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PATHOLOGIES IN THE ED-REGRESSIVE SETS OF ORDER 2

SETH CATLIN

1 *Introduction* Ed-regressive sets of order n were introduced in [1]. Concerning ed-regressive sets of order 2, it is natural to ask which properties they share with the infinite regressive sets. In this paper, six of the well-known properties of (infinite) regressive sets and (infinite) regressive isols are shown not to hold for the two-dimensional case. They are:

(1) Every (infinite) retraceable set is the range of exactly one retraceable function.
(2) Every (infinite) separable subset of a regressive set is regressive.
(3) If A is a (infinite) regressive isol, then so is $\mathrm{A} - 1$.
(4) If α is retraceable and β is an infinite separable subset of α, then α and β are Turing equivalent.
(5) If α and β are infinite regressive sets, and $\alpha \subseteq \beta$, then $\alpha \leqslant_* \beta$.
(6) If T is a (infinite) regressive isol, and a_n is a recursive function, then $\sum_{\mathrm{T}} a_n$ is a regressive isol.

2 *Preliminaries* It is assumed that the reader is familiar with degrees of unsolvability and the main properties of regressive sets. The set of non-negative integers will be denoted by E. For $m \in E$, $\nu(m)$ will be $\{0, 1, \ldots, m - 1\}$. For any function b from E^n into E, ρb will denote the range of b. For functions f and g, $fg(x)$ will denote $f(g(x))$. Define $j(x, y) = (x + y)(x + y + 1)/2 + x$. It is well known that j is one-one, recursive, and maps $E \times E$ onto E. Therefore, the functions $k(x)$ and $l(x)$, defined by $j(k(x), l(x)) = x$ are well-defined and recursive. If we let $j_2 = j$, then, for $n \geqslant 2$, define j_{n+1} by

$$j_{n+1}(x_1, x_2, \ldots, x_{n+1}) = j(j_n(x_1, x_2, \ldots, x_n), x_{n+1}).$$

Then each j_n is recursive, one-one, and maps E^n onto E. Define, for $n \geqslant 2$, the functions $k_{n,1}(x), \ldots, k_{n,n}(x)$ by

$$j_n(k_{n,1}(x), k_{n,2}(x), \ldots, k_{n,n}(x)) = n.$$

$\mathrm{Req}\ \alpha$ will denote the recursive equivalence type of α. p^* is a function

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defined by $p^*(x) = (\mu n)(p^{n+1}(x) = p^n(x))$. If p is partial recursive, so is p^*. If α and β are sets, then $\alpha \leqslant_T \beta$ means α is Turing reducible to β, $\alpha \equiv_T \beta$ means $\alpha \leqslant_T \beta$ and $\beta \leqslant_T \alpha$, and $\alpha <_T \beta$ means $\alpha \leqslant_T \beta$ but not $\alpha \equiv_T \beta$. And $\alpha \leqslant_* \beta$ means that there is a partial recursive function $p(x)$ such that p is defined on α, $p(\alpha) = \beta$, and p is one-one on α.

A collection δ of ordered pairs is called an *initial set* if, given that $\langle x_1, y_1\rangle \leqslant \langle x_2, y_2\rangle$ and $\langle x_2, y_2\rangle \in \delta$, then $\langle x_1, y_1\rangle \in \delta$. A function a_{xy} is a *regressive function of order* 2 if a_{xy} is one-one, the domain of a_{xy} is an initial set, and there are partial recursive functions $p(x)$ and $q(x)$ such that $p(a_{xy}) = a_{x\dot{-}1,y}$ and $q(a_{xy}) = a_{x,y\dot{-}1}$ for all x and y for which a_{xy} is defined. Then the functions $p(x)$ and $q(x)$ are called *regressing functions for a*. A *regressive set of order* 2 is the range of a regressive function of order 2. A function is an *ed-regressive function of order* 2 if it is regressive of order 2 and has domain $E \times E$. A set is an *ed-regressive set of order* 2 if it is the range of an ed-regressive function of order 2. We will use the following notations:

$$\begin{aligned} \mathrm{reg}_2 &= \{\alpha\colon \alpha \text{ is a regressive set of order } 2\} \\ \mathrm{Edreg}_2 &= \{a\colon a \text{ is an ed-regressive function of order } 2\} \\ \mathrm{Edregsi}_2 &= \{a\colon a \in \mathrm{Edreg}_2 \text{ and } a \text{ is strictly increasing}\} \\ \mathrm{edreg}_2 &= \{\alpha\colon \alpha = \rho a \text{ for some } a \in \mathrm{Edreg}_2\} \\ \mathrm{edregsi}_2 &= \{\alpha\colon \alpha = \rho a \text{ for some } a \in \mathrm{Edregsi}_2\} \end{aligned}$$

3 *The theorems* In this section there will be six major theorems, each showing the failure of the two-dimensional analogue of the statement of the same number in the introduction.

Theorem 1 *If* $\beta \in \mathrm{edregsi}_2$, *then there are exactly* $\aleph_0$ *functions in* $\mathrm{Edregsi}_2$ *which have range* β.

Proof: Let $b \in \mathrm{Edregsi}_2$ with $\rho b = \beta$. Define functions x_n and y_n by

(1) $b_{x_1,y_1} < b_{x_2,y_2} < \ldots$

and $\beta = \{b_{x_1,y_1}, b_{x_2,y_2}, \ldots\}$. There is an infinite collection of pairs $\langle x_i, y_i\rangle$ such that neither $\langle x_i, y_i\rangle \leqslant \langle x_{i+1}, y_{i+1}\rangle$ nor $\langle x_{i+1}, y_{i+1}\rangle \leqslant \langle x_i, y_i\rangle$, since, otherwise, there would be a pair $\langle x_j, y_j\rangle$ with $x_j > 0$ and $\langle x_j, y_j\rangle \leqslant \langle x_{j+1}, y_{j+1}\rangle \leqslant \ldots$, which implies that the set $\{b_{0,n}\colon n \in E\}$ is finite, a contradiction. Say that these pairs are $\langle x_{i_1}, y_{i_1}\rangle, \langle x_{i_2}, y_{i_2}\rangle, \ldots$. Now, for each $j \geqslant 1$, define

$$b^{(j)}_{x,y} = \begin{cases} b_{x_{i_j+1}, y_{i_j+1}} & \text{if } \langle x, y\rangle = \langle x_{i_j}, y_{i_j}\rangle \\ b_{x_{i_j}, y_{i_j}} & \text{if } \langle x, y\rangle = \langle x_{i_j+1}, y_{i_j+1}\rangle \\ b_{x,y} & \text{otherwise} \end{cases}$$

It is clear that $\rho b^{(j)} = \beta$ $(j \geqslant 1)$. Moreover, since $m \neq n$ implies $\langle x_{i_m}, y_{i_m}\rangle \neq \langle x_{i_n}, y_{i_n}\rangle$, it is easily seen that the $b^{(j)}$ are distinct. Thus, if it can be shown that

(a) each $b^{(j)}$ is an increasing function,

and

(b) each $b^{(j)} \in \mathrm{Edreg}_2$, the proof that there are at least $\aleph_0$ such functions will be complete.

Re (a): Note that, by (1) and the definition of $b^{(j)}$,

$$
\begin{aligned}
(2)\quad & b^{(j)}_{x_1,y_1} < b^{(j)}_{x_2,y_2} < \ldots < b^{(j)}_{x_{i_j-1},y_{i_j-1}} < b^{(j)}_{x_{i_j+1},y_{i_j+1}} \\
& < b^{(j)}_{x_{i_j},y_{i_j}} < b^{(j)}_{x_{i_j+2},y_{i_j+2}} < \ldots .
\end{aligned}
$$

Let $\langle x_u, y_u\rangle < \langle x_v, y_v\rangle$. By (1) and the fact that b_{xy} is an increasing function, $u < v$. By (2), $b^{(j)}_{x_u,y_u} < b^{(j)}_{x_v,y_v}$ unless $u = i_j$ and $v = i_j + 1$. But $\langle x_{i_j}, y_{i_j}\rangle \not< \langle x_{i_j+1}, y_{i_j+1}\rangle$, so either $u \neq i_j$ or $v \neq i_j + 1$.

Re (b): It is clear from the definition of $b^{(j)}$ that it is everywhere defined. Let $p(x)$ and $q(x)$ be regressing functions for b_{xy}. Since b_{xy} and $b^{(j)}_{xy}$ are identical, except for a finite number of differences, it is clear that a finite number of modifications of each of $p(x)$ and $q(x)$ can be made to produce regressing functions for $b^{(j)}$.

We will now see that there are at most $\aleph_0$ functions in Edregsi$_2$ which have range β. Let a_{xy} and b_{xy} be distinct members of Edregsi$_2$ such that $\rho a = \rho b = \beta$, and let $\langle p, q\rangle$ be an ordered pair of regressing functions for a_{xy}. Since $a_{xy} \neq b_{xy}$, there are distinct ordered pairs $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$ such that $a_{x_1,y_1} = b_{x_2,y_2}$. If $\langle p, q\rangle$ is also a pair of regressing functions for b_{xy}, then

$$x_1 = p^*(a_{x_1,y_1}) = p^*(b_{x_2,y_2}) = x_2$$

and

$$y_1 = q^*(a_{x_1,y_1}) = q^*(b_{x_2,y_2}) = y_2,$$

a contradiction. Hence, distinct members of Edregsi$_2$, each with range β, have distinct ordered pairs of regressing functions. Since there are $\aleph_0$ ordered pairs of partial recursive functions, it follows that there can be at most $\aleph_0$ members of Edregsi$_2$ which have range β. Q.E.D.

In the above proof, if one deletes (a) and its proof, and replaces each occurrence of "Edregsi$_2$" with "Edreg$_2$", the proof becomes a proof of

If $\beta \in$ edreg$_2$, *then there are exactly* $\aleph_0$ *functions in* Edreg$_2$ *which have range* β,

a fact that is also true of infinite regressive sets.

Lemma 1 *Let* $f_1(x), f_2(x), \ldots, f_n(x)$ *be any* n *functions, and let* $\alpha_1, \alpha_2, \ldots, \alpha_m$ *be any* m *infinite sets. Then there exist elements* $x_1, \ldots, x_m$ *in* $\alpha_1, \ldots, \alpha_m$, *respectively, such that, for* $1 \leq i \leq m$, *we have*

(3) $x_i \notin \{f_j(x_k)\colon 1 \leq j \leq n,\ 1 \leq k \leq m,\ i \neq k,\ f_j(x_k)$ *defined*$\}$.

Proof: (by induction on m) If $m = 1$, then, for $1 \leq i \leq m$, the set on the right in (3) is empty, and any member x_1 of α_1 satisfies (3). Let $m = k + 1$ and assume that, given any k infinite sets $\beta_1, \ldots, \beta_k$, there are elements $x_1, \ldots, x_k$ in $\beta_1, \ldots, \beta_k$, respectively, such that, for $1 \leq i \leq k$,

(4) $x_i \notin \{f_j(x_l)\colon 1 \leq j \leq n,\ 1 \leq l \leq k,\ i \neq l,\ f_j(x_l)$ defined$\}$.

We will show next that there is an infinite sequence of mutually disjoint k-tuples, each satisfying (4). This will be done inductively as follows:

By the inductive hypothesis, there is a k-tuple $\langle x_1^1, \ldots, x_k^1\rangle$ satisfying (4). Now suppose that $\langle x_1^1, \ldots, x_k^1\rangle, \ldots, \langle x_1^j, \ldots, x_k^j\rangle$ is a collection of mutually disjoint k-tuples, each of which satisfies (4). Define the sets $\gamma_1, \ldots, \gamma_k$ by

$$\gamma_i = \alpha_i - \{x_s^t\colon 1 \leqslant s \leqslant k,\ 1 \leqslant t \leqslant j\},$$

for $1 \leqslant i \leqslant k$. Then $\gamma_1, \ldots, \gamma_k$ are all infinite, so by the inductive hypothesis, there is another k-tuple

$$\langle x_1^{j+1}, \ldots, x_k^{j+1}\rangle,$$

disjoint from the others, which satisfies (4) and where $x_i^{j+1} \in \gamma_i \subseteq \alpha_i$ $(1 \leqslant i \leqslant k)$. Thus, we have an infinite sequence $\langle x_1^1, \ldots, x_k^1\rangle, \langle x_1^2, \ldots, x_k^2\rangle, \ldots,$ of k-tuples which are mutually disjoint, where each satisfies (4), and where $x_i^j \in \alpha_i$ $(1 \leqslant i \leqslant k,\ j \geqslant 1)$.

Now consider the first $n + 1$ of these k-tuples, namely $\langle x_1^1, \ldots, x_k^1\rangle, \ldots, \langle x_1^{n+1}, \ldots, x_k^{n+1}\rangle$. Select x_{k+1} from α_{k+1} so that x_{k+1} is not a member of the set

$$\{f_j(x_t^i)\colon 1 \leqslant j \leqslant n,\ 1 \leqslant t \leqslant k,\ 1 \leqslant i \leqslant n + 1,\ f_j(x_t^i) \text{ defined}\}.$$

This is possible since α_{k+1} is infinite. The set δ, defined by $\delta = \{f_1(x_{k+1}), \ldots, f_n(x_{k+1})\}$ has at most n (defined) members, so at least one of the $n + 1$ mutually disjoint k-tuples, $\langle x_1^1, \ldots, x_k^1\rangle, \ldots, \langle x_1^{n+1}, \ldots, x_k^{n+1}\rangle$, will have no components in δ. Let $\langle x_1^q, \ldots, x_k^q\rangle$ be such a k-tuple. Then we have the following facts, for $1 \leqslant i \leqslant k$:

(a) $x_i^q \notin \{f_l(x_s^q)\colon 1 \leqslant l \leqslant n,\ 1 \leqslant s \leqslant k,\ i \neq s,\ f_l(x_s^q) \text{ defined}\}$,

since the k-tuple $\langle x_1^q, \ldots, x_k^q\rangle$ satisfies (4).

(b) $x_{k+1} \notin \{f_l(x_i^q)\colon 1 \leqslant l \leqslant n\}$,

since x_{k+1} was selected to have this property.

(c) $x_i^q \notin \{f_l(x_{k+1})\colon 1 \leqslant l \leqslant n\}$,

because of the manner in which q was selected.

Hence, if we let $x_1 = x_1^q, \ldots, x_k = x_k^q$, we have, combining (a), (b), and (c), that the numbers $x_1, x_2, \ldots, x_{k+1}$ satisfy (3), and this completes the proof. Q.E.D.

Definition: If $p_1(x), p_2(x), \ldots, p_n(x)$ are partial recursive functions, then the set $\{x_0, x_1, \ldots, x_m\}$ is said to be $(p_1, p_2, \ldots, p_n)$-*unrelated for* t *generations* $(t \geqslant 1)$, if, for $r, s \leqslant m$, one has $f_1 f_2 \ldots f_j(x_r) = x_s$ only if $r = s$, where each $f_i \in \{p_1, p_2, \ldots, p_n\}$ and $1 \leqslant j \leqslant t$.

Lemma 2 *Let a_{xy} be a regressive function of order 2 with regressing functions $p(x)$ and $q(x)$. If subset α of ρa is closed under the functions $p(x)$ and $q(x)$, and α has a subset β, consisting of n elements which are (p, q)-unrelated for $\binom{n+1}{2}$ generations, then α has at least $\binom{n+1}{2}$ elements.*

Proof: Let $\beta = \{a_{x_1,y_1}, \ldots, a_{x_n,y_n}\}$. We consider two cases:

Case (i): $x_i \leqslant x_j$ and $y_i \leqslant y_j$, for some $i \neq j$. Then, if we apply the regressing functions $p(x)$ and $q(x)$ to a_{x_j,y_j}, as many times and in whatever manner as is necessary to arrive at a_{x_i,y_i}, it follows that we will obtain along the way $\binom{n+1}{2}$ distinct elements of α, since α is closed under $p(x)$ and $q(x)$, and the numbers a_{x_i,y_i} and a_{x_j,y_j} are (p, q)-unrelated for $\binom{n+1}{2}$ generations.

Case (ii): $i \neq j$ implies that either both $x_i < x_j$ and $y_i > y_j$ or both $x_i > x_j$ and $y_i < y_j$. Define the sets β_i $(1 \leqslant i \leqslant n)$ by $\beta_i = \{a_{x_i,y_i}, a_{x_i,y_{i-1}}, \ldots, a_{x_i,0}\}$. The sets β_i are pairwise disjoint non-empty subsets of α with card $\beta_i = y_i + 1$, so card $\alpha \geqslant \sum_{i=1}^{n}$ card $\beta_i = \sum_{i=1}^{n} (y_i + 1) \geqslant \sum_{i=1}^{n} i = \binom{n+1}{2}$, the last inequality holding since the y_i's are distinct. Q.E.D.

Theorem 2 *There is a set in* edreg$_2$ *which has an infinite separable subset that is not in* edreg$_2$.

Proof: Let $\{\langle p_1, q_1\rangle, \langle p_2, q_2\rangle, \ldots\}$ be an enumeration of all ordered pairs of distinct partial recursive functions. Define the function c_{xy} as follows:

(i) $c_{00} = j_5(0, 0, 0, 0, 0)$, $c_{01} = j_5(0, 1, 0, 0, 0)$, $c_{10} = j_5(1, 0, 0, 0, 0)$.
(ii) Assume that, for $k \geqslant 1$, c_{xy} is defined for all x and y with $x + y \leqslant k$. Define c_{xy} $(x + y = k + 1)$ by

$$c_{xy} = j_5(x, y, d_{xy}, e_{xy}, b_{xy}),$$

where

$$d_{xy} = \begin{cases} c_{x-1,y} & \text{if } x > 0 \\ 0 & \text{if } x = 0, \end{cases}$$

$$e_{xy} = \begin{cases} c_{x,y-1} & \text{if } y > 0 \\ 0 & \text{if } y = 0, \end{cases}$$

and where the numbers b_{xy} (for $x + y = k + 1$) are chosen so that

(a) each of the numbers c_{xy} (for $x + y = k + 1$) is larger than all of the numbers $p_r(c_{uv})$ and $q_r(c_{uv})$, where $u + v \leqslant k$ and $r \leqslant k$,

and

(b) the numbers c_{xy} (for $x + y = k + 1$) are (p_k, q_k)-unrelated for $\binom{k+2}{n}$ generations.

That the above selection of the b_{xy} (for $x + y = k + 1$) is possible can be seen by appealing to Lemma 1, where the functions $f_1, \ldots, f_n$ are the functions $h_1 h_2 \ldots h_j$, such that $j \leqslant \binom{k+2}{2}$ and each h_i is either p_k or q_k, and the sets $\alpha_1, \ldots, \alpha_m$ are the sets ω_{xy} (for $x + y = k + 1$) defined by

$$\omega_{xy} = \{w: (\exists b)[w = j_5(x, y, d_{xy}, e_{xy}, b) \text{ and } w > \max\{x: (x = p_r(c_{uv}) \text{ or } x = q_r(c_{uv})) \text{ and } u + v \leq k \text{ and } r \leq k\}]\}.$$

The function $c_{xy} \in \mathrm{Edreg}_2$, since it is clearly everywhere defined, and the recursive functions $p(x)$ and $q(x)$, defined by

$$p(x) = \begin{cases} k_{5,3}(x) & \text{if } k_{5,1}(x) > 0 \\ x & \text{if } k_{5,1}(x) = 0, \end{cases}$$

and

$$q(x) = \begin{cases} k_{5,4}(x) & \text{if } k_{5,2}(x) > 0 \\ x & \text{if } k_{5,2}(x) = 0, \end{cases}$$

are readily seen to be regressing functions for c_{xy}.

Define $\alpha = \rho c - \{c_{00}\}$. Then α is clearly a separable subset of ρc. We will complete the proof by showing that $\alpha \notin \mathrm{edreg}_2$. Suppose that $\alpha \in \mathrm{edreg}_2$. Let $a_{xy} \in \mathrm{Edreg}_2$, such that $\alpha = \rho a$. Let p_n and q_n be regressing functions for a. Let $\alpha_n = \{c_{xy}: 0 < x + y \leq n\}$. Since each c_{xy} with $x + y > n$ is larger than any member of either $p_n(\alpha_n)$ or $q_n(\alpha_n)$, we see that the set α_n is closed under both p_n and q_n. Define $\alpha_n' = \{c_{xy}: x + y = n\}$. By the definition of c_{xy}, the $n + 1$ members of α_n' are (p_n, q_n)-unrelated for $\binom{n+2}{2}$ generations. By Lemma 2, α_n has at least $\binom{n+2}{2}$ elements. But $\mathrm{card}\, \alpha_n = \mathrm{card}\, \{c_{10}, c_{01}\} + \mathrm{card}\, \{c_{20}, c_{11}, c_{02}\} + \mathrm{card}\, \alpha_n' = 2 + 3 + \ldots + (n + 1) = \binom{n+2}{2} - 1$. This is a contradiction. Therefore, $\alpha \notin \mathrm{edreg}_2$. Q.E.D.

It is easy to show that, if $\alpha \in \mathrm{edreg}_2$ ($\alpha \in \mathrm{reg}_2$) and $\alpha \simeq \beta$, then $\beta \in \mathrm{edreg}_2$ ($\beta \in \mathrm{reg}_2$). Thus, we can define A to be an *ed-regressive* (*regressive*) *isol of order* 2 if it is a recursive equivalence type which is composed of immune ed-regressive (regressive) sets of order 2.

Theorem 3 *There is an ed-regressive isol* C *of order* 2, *such that* $\mathrm{C} - 1$ *is not an ed-regressive isol of order* 2.

Proof: Let $\{\gamma_1, \gamma_2, \ldots\}$ be an enumeration of the infinite r.e. sets. Then define c_{xy} as in the proof of Theorem 2, except that (a) is replaced by

(a′) each of the numbers c_{xy} (for $x + y = k + 1$) is larger than all of the numbers t_k, $p_r(c_{uv})$, $q_r(c_{uv})$, where $u + v \leq k$, $r \leq k$, and t_k is some member of γ_k which is larger than all of the numbers c_{mn} with $m + n \leq k$.

As is the proof of Theorem 2, $c_{xy} \in \mathrm{Edreg}_2$. Moreover, since none of the sets γ_k is a subset of ρc, ρc is immune. Let $\mathrm{C} = \mathrm{Req}\, \rho c$. Then C is an ed-regressive isol of order 2, but $\mathrm{C} - 1 = \mathrm{Req}\, (\rho c - \{c_{00}\})$ is not. Q.E.D.

In the proof of Theorem 2, supposing that $\alpha \in \mathrm{edreg}_2$ led to a contradiction. But all that was used was that $\alpha \in \mathrm{reg}_2$, so we have also proved

Theorem 2A *There is a set in* reg_2 *which has an infinite separable subset which is not in* reg_2.

Theorem 3A *There is a regressive isol* C *of order* 2, *such that* C - 1 *is not a regressive isol of order* 2.

Lemma 3 *If* $a_{xy} \in \mathrm{Edreg}_2$, $\alpha = \rho a$, *and* β *and* γ *are subsets of* α *such that*

(i) β *is a separable subset of* α,
(ii) *given* $a_{uv} \in \beta$, *there is an* $a_{rs} \in \gamma$ *such that* $u \leq r$ *and* $v \leq s$,
(iii) *there is an effective way of finding, for any* $x \in E$, *two numbers* u_x *and* v_x, *such that* $x \in \beta$ *iff* $x = a_{u_x, v_x}$,

then $\beta \leq_T \gamma$.

Proof: Let $x \in E$. Assume any question "$y \in \gamma$?" can be answered. Let u_x and v_x be the numbers such that $x \in \beta$ iff $x = a_{u_x, v_x}$. Let p and q be regressing functions for a. Start asking "$0 \in \gamma$?", "$1 \in \gamma$?", etc., until a number y has been found such that $y \in \gamma$, $p^*(y) \geq u_x$, and $q^*(y) \geq v_x$. Then it is clear that

$$x \in \beta \text{ iff } x \in \{p^m q^n(y) \colon m \leq p^*(y) \text{ and } n \leq q^*(y)\},$$

so $\beta \leq_T \gamma$. Q.E.D.

Theorem 4 *There exists a set* β *in* $\mathrm{edregsi}_2$ *which is the union of* $\aleph_0$ *pairwise disjoint, mutually separable subsets, each of which is in* $\mathrm{edresgi}_2$ *but is of lower Turing degree than* β.

Proof: Let $\alpha_0, \alpha_1, \ldots$ be a sequence of sets with the property that, for no $n \in E$ do we have $\alpha_{n+1} \leq_T \bigcup_{i=0}^{n} \alpha_i$. This is possible because there are uncountably many degrees, and each degree has at most countably many predecessors. Assume without loss of generality that, for each $n \in E$, $0 \notin \alpha_n$ and $1 \in \alpha_n$. For each $n \in E$, define δ_n by

$$\delta_n = \{j(n, x) \colon x \in \alpha_n - \{1\}\} \cup \{1\}.$$

Then it is clear that $\alpha_n \equiv_T \delta_n$ for each $n \in E$, and the sets $\delta_0 - \{1\}, \delta_1 - \{1\}, \ldots$ are mutually separable. For $i \in E$, define the function $d_n^{(i)}$ to be the strictly increasing total function that ranges over δ_i. Now define, for $i \in E$, the function $b_n^{(i)}$ by $b_0^{(i)} = d_0^{(i)} = 1$ and $b_{n+1}^{(i)} = j(b_n^{(i)}, d_{n+1}^{(i)})$. Then the function $f(x)$, defined by

$$f(x) = \begin{cases} k(x) & \text{if } x \neq 1 \\ x & \text{if } x = 1 \end{cases}$$

is a retracing function for each of the sets $\beta_n = \rho b^{(n)}$.
As in [6], Theorem T2, $\delta_n \equiv_T \beta_n$ for each $n \in E$, so the sets $\beta_0, \beta_1, \ldots$ have the property that

(5) for no $n \in E$ do we have $\beta_{n+1} \leq_T \bigcup_{i=0}^{n} \beta_i$.

Moreover, the sets $\beta_0 - \{1\}, \beta_1 - \{1\}, \ldots$ are mutually separable. Define the function c_{mn} inductively as follows:

$$\text{For } m = 0\colon\ c_{0,n} = j_5\,(0, n, b_n^{(0)}, b_n^{(0)}, b_n^{(0)})$$

$$\text{For } m > 0: \begin{cases} c_{m,0} = j_5(m, 0, c_{m-1,0}, c_{m-1,0}, b_0^{(m)}) \\ c_{m,n+1} = j_5(m, n+1, c_{m-1,n+1}, c_{m,n}, b_{n+1}^{(m)}). \end{cases}$$

Note that c_{mn} is everywhere defined and strictly increasing. Define $\gamma = \rho c$ and, for each $r \in E$, $\gamma_r = \{c_{mn} : m \leqslant r\}$. We prove the following statements:

(a) $c \in \mathrm{Edregsi}_2$ (and, thus, $\gamma \in \mathrm{edregsi}_2$),

(b) for $n \in E$, $\gamma_n \equiv_{\mathrm{T}} \bigcup_{i=0}^{n} \beta_i$,

(c) for $n \in E$, $\beta_n \leqslant_{\mathrm{T}} \bigcup_{i=0}^{n} \beta_i$,

(d) for $n \in E$, $\gamma_{n+1} \equiv_{\mathrm{T}} \gamma_{n+1} - \gamma_n$,

(e) for $n \in E$, $\gamma_n <_{\mathrm{T}} \gamma_{n+1} - \gamma_n$,

(f) $\gamma_0 <_{\mathrm{T}} \gamma_1 - \gamma_0 \equiv_{\mathrm{T}} \gamma_1 <_{\mathrm{T}} \gamma_2 - \gamma_1 \equiv_{\mathrm{T}} \gamma_2 <_{\mathrm{T}} \ldots.$

Re (a): Define the functions $p(x)$ and $q(x)$ by

$$p(x) = \begin{cases} k_{5,3}(x) & \text{if } k_{5,1}(x) \neq 0 \\ x & \text{if } k_{5,1}(x) = 0, \end{cases}$$

and

$$q(x) = \begin{cases} j_5(0, k_{5,2}(x) \dot{-} 1, fk_{5,3}(x), fk_{5,4}(x), fk_{5,5}(x)) \text{ if } k_{5,1}(x) = 0 \\ k_{5,4}(x) \text{ if } k_{5,1}(x) \neq 0 \text{ and } k_{5,2}(x) \neq 0 \\ x \text{ otherwise.} \end{cases}$$

Then $p(x)$ and $q(x)$ are partial recursive, and, as the reader can verify, are regressing functions for c_{mn}, so $c_{mn} \in \mathrm{Edregsi}_2$.

Re (b): Let $x, n \in E$, and let $r = k_{5,1}(x)$ and $s = k_{5,2}(x)$. If $r > n$, then $x \notin \gamma_n$. If $r \leqslant n$, assume one can answer any question "$y \in \bigcup_{i=0}^{n} \beta_i$?" Now ask "$0 \in \bigcup_{i=0}^{n} \beta_i$?", "$1 \in \bigcup_{i=0}^{n} \beta_i$?", etc., until all of the elements $b_v^{(u)}$ have been found such that $u \leqslant r$ and $v \leqslant s$. This is possible since, for $z = b_v^{(u)}$, $f^*(z) = v$, and the sets $\beta_0 - \{1\}, \beta_1 - \{1\}, \ldots, \beta_n - \{1\}$ are mutually separable. Now use the definition of the function c to construct $c_{r,s}$. Then $x \in \gamma_n$ iff $x = c_{r,s}$. Thus, $\gamma_n \leqslant_{\mathrm{T}} \bigcup_{i=0}^{n} \beta_i$. Now let $x, n \in E$, and assume we can answer any question "$y \in \gamma_n$?" If $f^{f^*(x)}(x) \neq 1$, then $x \notin \bigcup_{i=0}^{n} \beta_i$. If $x = 1$, then $x \in \bigcup_{i=0}^{n} \beta_i$. Otherwise, let $v = f^*(x)$. Let $u = kl(x)$. Then $x \in \bigcup_{i=0}^{n} \beta_i$ iff

$$v \neq 0,\ u \leqslant n, \text{ and } x = b_v^{(u)} = k_{5,5}(c_{uv}).$$

Since we can effectively find c_{uv} by asking "$0 \in \gamma_n$?", "$1 \in \gamma_n$?", etc., and using the functions p, q, $k_{5,1}$, and $k_{5,2}$, we have an effective test for deciding whether x is in $\bigcup_{i=0}^{n} \beta_i$. Thus, $\bigcup_{i=0}^{n} \beta_i \leqslant_{\mathrm{T}} \gamma_n$.

Re (c): This is immediate from

$$x \in \beta_n \text{ iff}$$

$$x \in \bigcup_{i=0}^{n} \beta_i \text{ and either } x = 1 \text{ or both } x \neq 1 \text{ and } kl(x) = n.$$

Re (d): This is an immediate consequence of Lemma 3.

Re (e): $\gamma_n \leq_{\mathrm{T}} \gamma_{n+1} - \gamma_n$ is a consequence of Lemma 3. Suppose $\gamma_n \equiv_{\mathrm{T}} \gamma_{n+1} - \gamma_n$. Then, by (b), (c), and (d),

$$\beta_{n+1} \leq_{\mathrm{T}} \bigcup_{i=0}^{n+1} \beta_i \leq_{\mathrm{T}} \gamma_{n+1} \leq_{\mathrm{T}} \gamma_{n+1} - \gamma_n \leq_{\mathrm{T}} \gamma_n \leq_{\mathrm{T}} \bigcup_{i=0}^{n} \beta_i,$$

which contradicts (5). Hence, $\gamma_n <_{\mathrm{T}} \gamma_{n+1} - \gamma_n$ for each $n \in E$.

Re (f): This is immediate from (d) and (e).

The sets $\gamma_0, \gamma_1 - \gamma_0, \gamma_2 - \gamma_1, \ldots$ form a denumerable collection of pairwise disjoint, mutually separable subsets of γ, and their union is γ. Each is retraceable (retracing function for each is $q(x)$) and, therefore, is in $\mathrm{edregsi}_2$ ([1], comment following Proposition 1). In view of (f) and the fact that each is separable in γ, they are all of lower degree than γ. Q.E.D.

Lemma 4 *If* $\alpha, \beta \in \mathrm{edregsi}_2$ *and* $\alpha \leq_* \beta$, *then* $\beta \leq_{\mathrm{T}} \alpha$.

Proof: Let $p(x)$ be a partial recursive function whose domain includes α, such that $p(x)$ is one-one on α. Assume we can answer any question "$y \in \alpha$?" Let $z \in E$. Since $\beta \in \mathrm{edregsi}_2$, there is a function b_{xy} in $\mathrm{Edregsi}_2$ such that $\beta = \rho b$. Define δ by

$$\delta = \{b_{xy}: x \leq z \text{ and } y \leq z\}.$$

Note that $z \in \beta$ iff $z \in \delta$. Answer the questions "$0 \in \alpha$?", "$1 \in \alpha$?", . . . until a number k is found such that $k \in \alpha$ and $p(k) = b_{uv}$ where $u \geq z$ and $v \geq z$. Then apply the regressing functions for b to b_{uv} in order to generate δ. Q.E.D.

Theorem 5 *There exist sets* $\alpha, \beta \in \mathrm{edreg}_2$ *such that* α *is an infinite separable subset of* β, *but* $\alpha \not\leq_* \beta$.

Proof: Let $\alpha = \gamma_0$ and $\beta = \gamma$ in the proof of Theorem 3. If $\alpha \leq_* \beta$, then $\beta \leq_{\mathrm{T}} \alpha$ by Lemma 4. Q.E.D.

If T is a regressive isol and t_n is a regressive function whose range is in T, then $\sum_{\mathrm{T}} a_n$ is defined [2], for any total function a_n, to be $\mathrm{Req} \bigcup_{n=0}^{\infty} j(t_n, \nu(a_n))$. A natural way to extend this definition to the order 2 case is:

If T is an ed-regressive isol of order 2, $t_{xy} \in \mathrm{Edreg}_2$, $\rho t \in \mathrm{T}$, and a_{xy} is a total function of two variables, define

$$\sum_{\mathrm{T}} a_{mn} = \mathrm{Req} \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{\infty} j(t_{mn}, \nu(a_{mn})).$$

This definition is easily seen to be independent of the choice of t_n.

Theorem 6 *It is not the case that, if b_{mn} is a recursive function, and* T *is an ed-regressive isol of order* 2, *then* $\sum_{\mathrm{T}} a_{mn}$ *is an ed-regressive isol of order* 2.

Proof: Let t_{xy} be the function c_{xy} in the proof of Theorem 2. Let $\mathrm{T} = \mathrm{Req}\,\rho t$. Define b_{mn} by

$$b_{mn} = \begin{cases} 1 \text{ if } m + n \neq 0 \\ 0 \text{ if } m + n = 0. \end{cases}$$

Then b_{mn} is recursive, and $\alpha \in \sum_{\mathrm{T}} b_{mn}$, where α is the set $\alpha = \{c_{xy}: x + y > 0\}$ in the proof of Theorem 2. Since $\alpha \notin \mathrm{edreg}_2$, we have that T is an ed-regressive isol of order 2, but $\sum_{\mathrm{T}} b_{mn}$ is not. Q.E.D.

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Eastern Oregon State College
LaGrande, Oregon