \tag{3.7}$$

for each $a \in S$. One also shows, beyond (3.6), *Glivenko's equation*

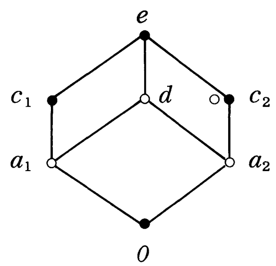
$$\gamma_z(a \wedge b) = \gamma_z(a) \wedge \gamma_z(b) \tag{3.8}$$

for each a, $b \in S$. (iii) $\Longrightarrow$ (ii): Suppose a, $b \leqslant z$. Since γ_z is a closure operator with least fixed point z, $\gamma_z(a \wedge b) = z$. (3.8) yields $\gamma_z(a) \wedge \gamma_z(b) = z$. By virtue of (3.7), $\gamma_z(a) \geqslant a$ and $\gamma_z(b) \geqslant b$. So z is meet-distributive. (ii) $\Longrightarrow$ (i): Suppose $a \wedge x \leqslant z$. So $b \wedge y = z$, for some $b \geqslant a$, $y \geqslant x$. So $b \wedge y = b \wedge z$ and $x \leqslant y \leqslant b \rightarrow z \leqslant a \rightarrow z$. Since $a \wedge (a \rightarrow z) \leqslant z$ anyway, (2.3) holds.

Corollary 3.4 *A meet-semilattice S is Brouwerian iff S is semi-Brouwerian and meet-distributive.*

Combining this with Proposition 2.3, we get Katriňák's result [18], 2.9; for lattices, *cf.* also Varlet [31], Théorème 3. Recall that in the lattice case, distributivity may be weakened to modularity.

Note: In order to test the meet-distributivity of an element z, one may restrict oneself to elements a, b such that $a \wedge b < z$. Also, if S is directed, one may assume $a, b \nleq z$, a, and b incomparable. Hence zero and the identity (whenever they exist) are meet-distributive. In the semi-Brouwerian (semi)lattice S of join-congruences of the Boolean lattice 2^2,



the meet-distributive (strongly right Brouwerian) elements are exactly those marked by $\bullet$. Here, γ_d: $S \rightarrow S$ is the closure operator associated with the subset $\{d, e\}$: (3.7) holds. So (3.8) cannot hold. Indeed, $\gamma_d(c_1 \wedge c_2) = d$, whereas $\gamma_d(c_1) \wedge \gamma_d(c_2) = e \wedge e = e$. On the other hand, γ_{a_1}: $S \rightarrow S$ is no closure operator at all: (3.7) does not hold. Indeed, $c_2 \nleq \gamma_{a_1}(c_2) = d$. However, γ_{a_1} is meet-preserving, i.e., satisfies (3.8). One may already find a counter-example of the second type in the non-modular 5-lattice. At any rate, (3.7) and (3.8) are logically independent.

Let us now look at the following "mixed" inequality ($\rightarrow$ being considered here really as a function of two variables):

(3.9) $$a \rightarrow (y \rightarrow z) \leqslant (a \wedge y) \rightarrow z.$$

Indeed, $(a \wedge y) \wedge (a \rightarrow (y \rightarrow z)) = a \wedge y \wedge z$, yielding (3.9). It is well-known that in the strong case equality takes place in (3.9). We even have

Proposition 3.5 *A semi-Brouwerian semilattice S is Brouwerian iff the equation*

(3.10) $$a \rightarrow (y \rightarrow z) = (a \wedge y) \rightarrow z$$

holds.

Proof: Suppose $y \leqslant z$. (3.10) yields

$$(a \rightarrow y) \rightarrow (a \rightarrow z) = (a \wedge (a \rightarrow y)) \rightarrow z) = (a \wedge y) \rightarrow z = e,$$

where $a \rightarrow y \leqslant a \rightarrow z$. By Proposition 2.1, S is Brouwerian.

As in the Brouwerian case (Rasiowa-Sikorski [25], Ch.I, 13.1), the filters of a semi-Brouwerian semilattice can still be characterized as those subsets F containing e and closed under *modus ponens*:

(3.11) *if a, $a \rightarrow z \in F$, then $z \in F$.*

In fact, let F be a filter and, $a, a \rightarrow z \in F$. Then $a \wedge z = a \wedge (a \rightarrow z) \in F$, where $z \in F$. Conversely, let $e \in F$ and (3.11) hold. Suppose $a \in F$ and $a \leqslant z$. So

$a \to z = e \in F$ and $z \in F$. Suppose a, $b \in F$. Since $b \wedge a = b \wedge (a \wedge b)$, $a \leqslant b \to (a \wedge b)$, where $b \to (a \wedge b) \in F$. By (3.11), we get $a \wedge b \in F$.

In any meet-semilattice with identity, we define the congruence $\bmod F$, where F is a filter, by

(3.12) $$y \equiv z \mod F \textit{ iff } a \wedge y = a \wedge z \textit{ for some } a \in F.$$

$\bmod F$ is the least congruence with *kernel* (= congruence class of the identity) F. The congruences of the form $\bmod F$ may be called *filter congruences*. It is well-known that in a Brouwerian semilattice, the filter congruences are exactly the congruences of the algebra $\langle S, \wedge, \to\rangle$. This turns out to be characteristic for Brouwerian semilattices in the larger variety of semi-Brouwerian semilattices. A meet-congruence of a semi-Brouwerian semilattice will be called a *left* (*right*) *congruence* once it is compatible with each of the left (right) "translations" $a \to$ (respectively $\to z$).

Theorem 3.6 *Each left (right) congruence $\equiv$ of a semi-Brouwerian semilattice $\langle S, \wedge, \to\rangle$ is a filter congruence. S is Brouwerian iff each filter congruence (it suffices: each congruence modulo a principal filter) is a left congruence (right congruence). As a matter of fact, each filter congruence is then a congruence of $\langle S, \wedge, \to\rangle$.*

Proof: Let $\equiv$ be a left congruence. $F = \{x \in S \mid x \equiv e\}$ is then a filter. If $y \equiv z$, then $a = y \leftrightarrow z \in F$, where

(3.13) $$y \leftrightarrow z = (y \to z) \wedge (z \to y).$$

Indeed, $y \to z \equiv y \to y = e \in F$, whence $y \to z \in F$. Likewise, $z \to y \in F$, whence $y \leftrightarrow z \in F$. We arrive at the same conclusion if $\equiv$ is a right congruence. But $(y \leftrightarrow z) \wedge y = y \wedge z = (y \leftrightarrow z) \wedge z$ anyway. Hence $y \equiv z \mod F$, making $\equiv$ the filter congruence $\bmod F$. Suppose now that each (meet-) congruence $\bmod [b)$ $(b \in S)$ is a right congruence. Since $a \wedge b \equiv a \mod [b)$, we get $(a \wedge b) \to z \equiv a \to z \mod [b)$, which is R3, making S Brouwerian. If each congruence $\bmod [b)$ is a left congruence, we conclude that $a \to (b \wedge z) \equiv a \to z \mod [b)$, i.e.,

$$b \wedge (a \to (b \wedge z)) = b \wedge (a \to z).$$

$R3_s$ holding in a semi-Brouwerian semilattice anyway, we get R3: S is again Brouwerian.

We may say that in the variety of all semi-Brouwerian semilattices, the subvariety of Brouwerian semilattices behaves somewhat like the subvariety of ableian groups in the variety of all groups.

Let us denote by $\langle M\rangle$ the filter generated by M.

Theorem 3.7 *Let S be a semi-Brouwerian semilattice. Then S is Brouwerian iff the "deduction theorem" holds:*

(3.14) $$\langle M \cup \{a\}\rangle = \{z \in S \mid a \to z \in \langle M\rangle\}$$

for each $M \subset S$, $a \in S$.

Proof: It is well-known that $z \in \langle M \cup \{a\}\rangle$ iff $a \wedge y \leq z$ for some $y \in M$. In the Brouwerian case, the latter is equivalent with $a \to z \in \langle M\rangle$. Conversely, suppose (3.14) holds. Let $a \wedge y \leq z$. So $z \in \langle [y) \cup \{a\}\rangle$ and $a \to z \in [y)$, i.e., $y \leq a \to z$. Since $a \wedge (a - z) \leq z$ anyway, (2.3) holds.

For the role of the deduction theorem in Brouwerian semilattices, *cf.* also Fajtlowicz-Schmidt [11], Proposition 2.1.

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