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ON G. SPENCER BROWN'S LAWS OF FORM

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This paper clarifies the mathematical nature of G. Spencer Brown's concept of primary algebra by establishing its equivalence with Boolean algebra. More precisely, we show that *primary algebra is exactly the theory of join and addition (= symmetric difference) of Boolean algebras.*

The *primary arithmetic* introduced in [3] may be regarded as an algebraic system consisting of two elements and two binary operations: the elements are ┐ and ; the first operation, called condensation, is indicated by juxtaposition, and the second, called cancellation, by a kind of exponentiation, the explicitly stated laws being

┐┐ = ┐ and $\overline{┐}|$ = .

Combining these with the obvious outcome of performing these operations with as part of the argument we obtain the following tables

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for condensation and cancellation, respectively. Evidently, the usual operations v and + in the two-element Boolean algebra have exactly analogous tables:

v	0	1
0	0	1
1	1	1

+	0	1
0	0	1
1	1	0

Thus, *primary arithmetic is, up to isomorphism, given by join and addition in the two-element Boolean algebra.*

Primary algebra, as defined in [3], can be viewed as the equational theory of type (0, 1, v, +) determined by the following identities: first, the two that correspond to the "initials" given in [3], p. 28,

J1 $((x + 1) \vee x) + 1 = 0$
J2 $((x \vee y) + 1) \vee ((z \vee y) + 1) + 1 = (((x + 1) \vee (z + 1)) + 1) \vee y$;

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then an identity which expresses the operation + in terms of $\vee$ and addition by 1,

A $x + y = (((x + 1) \vee y) + 1) \vee ((x \vee (y + 1)) + 1)$;

and finally the identities $0 \vee x = x = x \vee 0$, corresponding to the meaning of 0 as , and the above $\vee$ and + tables for 0 and 1. The only point about this assertion which requires some clarification concerns the fact that the operation in [3] which originates from cancellation in primary arithmetic is restricted to occurrences of the form $(\ldots) + 1$ and not considered for arbitrary arguments. However, the map

$$(x, y) \rightsquigarrow (((x + 1) \vee y) + 1) \vee ((x \vee (y + 1)) + 1)$$

evidently extends + from its limited use in [3] to a proper binary operation; our identity A is to ensure that this is in fact what our + is.

In Appendix I of [3] it is shown that, for any primary algebra, the derived operation $(x, y) \rightsquigarrow (x \vee y) + 1$ satisfies Sheffer's axioms for his operation "stroke" [2]. The well-known relations between the latter and the usual operations in Boolean algebras (Birkhoff [1], p. 45) then lead to the conclusion that, in any primary algebra, the operations

$$(*) \qquad 0,\ 1,\ x \vee y,\ x \wedge y = ((x + 1) \vee (y + 1)) + 1,\ x' = x + 1$$

satisfy the Boolean laws for zero, unit, join, meet, and complement respectively. The calculations required to show this are as follows:

Join: $$(((x \vee y) + 1) \vee ((x \vee y) + 1)) + 1 = (((x + 1) \vee (x + 1)) + 1) \vee y$$
$$= ((x + 1) + 1) \vee y = x \vee y,$$

by J1, and C5 and C1 of [3] respectively which state that, for any primal algebra, $x \vee x = x$ and $(x + 1) + 1 = x$, for all x.

Meet: $(((x \vee x) + 1) \vee ((y \vee y) + 1)) + 1 = ((x + 1) \vee (y + 1)) + 1$,

by C5.

Complement: $(x \vee x) + 1 = x + 1$,

again by C5. Finally, the role of 0 as zero is clear, and that 1 is the unit in C3 of [3].

Conversely, the laws of primary algebra listed above are satisfied in any Boolean algebra if 0, 1, and $\vee$ are taken as given and + as sum: J1 holds since $x + 1$ is the complement of x, J2 is a disguised form of the distributivity law

$$(x \vee y) \wedge (z \vee y) = (x \wedge z) \vee y,$$

A is just the definition of sum, and the rest is obvious. Furthermore, the passage from a Boolean to a primary algebra and the reverse, the latter given by (*) and the former as just described, are obviously inverse to each other. This establishes the assertion we set out to prove.

In conclusion, we note that primary algebra could also be interpreted as having type $(0, 1, \vee, ')$, $'$ a unary operation, the identities then taking the form

$$(x' \vee x)' = 0$$
$$((x \vee y)' \vee (z \vee y)')' = (x' \vee z')' \vee y$$
$$0 \vee x = x = x \vee 0$$

together with the tables for join and complement for the elements 0 and 1 of the two-element Boolean algebra. It immediately follows from the above discussion that this describes primary algebra as the theory of Boolean join and complementation. Perhaps one could argue that this is a more natural interpretation of the formalism presented in [3]; however from this point of view primary arithmetic and primary algebra have different type, and the former no longer generates, in the sense of generation of equational classes, the latter.

REFERENCES

[1] Birkhoff, G., *Lattice Theory*, third (new) edition, American Mathematical Colloquium Publications, Providence, Rhode Island, vol. XXV (1967).

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