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A TERNARY UNIVERSAL DECISION ELEMENT

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1 *Introduction* A universal decision element is one which may be used to define any of the functors of one or two arguments by the substitution of variables x, y, etc., or constants in its arguments, the universal element only being used once in any definition. In the 2-valued case, Sobociński [8] has shown that there exists such a functor with four arguments and several authors have developed the 2-valued situation in detail. (See [1], [2], [4], [5], [6].) In the 3-valued case, Rose [7] has given a particular universal element as have the authors in [3].

In the present paper we define a seven input device which will act as a ternary universal decision element. In addition to the variables and the constants 0, 1, 2 we assume that we have both the Post negations of one of the variables available as inputs. The heart of the element is a two-place functor which may be used to generate all the one-place functions using only the variable, its Post negations and the constants, and not requiring any negations of the output. This compares with Rose's approach [7] except that, for his one-place generator, he uses three given one-place functions which may act on both the inputs and output. It is a considerable improvement over the device in [3] which used ten inputs. The technique for generating the two-place functions is to generate each of the rows of the defining truth table separately and combine them using a modified disjunction. This means that it is straightforward to use the element, not requiring complex table look-up to decide the inputs to generate a particular functor.

0, 1, 2 will be used to denote the three values of the ternary system and they will also be used for the three logical constants of the system. Variables will be denoted by x, y, z, etc. We shall identify the one-place functions by their value sequences, that is $\langle abc\rangle$ is the value sequence for the one-place function Kx which is such that $Kx = a, b,$ or c according as $x = 0, 1,$ or 2 respectively. This will be written $Kx = \langle abc\rangle$. The two Post negations of x will be denoted by Px and Qx, that is $Px = \langle 120\rangle$ and $Qx = \langle 201\rangle$. Note that $PQx = QPx = x$, $PPx = Qx$ and $QQx = Px$.

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2 *The one-place generator* Consider the function Dxy specified by the truth table shown. We shall show that all 27 one-place functions may be generated using a single occurrence of D, so long as both Post negations of the variable and the constant functions are available.

Dxy	0	1	2 (y)
0	0	1	2
x 1	1	1	0
2	2	2	0

Initially we may exclude $\langle 000\rangle$, $\langle 111\rangle$, $\langle 222\rangle$ since these may be defined by $D00$, $D01$, and $D02$ respectively. The remaining 24 one-place functions are divided into eight classes:

$$\begin{aligned}
\mathbf{E}_1 &= \{\langle 012\rangle, \langle 120\rangle, \langle 201\rangle\},\\
\mathbf{E}_2 &= \{\langle 102\rangle, \langle 021\rangle, \langle 210\rangle\},\\
\mathbf{E}_3 &= \{\langle 001\rangle, \langle 010\rangle, \langle 100\rangle\},\\
\mathbf{E}_4 &= \{\langle 002\rangle, \langle 020\rangle, \langle 200\rangle\},\\
\mathbf{E}_5 &= \{\langle 110\rangle, \langle 101\rangle, \langle 011\rangle\},\\
\mathbf{E}_6 &= \{\langle 112\rangle, \langle 121\rangle, \langle 211\rangle\},\\
\mathbf{E}_7 &= \{\langle 220\rangle, \langle 202\rangle, \langle 022\rangle\},\\
\mathbf{E}_8 &= \{\langle 221\rangle, \langle 212\rangle, \langle 122\rangle\},
\end{aligned}$$

It will suffice to prove that one element from each $\mathbf{E}_i$ $(1 \leqslant i \leqslant 8)$ may be defined using Dxy. This follows since if we can define $Kx = \langle pqr\rangle$ $(p, q, r \in \{0, 1, 2\})$ by Da_1a_2, that is

$$Da_1a_2 = \langle pqr\rangle,\ a_1, a_2 \in \{0, 1, 2, x, Px, Qx\},$$

then

$$Da_1^!a_2^! = \langle qrp\rangle \text{ where, for } i = 1, 2,$$

$$a_i^! = \begin{cases} a_i \text{ if } a_i \in \{0, 1, 2\},\\ Pa_i \text{ if } a_i \in \{x, Px, Qx\}.\end{cases}$$

Similarly $\langle rpq\rangle$ may also be defined by a repetition of the above process. Hence if $\langle pqr\rangle$ can be defined then so can $\langle qrp\rangle$ and $\langle rpq\rangle$.

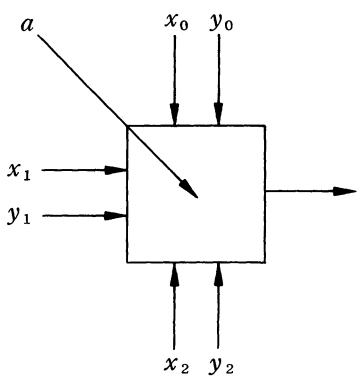
Now $Dx0$, $Dx1$, $Dx2$, $D1x$, $D2x$, Dxx, $DxPx$, $DxQx$ define elements from $\mathbf{E}_1$, $\mathbf{E}_6$, $\mathbf{E}_4$, $\mathbf{E}_5$, $\mathbf{E}_7$, $\mathbf{E}_3$, $\mathbf{E}_2$, and $\mathbf{E}_8$ respectively. This demonstrates that a single use of D is sufficient to generate all 27 one-place functions. A complete table of the assignments for D for the 24 nonconstant one-place functions is given below, see p. 634. The assignment listed gives the one-place functions in the second column. The replacement of Px for x in the assignments will give the rotation 1 column, while Qx for x will give the rotation 2 column.

The function D is typical of a particular class of two-place functions which are adequate to generate all the one-place functions. A complete characterization of this class is currently being investigated by one of the authors. It is clear, for example, that any function deducible from Dxy by a rotation of the columns or a rotation of the rows of the defining table for D will also be adequate. If a function were to be used solely to generate

Dxy Assignment x	y	One-Place Function	Rotation 1	Rotation 2
x	0	$\langle 012\rangle$	$\langle 120\rangle$	$\langle 201\rangle$
x	1	$\langle 112\rangle$	$\langle 121\rangle$	$\langle 211\rangle$
x	2	$\langle 200\rangle$	$\langle 002\rangle$	$\langle 020\rangle$
1	x	$\langle 110\rangle$	$\langle 101\rangle$	$\langle 011\rangle$
2	x	$\langle 220\rangle$	$\langle 202\rangle$	$\langle 022\rangle$
x	x	$\langle 010\rangle$	$\langle 100\rangle$	$\langle 001\rangle$
x	Px	$\langle 102\rangle$	$\langle 021\rangle$	$\langle 210\rangle$
x	Qx	$\langle 212\rangle$	$\langle 122\rangle$	$\langle 221\rangle$

one-place functions it would not be necessary for it to generate $\mathbf{E}_1$, but this is required for D to be used as part of the universal element below.

3 *The universal decision element* The universal element will be called M and denoted by $[a; x_0, y_0; x_1, y_1; x_2, y_2]$. We shall use the diagrammatic representation shown for M.



Initially M will be defined in terms of number of other functions, and a complete specification for M will follow.

Define

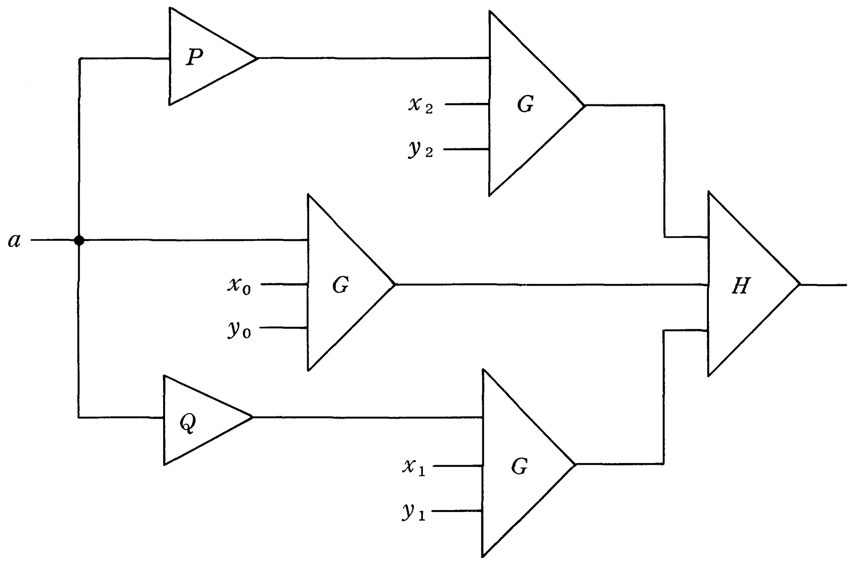
(i) $Gxyz = \begin{cases} Dxy, \text{ if } z = 0; \\ 0, \text{ otherwise.} \end{cases}$

(ii) $Hxyz = \max(x, y, z)$ iff two of x, y, z assume the value 0. Otherwise H is unspecified.

Then

$$[a; x_0, y_0; x_1, y_1; x_2, y_2] = HG\ x_0 y_0 a\, G x_1 y_1\, Qa\, G\, x_2 y_2\ Pa.$$

This is shown in the diagram below:



Now we may define a two-place function Fxy by

$$Fxy = [x;\, x_0,\, y_0;\, x_1,\, y_1;\, x_2,\, y_2]$$

where for each $k = 0, 1, 2,$ x_k and y_k are such that

$$Dx_k y_k = Fky, \quad x_k, y_k \in \{0, 1, 2, y, Py, Qy\}.$$

For example, consider Fxy defined by the truth table shown. Since

	Fxy	0	1	2
				y
	0	0	1	0
x	1	2	1	1
	2	2	0	1

$\langle 010\rangle = Dyy,$
$\langle 211\rangle = DQy1,$
$\langle 201\rangle = DQy0,$

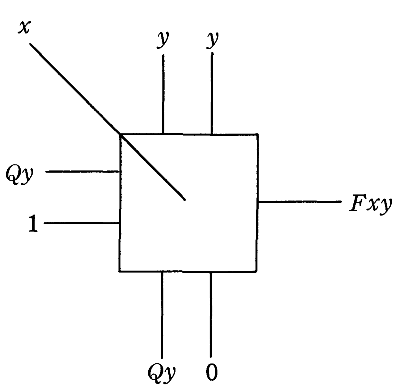
we may take

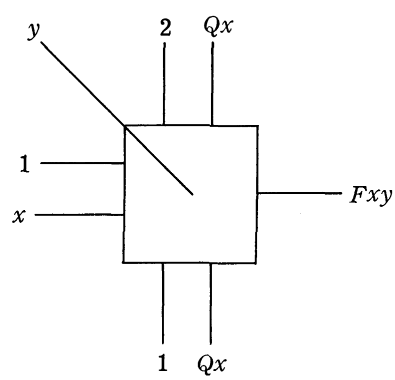
$$Fxy = [x;\, y,\, y;\, Qy,\, 1;\, Qy,\, 0]$$

or, if we partition F on columns instead of rows, we may use

$$Fxy = [y;\, 2,\, Qx;\, 1,\, x;\, 1,\, Qx].$$

The two alternatives are shown in the diagrams below:





Rather than showing M as a composition of other functions it is straightforward to define it directly by listing all the input configurations for which M takes the values 0, 1, or 2. These are listed respectively in the 0-array, the 1-array and the 2-array, the entries being in the generalized cube notation of [3]. Define $W = \{0, 1, 2\}$, $X = \{0, 1\}$, $Y = \{0, 2\}$, and $Z = \{1, 2\}$. Then for a four-variable function $Bpqrs$ the presence of a cube $OW1X$ in the 1-array indicates that $B0010 = B0011 = B0110 = B0111 = B0210 = B0211 = 1$. For M the specification is:

0-array

a	x_0	y_0	x_1	y_1	x_2	y_2
0	0	0	W	W	W	W
0	Z	2	W	W	W	W
1	W	W	0	0	W	W
1	W	W	Z	2	W	W
2	W	W	W	W	0	0
2	W	W	W	W	Z	2

1-array

a	x_0	y_0	x_1	y_1	x_2	y_2
0	X	1	W	W	W	W
0	1	X	W	W	W	W
1	W	W	X	1	W	W
1	W	W	1	X	W	W
2	W	W	W	W	X	1
2	W	W	W	W	1	X

2-array

a	x_0	y_0	x_1	y_1	x_2	y_2
0	0	2	W	W	W	W
0	2	X	W	W	W	W
1	W	W	0	2	W	W
1	W	W	2	X	W	W
2	W	W	W	W	0	2
2	W	W	W	W	2	X

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