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THE WEAK TOPOLOGY ON LOGICAL CALCULI

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1 *Introduction* The basis of this note is the thought that the discrete topology on $\mathcal{Q} = \{0, 1\}$, the topology generally used for mathematical statements about logical calculi, "throws away too much" of what is available by retaining an ordering on two elements. Perhaps it is still possible to say something about propositional calculus, and even predicate calculus, by regarding $\mathcal{Q}$ as an ordered set.

The first section below deals with a topology induced on the propositional calculus of a set of variables which arises by making each of the usual realizations in $\mathcal{Q}$, topologized, continuous. According to Theorem 1, this is the smallest topology for which consequence-closed sets are always closed. Theorem 2 pertains to the theory induced by a set of propositional formulas, the Lindenbaum algebra of the calculus, quotient topologies and an embedding of the Lindenbaum algebra in a product of $\mathcal{Q}$'s. In the second section below, dealing with first order predicate languages, a weak topology on the formulas of such a language is induced, with the object of obtaining the first order analogue of Theorem 1. In effect, for first order languages, the topology naturally associated with the "external" or semantic notion of consequence is characterized "internally," in terms of canonical realizations only. The propositional calculus (with its own weak topology) on the atomic formulas of the language is homeomorphically embedded in the larger space of all formulas, and the new topology is the smallest fulfilling a natural satisfiability condition expressed in terms of satisfiability in canonical realizations.

2 *Propositional Calculus* Let $\mathcal{Q}$ be endowed with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$. Let P be an infinite set of propositional variables and $\mathrm{Prop}(P)$ the propositional calculus on P. Let $\mathrm{Hom}\,(\mathrm{Prop}(P), \mathcal{Q})$ be the set of realizations $\rho\colon \mathrm{Prop}(P) \rightarrow \mathcal{Q}$. Now, let $\mathrm{Prop}(P)$ be given the weak topology, $\mathcal{W}$, induced by the realizations ρ. Observe that, for $A \in \mathrm{Prop}(P)$, $\mathrm{Cl}_{\mathcal{W}}(\{A\}) = \{B \in \mathrm{Prop}(P) \mid A \rightarrow B$ is a theorem$\}$. (If one had chosen $\{0\}$, rather than $\{1\}$, to be closed in $\mathcal{Q}$, $\mathrm{Cl}_{\mathcal{W}}(\{A\})$ would have been $\{B \mid B \rightarrow A$ is a theorem$\}$.) As usual, define $\mathrm{Con}(S) = \{A \in \mathrm{Prop}(P) \mid A$ is a consequence of $S\}$, $S \subset \mathrm{Prop}(P)$. The discrete

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topology has the property that $\mathrm{Con}(S)$ is closed, each $S \subset \mathrm{Prop}(P)$, while the indiscrete topology does not. The weak topology is the "breaking point."

We shall call a topology, τ, on $\mathrm{Prop}(P)$ *satisfactory* (after "satisfiable") if for each $S \subset \mathrm{Prop}(P)$ such that $S \subset \rho^{-1}(\{1\})$, some realization ρ, then $\perp \notin \mathrm{Con}(\mathrm{Cl}_\tau(S))$. The weak topology on $\mathrm{Prop}(P)$ is satisfactory: Suppose $S \subset \rho^{-1}(\{1\})$, some ρ, and suppose $\perp \in \mathrm{Con}(\mathrm{Cl}_{\mathcal{W}}(S))$. Then $\mathrm{Cl}_{\mathcal{W}}(S) \subset \mathrm{Cl}_{\mathcal{W}}(\rho^{-1}(\{1\})) = \rho^{-1}(\{1\})$, $\mathrm{Con}(\mathrm{Cl}_{\mathcal{W}}(S)) \subset \mathrm{Con}(\rho^{-1}(\{1\})) = \rho^{-1}(\{1\})$, and $\perp \in \rho^{-1}(\{1\})$, an impossibility.

Proposition 1 *If the topology, τ, on* $\mathrm{Prop}(P)$ *is such that* $\mathrm{Cl}_\tau(\mathrm{Con}(S)) = \mathrm{Con}(S)$, *each $S \subset \mathrm{Prop}(P)$, then τ is satisfactory.*

Proof: Suppose τ satisfies the hypothesis but is not satisfactory. Then there are ρ and S such that $S \subset \rho^{-1}(\{1\})$ and $\perp \in \mathrm{Con}(\mathrm{Cl}_\tau(S))$. Now, $\mathrm{Con}(S) = \mathrm{Cl}_\tau(\mathrm{Con}(S)) \supset \mathrm{Cl}_\tau(S)$ and hence $\mathrm{Con}(S) \supset \mathrm{Con}(\mathrm{Cl}_\tau(S))$ so that $\perp \in \mathrm{Con}(\mathrm{Cl}_\tau(S)) \subset \mathrm{Con}(S) \subset \mathrm{Con}(\rho^{-1}(\{1\})) = \rho^{-1}(\{1\})$, an impossibility.

Proposition 2 *$\mathcal{W}$ is the smallest satisfactory topology on* $\mathrm{Prop}(P)$.

Proof: Let τ be a satisfactory topology on $\mathrm{Prop}(P)$. We claim $\mathrm{Cl}_\tau(\rho^{-1}(\{1\})) = \rho^{-1}(\{1\})$, each ρ: Suppose $A \in [\mathrm{Cl}_\tau(\rho^{-1}(\{1\})) - \rho^{-1}(\{1\})]$. Then $\neg A \in \rho^{-1}(\{1\})$ and $\perp \in \mathrm{Con}(\mathrm{Cl}_\tau(\rho^{-1}(\{1\})))$. Thus, for each ρ, $(\rho \cdot \mathrm{id})^{-1}(\{1\}) = \rho^{-1}(\{1\}) = \mathrm{Cl}_\tau(\rho^{-1}(\{1\}))$, the identity map id:

$$(\mathrm{Prop}(P), \tau) \rightarrow (\mathrm{Prop}(P), \mathcal{W}) \text{ is continuous and } \tau \supset \mathcal{W}.$$

Theorem 1 *$\mathcal{W}$ is the smallest topology, τ, for which* $\mathrm{Cl}_\tau(\mathrm{Con}(S)) = \mathrm{Con}(S)$, *each $S \subset \mathrm{Prop}(P)$.*

Disjunction, as a function from $(\mathrm{Prop}(P), \mathcal{W}) \times (\mathrm{Prop}(P), \mathcal{W})$ into $(\mathrm{Prop}(P), \mathcal{W})$, is continuous, while negation, as a single-variable function is not. Letting $A \sim B$ when $\mathrm{Cl}_{\mathcal{W}}(\{A\}) = \mathrm{Cl}_{\mathcal{W}}(\{B\})$, $\mathfrak{A} := \mathrm{Prop}(P)/\sim$ is the set of elements of the Lindenbaum algebra of $\mathrm{Prop}(P)$, and $\mathfrak{A}$, with the quotient topology induced by the natural map $q: A \rightarrow |A|$, is T_0. Since open sets in $(\mathrm{Prop}(P), \mathcal{W})$ are saturated with respect to $\sim$, q is both open and closed and $\mathfrak{A}$ is an upper semi-continuous decomposition.

For $|A| \in \mathfrak{A}$, define, slightly abusing notation, $\rho(|A|) := \rho(A)$. Each (new) ρ is compatible with the usual operations defined on $\mathfrak{A}$ ($\rho(\neg|A|) = \rho(|\neg A|) = \rho(\neg A)$, etc.) and continuous from $\mathfrak{A}$ to $\mathcal{Q}$. Together they comprise $\mathrm{Hom}(\mathfrak{A}, \mathcal{Q})$. $\mathrm{Hom}(\mathfrak{A}, \mathcal{Q})$ separates points of $\mathfrak{A}$ and the evaluation map $e: \mathfrak{A} \rightarrow \mathcal{Q}^{\mathrm{Hom}(\mathfrak{A}, \mathcal{Q})}$ is an embedding. Since P is infinite, $e(\mathfrak{A})$ is a proper, dense, (trivially) compact and connected (since $|\top|$ is in each closed set) subset of $\mathcal{Q}^{\mathrm{Hom}(\mathfrak{A}, \mathcal{Q})}$.

Naturally, all that has been said about $\mathfrak{A}$ above applies to the theory of Boolean algebras generally.

One might wonder if the quotient weak topology on $\mathfrak{A}$ can be more simply described as that topology, $\mathcal{T}$, generated by principal upper ends in $\mathfrak{A}$, given the ordering $|A| \leq |B|$ if $(A \rightarrow B)$ is a theorem of $\mathrm{Prop}(P)$. The closed sets then would be intersections of sets of the form $(|A_1| \wedge \ldots \wedge |A_n|)$.

However, $\mathrm{Cl}_{\sigma}(\mathrm{Con}(S)) \not\subset \mathrm{Con}(S)$ (defined in the natural way for $S \subset \mathfrak{A}$) for all S, as one sees by taking $S = \{|p| \mid p \in P\}$.

One might likewise wonder whether topologizing $\mathfrak{A}$ by decreeing non-empty sets to be closed if they are both upper ends and contain infima (where they exist) of their subsets, yields the weak topology on $\mathfrak{A}$. As before, however, letting $S = \{|p| \mid p \in P\}$ gives $|\bot| = \mathrm{inf}(S)$ but $|\bot| \notin \mathrm{Con}(S)$.

Since the product topology for $\mathcal{Q}^{\mathrm{Hom}(\mathfrak{A}, \mathcal{Q})}$ is the Scott (induced) topology[1] (*cf.* [2]), $e(\mathfrak{A})$ carries the relativized Scott topology. One may think of the "theory of S" as $\mathrm{Con}(S)$, in $\mathfrak{A}$. Alternatively, let for $|A|$, $|B| \in \mathfrak{A}$, $|A| \,\widetilde{S}\, |B|$ if $\mathrm{Con}(\{|A|\} \cup S) = \mathrm{Con}(\{|B|\} \cup S)$. Denoting the equivalence classes of $\widetilde{S}$ by $|A|_S$, the theory of S may be identified with $\mathfrak{A}/\widetilde{S}$ on the basis of:

Proposition 3 *Given* $S = \mathrm{Con}(S)$ *and* $T = \mathrm{Con}(T)$, $S = T$ *if and only if* $|A|_S = |A|_T$, *each* $|A| \in \mathfrak{A}$.

Now, let $\mathfrak{A}/\widetilde{S}$ be given the quotient topology and note that for $S = \mathrm{Con}(S)$, if $|\bot| \in S$, $||\mathfrak{A}/\widetilde{S}|| = 1$. So, for $|\bot| \notin S = \mathrm{Con}(S)$, let $\mathcal{R}(S)$ ("$\mathcal{R}$" for "realizations") be the filter on $\mathcal{P}(\mathrm{Hom}(\mathfrak{A}, \mathcal{Q}))$ generated by $\{\rho \in \mathrm{Hom}(\mathfrak{A}, \mathcal{Q}) \mid \rho$ is a model for $S\}$. Define $\widetilde{\mathcal{R}(S)}$ on $e(\mathfrak{A})$ by $e(|A|) \,\widetilde{\mathcal{R}(S)}\, e(|B|)$ if $\{\rho \in \mathrm{Hom}(\mathfrak{A}, \mathcal{Q}) \mid \rho(A) = \rho(B)\} \in \mathcal{R}(S)$. Observe that $e(|A|) \,\widetilde{\mathcal{R}(S)}\, e(|B|)$ if and only if the set of realizations with respect to which $|A|$ and $|B|$ agree includes the models of S. If $|\bot| \in S$, there are no realizations of S; one defines all elements of $e(\mathfrak{A})$ to be $\widetilde{\mathcal{R}(S)}$-equivalent, and $||e(\mathfrak{A})/\widetilde{\mathcal{R}(S)}|| = 1$. In any event:

Theorem 2 *If* $S = \mathrm{Con}(S)$, $\mathfrak{A}/\widetilde{S}$ *is homeomorphic to* $e(\mathfrak{A})/\widetilde{\mathcal{R}(S)}$, *also with the quotient topology.*

Proof: Define $h\colon \mathfrak{A}/\widetilde{S} \to e(\mathfrak{A})/\widetilde{\mathcal{R}(S)}$ by $h(|A|_S) = e(|A|)/\mathcal{R}(S)$, the $\widetilde{\mathcal{R}(S)}$-equivalence class of $e(|A|)$. The function h is well defined: Suppose $|A|_S = |B|_S$ and $e(|A|)/\mathcal{R}(S) \cap e(|B|)/\mathcal{R}(S)$. Then there is a ρ satisfying S but not, say, $|B|$. Hence, $|B| \notin \mathrm{Con}(\{|A|\} \cup S)$, a contradiction.

The function h is clearly onto and is also 1-1: Suppose $e(|A|)/\mathcal{R}(S) = e(|B|)/\mathcal{R}(S)$ but $|\mathfrak{A}|_S \neq |B|_S$. Then one may assume, without loss of generality, that there is a $|C| \in [\mathrm{Con}(\{|A|\} \cup S) - \mathrm{Con}(\{|B|\} \cup S)]$ and thus that $\rho(|C|) = 1$ for every model, ρ, of $\{|A|\} \cup S$. However, there is a model, ρ_C, of $\{|B|\} \cup S$ such that $\rho_C(|C|) = 0$. Hence, $\rho_C(|B|) = 1$, and, necessarily then, $\rho_C(|A|) = 0$. Therefore, $\{\rho \in \mathrm{Hom}(\mathfrak{A}, \mathcal{Q}) \mid \rho(|A|) = \rho(|B|)\}$ does not contain the set of models of S, and $e(|A|)/\mathcal{R}(S) \neq e(|B|)/\mathcal{R}(S)$.

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1. Naturally, since $\{0\}$ is open instead of $\{1\}$, the order in [2] is reversed and the closed sets in $\mathcal{Q}^{\mathrm{Hom}(\mathfrak{A}, \mathcal{Q})}$ are those sets C such that:

   (i) C is an upper end,

   and

   (ii) If $D \subset C$ is down-directed and $\mathrm{inf}(D)$ exists, then $\mathrm{inf}(D) \in C$.

Finally, h is continuous and open: Note that $Q \subset \mathfrak{A}/\widetilde{S}$ is open if and only if $\bigcup_{|A|_S \in Q} e(|A|_S)$ is open in $e(\mathfrak{A})$. Further, $\bigcup_{|A|_S \in Q} e(|A|_S) = \bigcup_{e(|A|)/\mathcal{R}(S) \in h(Q)} e(|A|)/\mathcal{R}(S)$. Then by the definition of the $\widetilde{\mathcal{R}(S)}$-quotient topology, this last is open if and only if $h(Q) = \bigcup_{e(|A|)/\mathcal{R}(S) \in h(Q)} \{e(|A|)/\mathcal{R}(S)\}$ is open in $e(\mathfrak{A})/\widetilde{\mathcal{R}(S)}$.

3 *First Order Predicate Calculus* One can impose on the formulas of a first order language a weak topology, $\mathcal{W}_c$, induced by the canonical realizations, which is the smallest for which consequence-closed sets are closed. The formulas of the propositional calculus on the atomic formulas of the language, when endowed with the topology $\mathcal{W}$, embed in the space of all formulas. Finally, $\mathcal{W}_c$ is the smallest topology on the formulas of the language fulfilling a natural satisfiability condition involving the consequence operation and canonical realizations.

Let $\mathsf{T}_{\mathcal{L}}$ be the set of terms of the first order predicate language, $\mathcal{L}$, $\mathsf{F}_{\mathcal{L}}$ its set of formulas, $\mathsf{At}_{\mathcal{L}}$ its set of atomic formulas, and $\mathsf{V}_{\mathcal{L}}$ its infinite set of variables.[2]

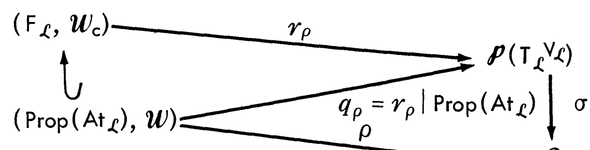
For $\rho \in \mathrm{Hom}(\mathrm{Prop}(\mathsf{At}_{\mathcal{L}}), \mathcal{Q})$, let r_ρ: $\mathsf{F}_{\mathcal{L}} \to \mathcal{P}(\mathsf{T}_{\mathcal{L}}{}^{\mathsf{V}_{\mathcal{L}}})$ be defined inductively (*cf.* [1], p. 16) beginning with, for an n-ary relational symbol P, $\overline{P}^{(r_\rho)} = \{(t_1, \ldots, t_n) \in \mathsf{T}_{\mathcal{L}}^n | \rho(Pt_1 \ldots t_n) = 1\}$. Then $\overline{Pt_1 \ldots t_n}^{(r_\rho)} = \{\partial \in \mathsf{T}_{\mathcal{L}}{}^{\mathsf{V}_{\mathcal{L}}} | \rho(P\partial t_1 \ldots \partial t_n) = 1\}$ where ∂t_i is the term derived from the term t_i when the variables take the values given by ∂. Observe that for $A \in \mathrm{Prop}(\mathsf{At}_{\mathcal{L}})$, $\rho(A) = 1$ if and only if $\mathrm{id} \in \overline{A}^{(r_\rho)}$, where $\mathrm{id} \in \mathsf{T}_{\mathcal{L}}{}^{\mathsf{V}_{\mathcal{L}}}$ is the identity map on $\mathsf{V}_{\mathcal{L}}$.

Further, define σ: $\mathcal{P}(\mathsf{T}_{\mathcal{L}}{}^{\mathsf{V}_{\mathcal{L}}}) \to \mathcal{Q}$ by $\sigma(S) = 1$ if $\mathrm{id} \in S$ and $\sigma(S) = 0$ otherwise. For $\rho \in \mathrm{Hom}(\mathrm{Prop}(\mathsf{At}_{\mathcal{L}}), \mathcal{Q})$ $(\sigma r_\rho)^{-1}(\{1\})$ is a prime filter in $(r_\rho(\mathsf{F}_{\mathcal{L}}) \cap \mathcal{P}(\mathsf{T}_{\mathcal{L}}{}^{\mathsf{V}_{\mathcal{L}}}), \subseteq)$. Note also that $\sigma q_\rho = \rho$, where $q_\rho = r_\rho | \mathrm{Prop}(\mathsf{At}_{\mathcal{L}})$.

We pause here to remark on the 1-1 correspondence[3] between canonical realizations, r, on $\mathsf{F}_{\mathcal{L}}$ and propositional realizations, ρ, on $\mathrm{Prop}(\mathsf{At}_{\mathcal{L}})$. Obviously, each r determines a unique ρ_r: $\mathrm{Prop}(\mathsf{At}_{\mathcal{L}}) \to \mathcal{Q}$ defined, starting with n-ary predicate letters P, by $\rho_r(Pt_1 \ldots t_n) = 1$ if and only if $\mathrm{id} \in \overline{Pt_1 \ldots t_n}^{(r)}$. Since $\rho_{r_\rho} = \rho$ and $r_{\rho_r} = r$, we can label each canonical realization as some r_ρ (with $\rho = \rho_r$).

Next, we topologize $\mathsf{F}_{\mathcal{L}}$ with $\mathcal{W}_c$, the weak topology induced by the maps σr_ρ: $\mathsf{F}_{\mathcal{L}} \to \mathcal{Q}$, $\rho \in \mathrm{Hom}(\mathrm{Prop}(\mathsf{At}_{\mathcal{L}}), \mathcal{Q})$. (The subscript "c" stands for "canonical.")

As a summarizing diagram:



2. Terminology in this section is that of [1].

3. *Cf.* [1], p. 23.

Proposition 4 $(\mathrm{Prop}(\mathrm{At}_{\mathcal{L}}), \mathcal{W})$ *is embedded in* $(\mathrm{F}_{\mathcal{L}}, \mathcal{W}_c)$.

Proof: Subbasic open sets in $\mathcal{W}$ are of the form $\rho^{-1}(\{0\}) = (\sigma q_\rho)^{-1}(\{0\}) = (\sigma r_\rho)^{-1}(\{0\}) \cap \mathrm{Prop}(\mathrm{At}_{\mathcal{L}})$ and $(\sigma r_\rho)^{-1}(\{0\})$ is a subbasic open set in $\mathcal{W}_c$.

Is $\mathcal{W}_c$ the smallest topology on $\mathrm{F}_{\mathcal{L}}$ in which $(\mathrm{Prop}(\mathrm{At}_{\mathcal{L}}), \mathcal{W})$ embeds?
Is $\mathrm{Prop}(\mathrm{At}_{\mathcal{L}})$ first category in $(\mathrm{F}_{\mathcal{L}}, \mathcal{W}_c)$?

As usual, we define $\mathrm{Con}(S)$, for $S \subset \mathrm{F}_{\mathcal{L}}$, to be $\left\{A \in \mathrm{F}_{\mathcal{L}} \mid \bigcap_{\bar{B} \in \bar{S}} \overline{B}^{(r)} \subset \overline{A}^{(r)} \text{ each realization, } r\right\}$.

Proposition 5 *For each* $\rho \in \mathrm{Hom}(\mathrm{Prop}(\mathrm{At}_{\mathcal{L}}), \mathcal{Q})$, $(\sigma r_\rho)^{-1}(\{1\}) = \mathrm{Con}((\sigma r_\rho)^{-1}(\{1\}))$.

Paralleling section 2, we shall call a topology, τ, on $\mathrm{F}_{\mathcal{L}}$ *satisfactory* if for each $S \subset \mathrm{F}_{\mathcal{L}}$ such that $S \subset (\sigma r_\rho)^{-1}(\{1\})$, for some canonical realization r_ρ, (i.e., such that the identity mapping on the variables is in the r_ρ-realized value of each formula of S), $\perp \notin \mathrm{Con}(\mathrm{Cl}_\tau(S))$. The weak topology is satisfactory, and substituting "σr_ρ" for "ρ" in the proofs of Propositions 1 and 2, we get:

Proposition 6 *If the topology,* τ, *on* $\mathrm{F}_{\mathcal{L}}$ *is such that* $\mathrm{Cl}_\tau(\mathrm{Con}(S)) = \mathrm{Con}(S)$, *each* $S \subset \mathrm{F}_{\mathcal{L}}$, *then* τ *is satisfactory.*

Proposition 7 $\mathcal{W}_c$ *is the smallest satisfactory topology on* $\mathrm{F}_{\mathcal{L}}$.

Theorem 3 $\mathcal{W}_c$ *is the smallest topology,* τ, *on* $\mathrm{F}_{\mathcal{L}}$ *such that* $\mathrm{Cl}_\tau(\mathrm{Con}(S)) = \mathrm{Con}(S)$, *each* $S \subset \mathrm{F}_{\mathcal{L}}$.

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