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A THEORY OF RESTRICTED VARIABLES WITHOUT EXISTENCE ASSUMPTIONS

MARGARET MURPHY PRULLAGE

1 *Introduction* The individual variables of ordinary first-order logic are generally thought of as ranging over all the objects in a certain set, the universe or domain of discourse, with no discrimination made among the variables. In everyday mathematics, however, this is often not the case, and some variables, usually distinguished by the use of different letters, are restricted in their signification to some proper subset of the domain of discourse. For example, the letters x, y, and z may refer to real numbers in a formula such as $x^2 - y^2 = (x + y)(x - y)$, but there may also be formulas of the sort "For all integers m, ..." or "There is a positive prime p, such that . . .." Thus it is useful to formulate a logic which allows for the restriction of variables to certain ranges as well as for the general interpretation of variables.*

Bourbaki, in his treatment of logic in [1], allows for restricted quantification by defining quantifiers $\exists_A x$ and $\forall_A x$ in terms of the existential quantifier $\exists$, $(\exists_A x)R$ being defined as $(\exists x)(A \,\&\, R)$. Intuitively, if A and R are formulas expressing properties of x, then $(\exists x)(A \,\&\, R)$, meaning "There is an x, such that A and R hold," is equivalent to $(\exists_A x)R$, interpreted as "There is an x of kind A, such that R holds." In $(\exists_A x)R$, x is restricted to objects satisfying A by the symbol $\exists_A$. $(\forall_A x)R$ is defined to be $\neg(\exists_A x)\neg R$. The symbols $\exists_A$ and $\forall_A$ might be used in a demonstration if one is interested only in objects satisfying A, where A might express the property of being an integer, or a positive prime. In [13], Rosser discusses restricted variables in some detail; his approach appears to differ from Bourbaki's since he considers restricted variables rather than restricted quantifiers. He uses Greek letters to refer to restricted variables; α might

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designate a variable subject to the restriction K, and the formula $(\forall\alpha)F(\alpha)$ is viewed as an abbreviation for the formula $(\forall x)(K(x) \rightarrow F(x))$. The letter β might denote a variable subject to the restriction L, and $(\forall\beta)F(\beta)$ would mean $(\forall x)(L(x) \rightarrow F(x))$. While Rosser, unlike Bourbaki, ostensibly attaches the restriction to the variable, in effect he associates it with the quantifier by declining to give any special significance to α when it occurs free in a formula. Rosser points out that this is contrary to the practice in ordinary mathematics, but that he considers it undesirable to associate a restriction with α in free occurrences because of some ambiguity in its interpretation.

In two papers published in 1957, *cf.* [3], Hailperin formulated a logical theory $\mathcal{Q}\mathcal{I}_\nu$ which included restricted variables among the symbols of the language and provided for their use in all places normally occupied by the ordinary variables of predicate calculus, hereafter called proper variables. This was accomplished by including the symbol $\boldsymbol{\nu}$ among the primitive symbols as a term-builder, and allowing for the formation, from a proper variable x and a formula P, of the expression $\boldsymbol{\nu} xP$, called a restricted variable. It was to be thought of as referring to "some individual such that P." All free occurrences of x in P are bound by $\boldsymbol{\nu} x$. There are other examples of variable-binding operators used to construct terms in logic; the best known are the $\boldsymbol{\varepsilon}$-symbol of Hilbert and Russell's definite description operator $\boldsymbol{\iota}$. Both $\boldsymbol{\varepsilon} xP$ and $\boldsymbol{\iota} xP$ are meant to denote an individual satisfying P, but they differ from each other and from $\boldsymbol{\nu} xP$ in some respects. $\boldsymbol{\varepsilon} xP$ is a constant, or at least a constant-form in case P has free variables other than x; it designates a definite though unspecified member of the set of all individuals in the domain satisfying P. The $\boldsymbol{\varepsilon}$-symbol is appropriately interpreted by a choice function on the collection of all subsets of the domain of discourse. A logic which includes the definite description operator $\boldsymbol{\iota}$ must also include the equality predicate; in case there is a unique individual satisfying P, $\boldsymbol{\iota} xP$ denotes that individual, and equality is needed to express that uniqueness. In case there is more than one individual satisfying P, or no such individual, $\boldsymbol{\iota} xP$ is an improper description and must be handled in some fashion. Other term-building symbols, less well-known, are Hilbert's $\boldsymbol{\eta}$-symbol, very similar to $\boldsymbol{\varepsilon}$ but differing from it in that the formula $\exists xP$ is required as an assumption or derived formula before $\boldsymbol{\eta} xP$ can be introduced as a term, and Bourbaki's $\boldsymbol{\tau}$-symbol, which is like the $\boldsymbol{\varepsilon}$-symbol and is used by Bourbaki to define the quantifiers $\exists$ and $\forall$.

In all these cases the expression built from the initial symbol ($\boldsymbol{\varepsilon}$, $\boldsymbol{\iota}$, $\boldsymbol{\tau}$, or $\boldsymbol{\eta}$), operator variable and formula is a constant or a constant-form which becomes a constant when all its free variables are bound or replaced by constants. They are thus not available for use as quantifier variables. $\boldsymbol{\nu}$-expressions, however, are variables, and can be used in argument places, in quantifiers and also as operator variables; though Hailperin does not make use of them in the last-named context, they are so used in the system which follows.

The way in which restricted variables are formed leads in a natural way to a theory in which some variables may fail to denote actual objects;

clearly, this occurs when the formula P in $\boldsymbol{\nu} xP$ is a contradiction, a possibility which is not ruled out by the rules of formation. Thus, a logical theory comprising restricted variables is best developed as a free logic, that is, a logic in which some or all of the variables may fail to have values. Traditionally, logic has excluded the empty set from the possible domains of interpretation, but within the past twenty-five years the question of the empty domain has begun to receive some attention. One of the first papers to investigate those formulas of quantification theory valid in all domains was by Mostowski. His system lacked several desirable features and was modified by Hailperin, and later revised by Quine. The end result was a standard logic, all the axioms of which were closed formulas valid in all domains. The intended interpretation was that all universally generalized statements came out true in the empty domain, all existential ones false. Next to receive attention were theories in which some of the individual terms might not designate actual objects, although all individual variables denoted members of the universe. The first system which allowed for reasoning with names which might not denote was presented by Hailperin and Leblanc in [4]; this was followed by the work of Hintikka, [5], Lambert, [6], Schock, [14], van Fraassen, [16], and others. Some of these logics were also valid for the empty domain, but all of them, with the exception of that of Leblanc and Meyer, made use of a non-standard predicate, either equality or the existence predicate. In [8] Leblanc and Meyer proved that their system was sound and complete, and it is their system which has been adapted in the following to the needs of a restricted quantification theory. Hailperin in his original paper, [3], in effect excluded empty ranges for variables by defining a deduction so that it included a formula guaranteeing a non-empty range for each variable appearing in the deduction.

In the next section formal definitions of essential notions are given. Under the rules of formation which I adopt, an anomalous situation arises in connection with the free variables of P, other than x, when $\boldsymbol{\nu} xP$ is used in a formula. These variables are not bound by $\boldsymbol{\nu} x$, nor are they free in the usual sense, since once a meaning is given to $\boldsymbol{\nu} xP$, the meaning of these other variables is tied to the meaning of $\boldsymbol{\nu} xP$. I thus distinguish a third sort of occurrence of a variable in a formula, which is neither bound nor free. This accomplishes the same purpose as Hailperin's notion of subordination, but puts no restrictions on the formation of terms and formulas. This is all spelled out in the rules of formation for the expressions of the system I call $\mathcal{QR}_\nu$. The axioms and single rule of inference of $\mathcal{QR}_\nu$ are set forth, and several theorems are proved, most of them in preparation for the completeness proof, many of them showing the relations existing between different restricted variables.

Section 3 gives a semantic basis for $\mathcal{QR}_\nu$. A model for $\mathcal{QR}_\nu$ is considerably more complex than usual because of the potentially complicated structure of the restricted variable, and it is defined inductively on the structure of terms and formulas. Some facts about the relations between variables in the model are proven. Finally, the notion of a valid formula is defined. In the final section the soundness and completeness of $\mathcal{QR}_\nu$ are demonstrated. $\mathcal{QR}_\nu$ is shown to be sound when it has been proven that every axiom of $\mathcal{QR}_\nu$ is valid and that the rules of inference preserve validity. The demonstration of completeness uses a Henkin-type proof; the main result is that every consistent set of formulas of $\mathcal{QR}_\nu$ is verifiable, that is, there is an interpretation under which every formula in the set comes out true. In order to do this the language of $\mathcal{QR}_\nu$ is extended by the addition of constants. This is accomplished by means of a "constant-building" operator $\boldsymbol{\sigma}$. The consistent set of formulas S is transformed into a consistent set S^σ by replacing free restricted variables by appropriate constants, then S^σ is extended to a maximal consistent set of formulas in a systematic way, so that if any closed formula is not in this maximized set, its negation is provable. The set thus constructed is used to define a model for $\mathcal{QR}_\nu$ in which every formula of S^σ is true; its domain is the set of constants with existence condition deducible from the maximal consistent set, and truth values for atomic formulas are determined by whether they are deducible or not. Finally, a theorem on the eliminability of restricted quantifiers is proved.

2 *Syntax* The primitive signs of $\mathcal{QR}_\nu$ are: a countable number of *proper variables* (the ordinary variables of first-order predicate calculus); for each m, $m = 0, 1, 2, \ldots$, a countable number of m-place predicate variables, among them the 0-place P_0; the logical connectives $\neg$ and $\rightarrow$; the indefinite description operator $\boldsymbol{\nu}$; the universal quantifier $\forall$; parentheses (and); and comma ,. The letters x, y, z, sometimes with primes or subscripts, are used to refer to proper variables, and Greek letters α, β, γ, δ, again sometimes primed or subscripted, to refer to variables in general. P and Q, with or without superscripts, indicate m-place predicates.

The *expressions* of the language, which include both variables and formulas, are defined as follows:

(i) A proper variable is a variable.
(ii) If P^n is an n-place predicate, and $\alpha_1, \ldots, \alpha_n$ are variables, then $P^n(\alpha_1, \ldots, \alpha_n)$ is an atomic formula. An atomic formula is a formula.
(iii) If A is a formula, then $\neg A$ is a formula.
(iv) If A and B are formulas, then $(A \rightarrow B)$ is a formula.
(v) If α is a variable and A is a formula, then $\boldsymbol{\nu}\alpha A$ is a *restricted variable*. A restricted variable is a variable.
(vi) If α is a variable and A is a formula, then $(\forall\alpha)A$ is a formula.
(vii) Only those strings of symbols which are variables or formulas by virtue of (i)-(vi) are expressions.

Occasionally, for the sake of convenience, $(A \,\&\, B)$ is used to abbreviate the formula $\neg(A \rightarrow \neg B)$, $(A \vee B)$ to abbreviate $(\neg A \rightarrow B)$, and $(\exists\alpha)A$ for $\neg(\forall\alpha)\neg A$. The formula $(P_0 \rightarrow P_0)$ is indicated by t, $\neg(P_0 \rightarrow P_0)$ by f. Parentheses are omitted when no confusion results from their omission, in accordance with the usual conventions. When a symbol appears as one of the signs of an expression, such an appearance is called an *occurrence*.

Similarly, an expression may *occur* in a second expression when the symbols of the first occur concurrently in the second. The *length* of an expression is the number of primitive signs occurring in it, a symbol being counted each time it occurs.

As indicated in the introductory section, three kinds of occurrence of a variable in an expression are distinguished: free, adherent and bound. *Adherent* occurrences result when there are free variables other than x in P, and the restricted variable $\boldsymbol{\nu}xP$ is formed; the free variables of P other than x are adherent in $\boldsymbol{\nu}xP$, and in expressions containing $\boldsymbol{\nu}xP$. Adherent variables cannot be bound by $\forall$ or $\boldsymbol{\nu}$, but they are freed from adherence when the restricted variable in which they adhere is bound by the universal quantifier. The formal definition is as follows:

(i) An occurrence of a variable in an expression is either free, adherent or bound, and these three classes of occurrence are mutually exclusive.
(ii) An occurrence of a variable in an argument place of a predicate is a free occurrence of that variable.
(iii) Each free occurrence of a variable α in a formula A is a bound occurrence in the expression $\boldsymbol{\nu}\alpha A$. Each adherent (respectively, bound) occurrence of a variable β in A or in α is an adherent (respectively, bound) occurrence in $\boldsymbol{\nu}\alpha A$. If β is not α, then each free occurrence of β in A is an adherent occurrence in $\boldsymbol{\nu}\alpha A$.
(iv) If α is a variable and B is a formula, then every free occurrence of α in B is bound in $(\forall\alpha)B$. If β is different from α, then a free (adherent, bound) occurrence of β in B is free (adherent, bound, respectively) in $(\forall\alpha)B$, except that when α is $\boldsymbol{\nu}\ldots\boldsymbol{\nu}\gamma C\ldots A$ (there may be only one $\boldsymbol{\nu}$ preceding γ), and β is free in C (and thus adherent in α), such adherent occurrences of β are free in $(\forall\alpha)B$.
(v) Free, adherent, and bound occurrences of a variable in $\neg A$ and $(A \rightarrow B)$ correspond respectively to free, adherent and bound occurrences in A and B.

To aid in defining the closure of a formula, and for later use in the inductive definition of a model, the notion of the *level* of an expression is defined as follows:

(i) If α is a proper variable, then the level of α, $l(\alpha)$, is 0.
(ii) If α is a restricted variable $\boldsymbol{\nu}\beta B$, then $l(\alpha)$ is $1 + \max\{l(\alpha_i)\colon \alpha_i$ occurs in $\boldsymbol{\nu}\beta B\}$.
(iii) If B is a formula, then $l(B) = \max\{l(\alpha_i)\colon \alpha_i$ occurs in $B\}$.

The level of a formula is an indicator of the complexity of the structure of the variables occurring in the formula. It aids in the definition of the closure of a formula, as follows: a *closure* for a formula B is the formula $(\forall\alpha_1)(\forall\alpha_2)\ldots(\forall\alpha_n)B$, with $\alpha_1, \alpha_2, \ldots, \alpha_n$ a complete list of all variables occurring free or adherent in B, in non-decreasing order of level. Arranging the variables in non-decreasing order according to level guarantees that in the closure no variable has a free or adherent occurrence since, for example, a variable α_{k-1} adherent in α_k has level less than $l(\alpha_k)$, and when it is freed by $(\forall\alpha_k)$ it is bound by the earlier quantifier $(\forall\alpha_{k-1})$. The interchange of quantifiers cannot in general be allowed, although quantifiers on the same level may be permuted; as an example, $(\forall x)(\forall \boldsymbol{\nu} yP(x, y))Q$ and $(\forall \boldsymbol{\nu} yP(x, y))(\forall x)Q$ are not equivalent, since x has a free occurrence in the second formula but not in the first. An expression is said to be *closed* if no variable has a free or adherent occurrence in it. Closed $\boldsymbol{\nu}$-expressions will be referred to as $\boldsymbol{\nu}$-*terms*.

The variable α is said to be *substitutible for* β in an expression E if the occurrences of α in E resulting from the replacement of β by α are free where and only where β is free in E. If α is substitutible for β in a formula B, then the formula resulting from the replacement of β in each free occurrence by α is denoted by $B_\beta[\alpha]$. Generally, when this notation is used it is assumed that the substitution is permissible. In case B has no free occurrences of β, $B_\beta[\alpha]$ is B; and when α is substitutible for β, and γ for α, $(B_\beta[\alpha])_\alpha[\gamma]$ is $B_\beta[\gamma]$.

Finally, I need to define the notion of an *alphabetic variant*; two variables are alphabetic variants if either (i) they are both proper variables, or (ii) they are restricted variables of the form $\boldsymbol{\nu}\alpha A$ and $\boldsymbol{\nu}\beta B$, such that α and β are alphabetic variants and B is $A_\alpha[\beta]$.

The axioms of $\mathcal{QR}_\nu$ are as follows:

A1 *If B is a tautology, then B is an axiom.*

A2 $A_\alpha[\boldsymbol{\nu}\alpha\neg A] \rightarrow (\forall\alpha)A$.

A3 $(\forall\alpha)(A \rightarrow B) \rightarrow ((\forall\alpha)A \rightarrow (\forall\alpha)B)$.

A4a $(\forall\alpha)B \rightarrow (\neg(\forall\beta)f \rightarrow B_\alpha[\beta])$, *where* β *is an alphabetic variant of* α.

A4b $(\forall\alpha)B \rightarrow (\neg(\forall\alpha)\neg A \rightarrow B_\alpha[\boldsymbol{\nu}\alpha A])$.

A5 $(\forall\boldsymbol{\nu}\alpha B)\neg(\forall\alpha)\neg B$.

A6 $(\forall\boldsymbol{\nu}\alpha A)B \rightarrow (\forall\beta)(A_\alpha[\beta] \rightarrow B_{\nu\alpha A}[\beta])$, *where* β *is an alphabetic variant of* α *not occurring free in* $(\forall\boldsymbol{\nu}\alpha A)B$.

A7 *If A is an axiom, then* $(\forall\alpha)A$ *is an axiom.*

The only rule of inference is *modus ponens*, **MP**: from A and $A \rightarrow B$, infer B.

A *derivation* of a formula A from a (possibly empty) set of formulas S, is a finite sequence of formulas $B_1, \ldots, B_n$, such that B_n is A, and for each i, either B_i is an axiom, a member of S, or the result of applying **MP** to two earlier formulas in the sequence. If S is empty, the derivation of A is called a *proof*, and A is a *theorem*. The notation $S \vdash A$ means that there is a derivation of A from S; $\vdash A$ means that A is a theorem. It is clear that if A is an axiom, or a member of S, then $S \vdash A$.

A set of formulas S is said to be *inconsistent* if $S \vdash f$; S is *consistent* if it is not inconsistent. The following two theorems concerning consistency are standard, and their proofs are omitted. They are included here for later use in the completeness proof.

Theorem **2.1** *If* $S \cup \{\neg A\}$ *is inconsistent, then* $S \vdash A$.

Theorem **2.2** *If* $S \vdash A$ *and* $S \vdash \neg A$, *then S is inconsistent.*

Other standard theorems which will be used are the following:

Theorem **2.3** (The Deduction Theorem) *If S is a set of formulas, and A and B formulas such that $S \cup \{A\} \vdash B$, then $S \vdash A \rightarrow B$.*

Since the only rule of inference is **MP**, the proof of the deduction theorem presents no special difficulties and is omitted, as are the next two proofs.

Theorem **2.4** *If $S \vdash A \rightarrow B$ and $S \vdash B \rightarrow C$, then $S \vdash A \rightarrow C$.*

Theorem **2.5** *If $S \vdash (\forall\alpha)(A \rightarrow B)$ and $S \vdash (\forall\alpha)A$, then $S \vdash (\forall\alpha)B$.*

I also will want to use a simple version of the generalization rule:

Theorem **2.6** *If $\vdash A$, then $\vdash (\forall\alpha)A$, for any variable α.*

Since there is not a set S of assumption formulas which might impose conditions on the variable α, this is an almost immediate consequence of A7.

The following theorems have special reference to restricted quantification theory:

Theorem **2.7** *If $S \vdash (\forall\alpha)f$, then $S \vdash (\forall\alpha)B$ for any formula B.*

Proof: $f \rightarrow B$ is an axiom because it is a tautology, therefore, by A7, $(\forall\alpha)(f \rightarrow B)$ is an axiom, so $S \vdash (\forall\alpha)(f \rightarrow B)$. Then, if $S \vdash (\forall\alpha)f$, it follows by **2.5** that $S \vdash (\forall\alpha)B$.

Theorem **2.8** *If α and α' are alphabetic variants, and α' does not occur free in $(\forall\alpha)B$, then $\vdash (\forall\alpha')((\forall\alpha)B \rightarrow B_\alpha[\alpha'])$.*

Proof: $(\neg B_\alpha[\alpha'] \rightarrow \neg(\forall\alpha)B) \rightarrow ((\forall\alpha)B \rightarrow B_\alpha[\alpha'])$ is a tautology, hence an axiom. By A7, its generalization

$$(\forall\alpha')((\neg B_\alpha[\alpha'] \rightarrow \neg(\forall\alpha)B) \rightarrow ((\forall\alpha)B \rightarrow B_\alpha[\alpha']))$$

is an axiom; also,

$$(\forall \boldsymbol{\nu} \alpha \neg B)(\neg(\forall\alpha)B) \rightarrow (\forall\alpha')(\neg B_\alpha[\alpha'] \rightarrow \neg(\forall\alpha)B)$$

is an instance of A6, and since $\vdash (\forall \boldsymbol{\nu} \alpha \neg B)\neg(\forall\alpha)B$ by A5, $\vdash (\forall\alpha')(\neg B_\alpha[\alpha'] \rightarrow \neg(\forall\alpha)B)$ follows by **MP**. Then, $\vdash (\forall\alpha')((\forall\alpha)B \rightarrow B_\alpha[\alpha'])$ follows from

$$\vdash (\forall\alpha')((\neg B_\alpha[\alpha'] \rightarrow \neg(\forall\alpha)B) \rightarrow ((\forall\alpha)B \rightarrow B_\alpha[\alpha']))$$

and

$$\vdash (\forall\alpha')(\neg B_\alpha[\alpha'] \rightarrow \neg((\forall\alpha)B))$$

by **2.5**.

This is the restricted variable analogue of Leblanc and Meyer's *Axiom of Specification,* adopted from Lambert, which distinguishes their free logic from other theories.

Theorem **2.9** *If α' is an alphabetic variant of α not occurring free in $(\forall\alpha)B$, then $\vdash (\forall\alpha)B \rightarrow (\forall\alpha')B_\alpha[\alpha']$.*

Proof: By **2.8**, $\vdash (\forall\alpha')((\forall\alpha)B \to B_\alpha[\alpha'])$. From A3 by **MP**, $\vdash(\forall\alpha')(\forall\alpha)B \to (\forall\alpha')B_\alpha[\alpha']$. Since α' does not occur free in $(\forall\alpha)B$, $((\forall\alpha)B)_{\alpha'}[\boldsymbol{\nu}\alpha\neg(\forall\alpha)B]$ is $(\forall\alpha)B$, and thus $(\forall\alpha)B \to (\forall\alpha')(\forall\alpha)B$ is an instance of A2. From above, $\vdash(\forall\alpha')(\forall\alpha)B \to (\forall\alpha')B_\alpha[\alpha']$, therefore, by **2.4**, $\vdash(\forall\alpha)B \to (\forall\alpha')B_\alpha[\alpha']$.

Theorem **2.10** *If* $\boldsymbol{\nu}\alpha A$ *does not occur free in* $(\forall\alpha)B$, *then* $\vdash(\forall\alpha)B \to (\forall\boldsymbol{\nu}\alpha A)B_\alpha[\boldsymbol{\nu}\alpha A]$.

Proof:

$$\vdash((\forall\alpha)B \to (\neg(\forall\alpha)\neg A \to B_\alpha[\boldsymbol{\nu}\alpha A])) \to (\neg(\forall\alpha)\neg A \to ((\forall\alpha)B \to B_\alpha[\boldsymbol{\nu}\alpha A]))$$

because it is a tautology, and

$$(\forall\alpha)B \to (\neg(\forall\alpha)\neg A \to B_\alpha[\boldsymbol{\nu}\alpha A])$$

is an instance of A4, therefore

$$\vdash\neg(\forall\alpha)\neg A \to ((\forall\alpha)B \to B_\alpha[\boldsymbol{\nu}\alpha A])$$

by **MP**. By **2.6**,

$$\vdash(\forall\boldsymbol{\nu}\alpha A)(\neg(\forall\alpha)\neg A \to ((\forall\alpha)B \to B_\alpha[\boldsymbol{\nu}\alpha A]));$$

and $(\forall\boldsymbol{\nu}\alpha A)\neg(\forall\alpha)\neg A$ is an instance of A5, therefore, by **2.5**, $\vdash(\forall\boldsymbol{\nu}\alpha A)((\forall\alpha)B \to B_\alpha[\boldsymbol{\nu}\alpha A])$. Then by A3,

$$\vdash(\forall\boldsymbol{\nu}\alpha A)(\forall\alpha)B \to (\forall\boldsymbol{\nu}\alpha A)B_\alpha[\boldsymbol{\nu}\alpha A],$$

and by A2, $\vdash(\forall\alpha)B \to (\forall\boldsymbol{\nu}\alpha A)(\forall\alpha)B$, since $\boldsymbol{\nu}\alpha A$ does not occur in $(\forall\alpha)B$. Thus $\vdash(\forall\alpha)B \to (\forall\boldsymbol{\nu}\alpha A)B_\alpha[\boldsymbol{\nu}\alpha A]$.

As an immediate consequence of **2.9** and **2.10**,

Theorem **2.11** *If* α' *is a variant of* α, *and* B' *is* $B_\alpha[\alpha']$, *then* $\vdash(\forall\alpha)A \to (\forall\boldsymbol{\nu}\alpha' B')A_\alpha[\boldsymbol{\nu}\alpha' B']$.

The next theorem could be considered a converse of A6; it relates two levels of quantification.

Theorem **2.12** $\vdash(\forall\alpha)(A \to B) \to (\forall\boldsymbol{\nu}\alpha A)B_\alpha[\boldsymbol{\nu}\alpha A]$.

Proof: By **2.10**,

$$\vdash(\forall\alpha)(A \to B) \to (\forall\boldsymbol{\nu}\alpha A)(A \to B)_\alpha[\boldsymbol{\nu}\alpha A],$$

therefore,

$$\vdash(\forall\alpha)(A \to B) \to ((\forall\boldsymbol{\nu}\alpha A)A_\alpha[\boldsymbol{\nu}\alpha A] \to (\forall\boldsymbol{\nu}\alpha A)B_\alpha[\boldsymbol{\nu}\alpha A]).$$

Using tautologies,

$$\vdash(\forall\boldsymbol{\nu}\alpha A)A_\alpha[\boldsymbol{\nu}\alpha A] \to ((\forall\alpha)(A \to B) \to (\forall\boldsymbol{\nu}\alpha A)B_\alpha[\boldsymbol{\nu}\alpha A]).$$

$\neg A_\alpha[\boldsymbol{\nu}\alpha A] \to (\forall\alpha)\neg A$ is an instance of A2, hence $\vdash\neg(\forall\alpha)\neg A \to A_\alpha[\boldsymbol{\nu}\alpha A]$, and by **2.6**, $\vdash(\forall\boldsymbol{\nu}\alpha A)(\neg(\forall\alpha)\neg A \to A_\alpha[\boldsymbol{\nu}\alpha A])$. But $(\forall\boldsymbol{\nu}\alpha A)\neg(\forall\alpha)\neg A$ is an axiom, hence $\vdash(\forall\boldsymbol{\nu}\alpha A)A_\alpha[\boldsymbol{\nu}\alpha A]$, therefore,

$$\vdash(\forall\alpha)(A \to B) \to (\forall\boldsymbol{\nu}\alpha A)B_\alpha[\boldsymbol{\nu}\alpha A].$$

Before stating and proving the next theorem, I need to make some observations and introduce some notation for future use. If β is a restricted variable, it has the form $\boldsymbol{\nu}\alpha A$, where α may be either a proper or restricted variable. α is said to be *nested in* β in this case, and the *depth* of nesting is 1. If α in turn is a restricted variable $\boldsymbol{\nu}\gamma C$, then γ is nested in α and also in β, which has the form $\boldsymbol{\nu\nu}\gamma CA$. Whenever β can be written as a string of n $\boldsymbol{\nu}$-symbols, followed by a variable α, followed by a string of n formulas, α is said to be *nested in* β to *depth* n, and the notation $\boldsymbol{\nu}^n\alpha\{A_n\}$ is used to indicate the structure of β.

Theorem **2.13** $\vdash(\forall\alpha)B \rightarrow (\forall\beta)B_\alpha[\beta]$ *for every* β *in which* α *is nested.*

Proof: The proof is by induction on the depth to which α is nested in β, that is, on the length of the sequence of $\boldsymbol{\nu}$-symbols preceding α. If $n = 1$, then β is $\boldsymbol{\nu}\alpha A$ for some A, and $\vdash(\forall\alpha)B \rightarrow (\forall\boldsymbol{\nu}\alpha A)B_\alpha[\boldsymbol{\nu}\alpha A]$ by **2.10**. Assume as hypothesis for the induction, that the theorem holds for every variable in which α is nested to depth less than n. Let β be the variable $\boldsymbol{\nu}^n\alpha\{A_n\}$. This can be written as $\boldsymbol{\nu}(\boldsymbol{\nu}^{n-1}\alpha\{A_{n-1}\})A_n$ or, letting $\boldsymbol{\nu}^{n-1}\alpha\{A_{n-1}\}$ be γ, as $\boldsymbol{\nu}\gamma A_n$. By **2.10**,

$$\vdash(\forall\gamma)B_\alpha[\gamma] \rightarrow (\forall\boldsymbol{\nu}\gamma A_n)(B_\alpha[\gamma])_\gamma[\boldsymbol{\nu}\gamma A_n],$$

that is, $\vdash(\forall\gamma)B_\alpha[\gamma] \rightarrow (\forall\beta)B_\alpha[\beta]$. By the induction hypothesis, $\vdash(\forall\alpha)B \rightarrow (\forall\gamma)B_\alpha[\gamma]$, and, therefore, by **2.4**, $\vdash(\forall\alpha)B \rightarrow (\forall\beta)B_\alpha[\beta]$.

From the formation of restricted variables, it is clear that every restricted variable has a proper variable nested in it. By means of **2.11** the above result can be extended to every β in which an alphabetic variant of α is nested. In particular, I have the following theorem:

Theorem **2.14** *If* α *is a restricted variable, then* $\vdash(\forall x)A \rightarrow (\forall\alpha)A_x[\alpha]$.

Theorem **2.15** $\vdash(\forall\alpha)\neg A \rightarrow (\forall\boldsymbol{\nu}\alpha A)f$.

Proof: $\vdash\neg A \rightarrow (A \rightarrow f)$, therefore, $\vdash(\forall\alpha)(\neg A \rightarrow (A \rightarrow f))$, so by A3 and **MP**, $\vdash(\forall\alpha)\neg A \rightarrow (\forall\alpha)(A \rightarrow f)$.

$$\vdash(\forall\alpha)(A \rightarrow f) \rightarrow (\forall\boldsymbol{\nu}\alpha A)(A \rightarrow f)_\alpha[\boldsymbol{\nu}\alpha A],$$

and

$$\vdash(\forall\boldsymbol{\nu}\alpha A)(A \rightarrow f) \rightarrow ((\forall\boldsymbol{\nu}\alpha A)A_\alpha[\boldsymbol{\nu}\alpha A] \rightarrow (\forall\boldsymbol{\nu}\alpha A)f),$$

therefore,

$$\vdash(\forall\alpha)\neg A \rightarrow ((\forall\boldsymbol{\nu}\alpha A)A_\alpha[\boldsymbol{\nu}\alpha A] \rightarrow (\forall\boldsymbol{\nu}\alpha A)f).$$

But then,

$$\vdash(\forall\boldsymbol{\nu}\alpha A)A_\alpha[\boldsymbol{\nu}\alpha A] \rightarrow ((\forall\alpha)\neg A \rightarrow (\forall\boldsymbol{\nu}\alpha A)f),$$

and since $\vdash(\forall\boldsymbol{\nu}\alpha A)A_\alpha[\boldsymbol{\nu}\alpha A]$ by the proof of **2.12**, it follows that $\vdash(\forall\alpha)\neg A \rightarrow (\forall\boldsymbol{\nu}\alpha A)f$.

Theorem **2.16** $(\forall\boldsymbol{\nu}\alpha A)f \rightarrow (\forall\alpha)\neg A$.

Proof: $\vdash (A_\alpha[\alpha'] \to f) \to \neg A_\alpha[\alpha']$, because it is a tautology.

$$\vdash (\forall\alpha')((A_\alpha[\alpha'] \to f) \to \neg A_\alpha[\alpha']),$$

therefore,

$$\vdash (\forall\alpha')(A_\alpha[\alpha'] \to f) \to (\forall\alpha')\neg A_\alpha[\alpha'].$$

By A6,

$$\vdash (\forall\boldsymbol{\nu}\alpha A)f \to (\forall\alpha')(A_\alpha[\alpha'] \to f),$$

and thus

$$\vdash (\forall\boldsymbol{\nu}\alpha A)f \to (\forall\alpha')\neg A_\alpha[\alpha'].$$

But for suitable α and α', $\vdash (\forall\alpha')\neg A_\alpha[\alpha'] \to (\forall\alpha)\neg A$, therefore, $\vdash (\forall\boldsymbol{\nu}\alpha A)f \to (\forall\alpha)\neg A$.

3 *Semantics* A model for $\mathcal{QR}_\nu$ consists of a (possibly empty) set of individuals $\mathbf{D}$, called the *universe* or *domain of discourse*, together with a mapping Π which is defined for all m-place predicate variables, $m = 0, 1, 2, \ldots$, such that for each P^m, $\Pi(P^m)$ is a function from $\mathbf{D}^m$, the set of all m-tuples of elements of $\mathbf{D}$, to $\{0, 1\}$. Functions $\mathfrak{D}$ and $\mathfrak{I}$ are defined; $\mathfrak{D}$, called a *domain assignment*, assigns a subset of $\mathbf{D}$, possibly $\emptyset$, to each variable of $\mathcal{QR}_\nu$; $\mathfrak{I}$, called a *value assignment* or *interpretation*, assigns to each variable α for which $\mathfrak{D}(\alpha) \neq \emptyset$ a member of $\mathfrak{D}(\alpha)$, and to each formula of $\mathcal{QR}_\nu$ either 1 or 0, 0 standing for falsehood and 1 for truth. The functions $\mathfrak{D}$ and $\mathfrak{I}$ are defined inductively and simultaneously on the level of expressions. $\mathfrak{I}(\alpha)$ is defined only for variables with non-empty domains, called *designating variables*; when $\mathfrak{D}(\alpha) = \emptyset$ a formula containing α cannot be assigned a value in the usual fashion. In such cases atomic formulas with free α are assigned a value arbitrarily, or in accordance with extra-logical considerations, and the theory itself takes no stand as to the truth or falsity of statements with non-designating variables, except that once values have been assigned to atomic formulas, the values of composite formulas are determined as usual. As to a formula of the sort $(\forall\alpha)A$, it is assigned the truth value 1 when $\mathfrak{D}(\alpha) = \emptyset$. Thus in the empty domain a formula and its closure need not have the same value.

The definitions of $\mathfrak{D}$ and $\mathfrak{I}$ proceed as follows: Let $\mathbf{D}$ be a set of objects ($\mathbf{D}$ may be $\emptyset$), and Π a mapping which assigns to each m-place predicate a function from $\mathbf{D}^m$ to $\{0, 1\}$. (If $\mathbf{D}$ is empty then the only function from $\mathbf{D}^m$ to $\{0, 1\}$ is the empty function.) A sequence $\{\mathfrak{D}_n\}$ of domain assignments from variables to subsets of $\mathbf{D}$ is constructed, so that each $\mathfrak{D}_n$ is defined on all variables of level less than or equal to n. Simultaneously a sequence $\{\mathfrak{I}_j\}$ of value assignments is defined so that, if α is a variable of level less than or equal to n and $\mathfrak{D}_n \neq \emptyset$, then $\mathfrak{I}_n(\alpha)$ is a member of $\mathfrak{D}_n(\alpha)$, and if B is a formula of level less than or equal to n, $\mathfrak{I}_n(B)$ is either 0 or 1. $\mathfrak{I}_{j\,\mathbf{d}}^{\,\alpha}$ denotes that value assignment which assigns the same individual to every variable $\beta \neq \alpha$ whose level is less than or equal to j that $\mathfrak{I}_j$ assigns to β, and which assigns the individual $\mathbf{d}$ in $\mathfrak{D}_j(\alpha)$ to α. $\mathfrak{I}_{j\,\mathbf{d}}^{\,\alpha}$ also agrees with $\mathfrak{I}_j$

on all the atomic formulas of level less than or equal to j which contain a non-designating variable. The following restrictions need to be made on the values given to atomic formulas with non-designating variables:

Restriction R1 Let A be an atomic formula of level n in which a variable with empty domain occurs, and let α and β be variables of level less than or equal to n such that $\mathcal{I}_n(\alpha) = \mathcal{I}_n(\beta)$, and β is substitutible for α. Then both A and $A_\alpha[\beta]$ contain a non-designating variable, so they both must be assigned a value by $\mathcal{I}_n$; R1 requires that $\mathcal{I}_n(A) = \mathcal{I}_n(A_\alpha[\beta])$.

Restriction R2 Let B be an atomic formula of level n in which a variable with empty domain occurs, let α and β be variables of level less than or equal to n such that $\mathcal{I}_n(\alpha)$ and $\mathcal{I}_n(\beta)$ are defined, and $\mathcal{I}_n(\beta)$ is in $\mathcal{D}_n(\alpha)$. Suppose α has a free occurrence in B and that β is substitutible for α. $\mathcal{I}_{n\mathcal{I}_n(\beta)}^{\alpha}$ is the assignment which agrees with $\mathcal{I}_n$ on n or less-level expressions, except that it assigns to α the individual which $\mathcal{I}_n$ assigns to β. R2 requires that $\mathcal{I}_{n\mathcal{I}_n(\beta)}^{\alpha}(B) = \mathcal{I}_n(B_\alpha[\beta])$. That is, $B_\alpha[\beta]$ does not necessarily have the same value as B; what counts in determining its value is the individual to which β and α refer.

The sequences $\{\mathcal{D}_n\}$ and $\{\mathcal{I}_n\}$ are constructed as follows:

$\mathcal{D}_0(\alpha) = \mathbf{D}$, for every 0-level variable, i.e., for every proper variable.

If $\mathcal{D}_0(x) \neq \emptyset$, let $\mathcal{I}_0(x)$ be a member of $\mathcal{D}_0(x)$, for every proper variable x. Let B be a 0-level formula.

(i) If B is $P(x_1, \ldots, x_n)$ and $\mathcal{D}_0(x_i) \neq \emptyset$, then $\mathcal{I}_0(P(x_1, \ldots, x_n)) = \Pi(P)(\mathcal{I}_0(x_1), \ldots, \mathcal{I}_0(x_n))$. If $\mathcal{D}_0(x_i) = \emptyset$, then let $\mathcal{I}_0(P(x_1, \ldots, x_n))$ be either $\mathbf{0}$ or $\mathbf{1}$.
(ii) Suppose B is $\neg A$. Then $\mathcal{I}_0(B) = \mathbf{1}$ if and only if $\mathcal{I}_0(A) = \mathbf{0}$.
(iii) Suppose B is $A \rightarrow C$. Then $\mathcal{I}_0(A \rightarrow C) = \mathbf{0}$ if and only if $\mathcal{I}_0(A) = \mathbf{1}$ and $\mathcal{I}_0(C) = \mathbf{0}$.
(iv) Suppose B is $(\forall x)A$. If $\mathcal{D}_0(x) = \emptyset$ then $\mathcal{I}_0((\forall x)A) = \mathbf{1}$. If $\mathcal{D}_0(x) \neq \emptyset$, then $\mathcal{I}_0((\forall x)A) = \mathbf{1}$ if and only if $\mathcal{I}_{0\mathbf{d}}^{x}(A) = \mathbf{1}$ for every individual $\mathbf{d}$ in $\mathcal{D}_0(x)$.

$\mathcal{D}_0$ and $\mathcal{I}_0$ are thus defined for 0-level variables and formulas. Assume as hypothesis for the inductive definition, that a domain has been assigned to each variable of level less than or equal to n by $\mathcal{D}_n$, and that for each variable α of level less than or equal to n such that $\mathcal{D}_n(\alpha) \neq \emptyset$, $\mathcal{I}_n(\alpha)$ is an element of $\mathcal{D}_n(\alpha)$, and that for each formula B of level less than or equal to n, $\mathcal{I}_n(B)$ has been determined. Let β be a restricted variable of level $n + 1$. Supposing β to be $\boldsymbol{\nu}\alpha A$, the hypothesis of the induction applies to α and A, since $l(\alpha)$ and $l(A)$ are less than $n + 1$, and $\mathcal{D}_n(\alpha)$ is a subset, possibly empty, of $\mathbf{D}$. If $\mathcal{D}_n(\alpha) \neq \emptyset$, then $\mathcal{I}_n(\alpha)$ belongs to $\mathcal{D}_n(\alpha)$, and whether $\mathcal{D}_n(\alpha)$ is empty or not, $\mathcal{I}_n(A)$ is either $\mathbf{0}$ or $\mathbf{1}$. Define

$$\mathcal{D}_n^{n+1}(\boldsymbol{\nu}\alpha A) = \{\mathbf{d} \text{ in } \mathcal{D}_n(\alpha)\colon \mathcal{I}_{n\mathbf{d}}^{\alpha}(A) = \mathbf{1}\},$$

and let

$$\mathcal{D}_{n+1} = \mathcal{D}_n \cup \mathcal{D}_n^{n+1}.$$

Thus $\mathfrak{D}_{n+1}$ assigns a subset of $\mathbf{D}$ to every variable of level less than or equal to $n + 1$, and agrees with $\mathfrak{D}_i$, for i less than $n + 1$, on all variables of level i. If $\mathfrak{D}_n(\alpha) = \emptyset$, of course $\mathfrak{D}_{n+1}(\boldsymbol{\nu}\alpha A) = \emptyset$ also. For every β of level $n + 1$ such that $\mathfrak{D}_{n+1}(\beta) \neq \emptyset$, let $\mathcal{I}_{n+1}(\beta)$ be a member of $\mathfrak{D}_{n+1}(\beta)$, and for α of level n or less, let $\mathcal{I}_{n+1}(\alpha)$ be $\mathcal{I}_n(\alpha)$. Let B be a level-$n + 1$ formula:

(i) If B is $P(\alpha_1, \ldots, \alpha_k)$ and $\mathfrak{D}_{n+1}(\alpha_i) \neq \emptyset$ for $i = 1, \ldots, k$, then $\mathcal{I}_{n+1}(P(\alpha_1, \ldots, \alpha_k)) = \Pi(P)(\mathcal{I}_{n+1}(\alpha_1), \ldots, \mathcal{I}_{n+1}(\alpha_k))$. If there is an i such that $\mathfrak{D}_{n+1}(\alpha_i) = \emptyset$, then let $\mathcal{I}_{n+1}(P(\alpha_1, \ldots, \alpha_k))$ be $\mathbf{0}$ or $\mathbf{1}$, being sure to follow restriction R1.

(ii) Suppose B is $\neg A$. Then $\mathcal{I}_{n+1}(B) = \mathbf{1}$ if and only if $\mathcal{I}_{n+1}(A) = \mathbf{0}$.

(iii) Suppose B is $(A \rightarrow C)$. Then $\mathcal{I}_{n+1}(A \rightarrow C) = \mathbf{0}$ if and only if $\mathcal{I}_{n+1}(A) = \mathbf{1}$ and $\mathcal{I}_{n+1}(C) = \mathbf{0}$.

(iv) Suppose B is $(\forall\alpha)A$. If $\mathfrak{D}_{n+1}(\alpha) = \emptyset$ then $\mathcal{I}_{n+1}((\forall\alpha)A) = \mathbf{1}$. If $\mathfrak{D}_{n+1}(\alpha) \neq \emptyset$, then $\mathcal{I}_{n+1}((\forall\alpha)A) = \mathbf{1}$ if and only if $\mathcal{I}_{n+1}{}^{\alpha}_{\mathbf{d}}(A) = \mathbf{1}$ for every $\mathbf{d}$ in $\mathfrak{D}_{n+1}(\alpha)$. (In determining $\mathcal{I}_{n+1}{}^{\alpha}_{\mathbf{d}}(A)$ restriction R2 must be kept in mind.)

Finally, let $\mathfrak{D} = \bigcup_{n=0}^{\infty} \mathfrak{D}_n$ and $\mathcal{I} = \bigcup_{j=0}^{\infty} \mathcal{I}_j$.

It should be noted that whenever $\mathfrak{D}_n(\alpha) = \emptyset$ for some α of level n, there are different interpretations possible depending on whether a particular atomic formula containing α is assigned $\mathbf{0}$ or $\mathbf{1}$ by $\mathcal{I}_n$. For example, it would be possible for two interpretations $\mathcal{I}$ and $\mathcal{I}'$ to agree up to level k, but $\mathcal{I}_k$ to be different from $\mathcal{I}'_k$. Also, if two interpretations differ at some level, the domain assignments associated with each of them may also differ on some variables of higher level.

The following theorems give some information about models for $\mathcal{QR}_\nu$; some of them will be necessary for the completeness proof.

Theorem **3.1** *Let α and β be variables and A a formula such that β is substitutible for α in A, let $\mathbf{D} \neq \emptyset$, and let $\mathfrak{D}$ and $\mathcal{I}$ be such that $\mathfrak{D}(\alpha) \neq \emptyset$. Then $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(A) = \mathbf{1}$ if and only if $\mathcal{I}(A_\alpha[\beta]) = \mathbf{1}$.*

Proof: If α does not occur free in A, then $A_\alpha[\beta]$ is A, and since $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}$ and $\mathcal{I}$ agree on all the free variables of A as well as on all atomic formulas with non-designating variables, $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(A) = \mathcal{I}(A_\alpha[\beta])$.

Suppose α occurs free in A. Let A be the atomic formula $P(\alpha_1, \ldots, \alpha_k)$. Since α occurs free in A, and an occurrence of α in some α_i would not be a free occurrence, some α_i must be α. If $\mathfrak{D}(\alpha_i) \neq \emptyset$ for $i = 1, \ldots, k$, then

$$\begin{aligned}\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(P(\alpha_1, \ldots, \alpha_k)) &= \mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(P(\alpha_1, \ldots, \alpha, \ldots, \alpha_k)) \\ &= \Pi(P)(\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(\alpha_1), \ldots, \mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(\alpha), \ldots, \mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(\alpha_k)) \\ &= \Pi(P)(\mathcal{I}(\alpha_1), \ldots, \mathcal{I}(\beta), \ldots, \mathcal{I}(\alpha_k)),\end{aligned}$$

because $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}$ agrees with $\mathcal{I}$ on all variables other than α, and $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(\alpha) = \mathcal{I}(\beta)$ in this case. Furthermore,

$$\begin{aligned}\Pi(P)(\mathcal{I}(\alpha_1), \ldots, \mathcal{I}(\beta), \ldots, \mathcal{I}(\alpha_k)) &= \mathcal{I}(P(\alpha_1, \ldots, \beta, \ldots, \alpha_k)) \\ &= \mathcal{I}((P(\alpha_1, \ldots, \alpha_k))_\alpha[\beta]).\end{aligned}$$

Therefore, $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(A) = \mathcal{I}(A_\alpha[\beta])$. If there is some i such that $\mathcal{D}(\alpha_i) = \emptyset$, then restrictions R1 and R2 insure that $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(A) = \mathcal{I}(A_\alpha[\beta])$ for an atomic formula A.

Assume that if B is a formula with fewer than n occurrences of $\neg$, $\rightarrow$, and $\forall$, then $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(B) = \mathcal{I}(B_\alpha[\beta])$. Let A be a formula with n occurrences of $\neg$, $\rightarrow$, and $\forall$. Suppose A is $\neg B$. $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(\neg B) = \mathbf{1}$ iff $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(B) = \mathbf{0}$ iff $\mathcal{I}(B_\alpha[\beta]) = \mathbf{0}$ iff $\mathcal{I}(\neg B_\alpha[\beta]) = \mathbf{1}$. Suppose A is $B \rightarrow C$. $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(B \rightarrow C) = \mathbf{0}$ iff $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(B) = \mathbf{1}$ and $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}(C) = \mathbf{0}$ iff $\mathcal{I}(B_\alpha[\beta]) = \mathbf{1}$ and $\mathcal{I}(C_\alpha[\beta]) = \mathbf{0}$ iff $\mathcal{I}(B_\alpha[\beta] \rightarrow C_\alpha[\beta]) = \mathbf{0}$ iff $\mathcal{I}((B \rightarrow C)_\alpha[\beta]) = \mathbf{0}$ iff $\mathcal{I}(A_\alpha[\beta]) = \mathbf{0}$. Suppose, lastly, that A is $(\forall\gamma)C$. Since α is free in $(\forall\gamma)C$, either α has an adherent occurrence in γ which is free in $(\forall\gamma)C$, or α has a free occurrence in C. If α is not adherent in γ, then the only free occurrences of α in A are its free occurrences in C. In this case, $((\forall\gamma)C)_\alpha[\beta]$ is $(\forall\gamma)C_\alpha[\beta]$. If $\mathcal{D}(\gamma) = \emptyset$ then $\mathcal{I}((\forall\gamma)C_\alpha[\beta]) = \mathbf{1}$ and $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}((\forall\gamma)C) = \mathbf{1}$. If $\mathcal{D}(\gamma) \neq \emptyset$, then $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}((\forall\gamma)C) = \mathbf{1}$ iff for every $\mathbf{d}$ in $\mathcal{D}(\gamma)$, $\mathcal{I}^{\alpha\ \gamma}_{\mathcal{I}(\beta)\mathbf{d}}(C) = \mathbf{1}$ iff for every $\mathbf{d}$ in $\mathcal{D}(\gamma)$ $\mathcal{I}^{\gamma\ \alpha}_{\mathbf{d}\,\mathcal{I}(\beta)}(C) = \mathbf{1}$ iff, by the induction hypothesis, $\mathcal{I}^{\gamma}_{\mathbf{d}}(C_\alpha[\beta]) = \mathbf{1}$ for every $\mathbf{d}$ in $\mathcal{D}(\gamma)$ iff $\mathcal{I}((\forall\gamma)C_\alpha[\beta]) = \mathbf{1}$. If α is adherent in γ and free in $(\forall\gamma)C$, then there is a variable $\boldsymbol{\nu}\delta D$ with α free in D, such that $\boldsymbol{\nu}\delta D$ is γ or nested in γ. Suppose $\boldsymbol{\nu}\delta D$ is γ. Then $((\forall\gamma)C)_\alpha[\beta]$ is $(\forall\boldsymbol{\nu}\delta D_\alpha[\beta])C_{\alpha,\gamma}[\beta, \boldsymbol{\nu}\delta D_\alpha[\beta]]$ or, for brevity, $(\forall\gamma')C'$. Now, $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}((\forall\gamma)C) = \mathbf{1}$ iff for every $\mathbf{d}$ in $\mathcal{D}^{\alpha}_{\mathcal{I}(\beta)}(\gamma)$, $\mathcal{I}^{\alpha\ \gamma}_{\mathcal{I}(\beta)\mathbf{d}}(C) = \mathbf{1}$, where $\mathcal{D}^{\alpha}_{\mathcal{I}(\beta)}$ denotes the domain assignment associated with $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}$. $\mathcal{I}^{\alpha\ \gamma}_{\mathcal{I}(\beta)\mathbf{d}}(C) = \mathbf{1}$ iff $\mathcal{I}^{\gamma\ \alpha}_{\mathbf{d}\,\mathcal{I}(\beta)}(C) = \mathbf{1}$ iff $\mathcal{I}^{\gamma}_{\mathbf{d}}(C_\alpha[\beta]) = \mathbf{1}$. $\mathcal{I}((\forall\gamma')C') = \mathbf{1}$ iff for every $\mathbf{d}$ in $\mathcal{D}(\gamma')$, $\mathcal{I}^{\gamma'}_{\mathbf{d}}(C') = \mathbf{1}$, that is, iff for every $\mathbf{d}$ in $\mathcal{D}(\boldsymbol{\nu}\delta D_\alpha[\beta])$, $\mathcal{I}^{\boldsymbol{\nu}\delta D_\alpha[\beta]}_{\mathbf{d}}(C_{\alpha,\gamma}[\beta, \boldsymbol{\nu}\delta D_\alpha[\beta]]) = \mathbf{1}$.

$$\begin{aligned}\mathcal{D}(\boldsymbol{\nu}\delta D_\alpha[\beta]) &= \{\mathbf{d}' \text{ in } \mathcal{D}(\delta)\colon\ \mathcal{I}^{\delta}_{\mathbf{d}'}(D_\alpha[\beta]) = \mathbf{1}\}\\ &= \{\mathbf{d}' \text{ in } \mathcal{D}^{\alpha}_{\mathcal{I}(\beta)}(\delta)\colon\ \mathcal{I}^{\delta\ \alpha}_{\mathbf{d}'\,\mathcal{I}(\beta)}(D) = \mathbf{1}\}\\ &= \{\mathbf{d}' \text{ in } \mathcal{D}^{\alpha}_{\mathcal{I}(\beta)}(\delta)\colon\ \mathcal{I}^{\alpha\ \delta}_{\mathcal{I}(\beta)\mathbf{d}'}(D) = \mathbf{1}\}\\ &= \mathcal{D}^{\alpha}_{\mathcal{I}(\beta)}(\boldsymbol{\nu}\delta D).\end{aligned}$$

That is, $\mathcal{D}^{\alpha}_{\mathcal{I}(\beta)}(\gamma) = \mathcal{D}(\gamma')$. Now, $\mathcal{I}^{\gamma}_{\mathbf{d}}(\gamma) = \mathcal{I}^{\gamma'}_{\mathbf{d}}(\gamma')$, and $\mathcal{I}^{\gamma}_{\mathbf{d}}(\beta) = \mathcal{I}^{\gamma'}_{\mathbf{d}}(\beta)$, so $\mathcal{I}^{\gamma}_{\mathbf{d}}(C_\alpha[\beta]) = \mathcal{I}^{\gamma'}_{\mathbf{d}}(C_{\alpha,\gamma}[\beta, \gamma'])$ and $\mathcal{D}^{\alpha}_{\mathcal{I}(\beta)}(\gamma) = \mathcal{D}(\gamma')$, therefore,

$$\{\mathbf{d} \text{ in } \mathcal{D}(\gamma')\colon\ \mathcal{I}^{\gamma'}_{\mathbf{d}}(C') = \mathbf{1}\} = \{\mathbf{d} \text{ in } \mathcal{D}^{\alpha}_{\mathcal{I}(\beta)}(\gamma)\colon\ \mathcal{I}^{\gamma}_{\mathbf{d}}(C_\alpha[\beta]) = \mathbf{1}\}.$$

Thus $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}((\forall\gamma)C) = \mathcal{I}((\forall\gamma')C')$, or $\mathcal{I}^{\alpha}_{\mathcal{I}(\beta)}((\forall\gamma)C) = \mathcal{I}(((\forall\gamma)C)_\alpha[\beta])$, when γ is $\boldsymbol{\nu}\delta D$. If $\boldsymbol{\nu}\delta D$ is nested in γ an inductive argument on the depth of $\boldsymbol{\nu}\delta D$ proves the theorem in this case.

Theorem **3.2** *If α and β are alphabetic variants, then $\mathcal{D}(\alpha) = \mathcal{D}(\beta)$.*

Proof: The proof is by induction on the structure of α and β. If α and β are both proper variables, then $\mathcal{D}(\alpha) = \mathbf{D} = \mathcal{D}(\beta)$. Suppose α is $\boldsymbol{\nu}\gamma C$ and β is $\boldsymbol{\nu}\delta D$. Since α and β are variants, γ is a variant of δ and D is $C_\gamma[\delta]$.

$$\mathcal{D}(\alpha) = \mathcal{D}(\boldsymbol{\nu}\gamma C) = \{\mathbf{d} \text{ in } \mathcal{D}(\gamma)\colon\ \mathcal{I}^{\gamma}_{\mathbf{d}}(C) = \mathbf{1}\}$$

and

$$\mathcal{D}(\beta) = \mathcal{D}(\boldsymbol{\nu}\delta D) = \{\mathbf{d}' \text{ in } \mathcal{D}(\delta)\colon\ \mathcal{I}^{\delta}_{\mathbf{d}'}(C_\gamma[\delta]) = \mathbf{1}\}.$$

By the induction hypothesis $\mathcal{D}(\gamma) = \mathcal{D}(\delta)$. For every $\mathbf{d}$ in $\mathcal{D}(\boldsymbol{\nu}\gamma C)$, $\mathcal{I}^{\gamma}_{\mathbf{d}}(C) = \mathbf{1}$, and since $\mathcal{I}^{\delta}_{\mathbf{d}}(\delta) = \mathbf{d}$, it follows that $\mathcal{I}^{\gamma}_{\mathcal{I}^{\delta}_{\mathbf{d}}(\delta)}(C) = \mathbf{1}$. Then by **3.1**, $\mathcal{I}^{\delta}_{\mathbf{d}}(C_\gamma[\delta]) = \mathbf{1}$, so d is in $\mathcal{D}(\boldsymbol{\nu}\,\delta D)$. A similar argument shows that $\mathcal{D}(\boldsymbol{\nu}\,\delta D) \subseteq \mathcal{D}(\boldsymbol{\nu}\,\gamma C)$ and, therefore, $\mathcal{D}(\alpha) = \mathcal{D}(\beta)$.

Theorem **3.3** *If α is nested in β, then $\mathcal{D}(\beta) \subseteq \mathcal{D}(\alpha)$.*

Proof: If β is $\boldsymbol{\nu}\alpha A$, then

$$\mathcal{D}(\beta) = \mathcal{D}(\boldsymbol{\nu}\alpha A) = \{\mathsf{d} \text{ in } \mathcal{D}(\alpha)\colon\ \mathcal{J}^{\alpha}_{\mathsf{d}}(A) = 1\}.$$

Thus any member of $\mathcal{D}(\boldsymbol{\nu}\alpha A)$ belongs to $\mathcal{D}(\alpha)$. Assume that if γ is $\boldsymbol{\nu}^{n-1}\alpha\{A_{n-1}\}$ then $\mathcal{D}(\gamma) \subseteq \mathcal{D}(\alpha)$. Now consider the variable $\boldsymbol{\nu}^{n}\alpha\{A_n\}$ which can be written in the form $\boldsymbol{\nu}\gamma A_n$ or $\boldsymbol{\nu}(\boldsymbol{\nu}^{n-1}\alpha\{A_{n-1}\})A_n$, and let β be $\boldsymbol{\nu}^{n}\alpha\{A_n\}$.

$$\mathcal{D}(\beta) = \mathcal{D}(\boldsymbol{\nu}\gamma A_n) = \{\mathsf{d} \text{ in } \mathcal{D}(\gamma)\colon\ \mathcal{J}^{\gamma}_{\mathsf{d}}(A_n) = 1\}$$

so if d is in $\mathcal{D}(\beta)$, then d belongs to $\mathcal{D}(\gamma)$. But by the induction hypothesis $\mathcal{D}(\gamma) \subseteq \mathcal{D}(\alpha)$, therefore, $\mathcal{D}(\beta) \subseteq \mathcal{D}(\alpha)$.

Theorem **3.4** *If $\mathcal{D}(\boldsymbol{\nu}\alpha A) \neq \emptyset$ then $\mathcal{J}(A_\alpha[\boldsymbol{\nu}\alpha A]) = 1$.*

Proof: If $\mathcal{D}(\boldsymbol{\nu}\alpha A) \neq \emptyset$, then $\mathcal{J}(\boldsymbol{\nu}\alpha A)$ must be a member of $\mathcal{D}(\boldsymbol{\nu}\alpha A)$ which is $\{\mathsf{d} \text{ in } \mathcal{D}(\alpha)\colon\ \mathcal{J}^{\alpha}_{\mathsf{d}}(A) = 1\}$. Then $\mathcal{J}(\boldsymbol{\nu}\alpha A)$ belongs to $\mathcal{D}(\alpha)$, and $\mathcal{J}^{\alpha}_{\mathcal{J}(\boldsymbol{\nu}\alpha A)}(A) = 1$, so by **3.1**, $\mathcal{J}(A_\alpha[\boldsymbol{\nu}\alpha A]) = 1$.

Finally, some essential definitions are included here. Let D be a set of individuals, and Π a mapping from m-place predicate variables to the set of functions from D^m to $\{0, 1\}$. A formula A is said to be *valid in* D if and only if $\mathcal{J}(A) = 1$ for every value assignment $\mathcal{J}$ defined over D. A is *valid* if it is valid in every domain D. A set S of formulas is *verifiable* if there is a domain D, and an assignment $\mathcal{J}$ defined over D, such that $\mathcal{J}(A) = 1$ for every A in S.

4 *The completeness of $\mathcal{QR}_\nu$* In this section $\mathcal{QR}_\nu$ is shown to be sound and semantically complete. A theory is *sound* if every theorem of the theory is a valid formula; $\mathcal{QR}_\nu$ is shown to be sound when it has been proved that every axiom is valid, and that the sole rule of inference, **MP**, preserves validity. A theory is *semantically complete* if every valid formula is provable.

First, the proof that the axioms of $\mathcal{QR}_\nu$ are valid:

1. Clearly, those axioms which are such by virtue of being tautologous are valid.
2. The formulas which are instances of $A_\alpha[\boldsymbol{\nu}\alpha\neg A] \rightarrow (\forall\alpha)A$ are valid in the empty domain, because in that case $\mathcal{D}(\alpha) = \emptyset$ and thus for any $\mathcal{J}$, $\mathcal{J}((\forall\alpha)A) = 1$, so whether $\mathcal{J}(A_\alpha[\boldsymbol{\nu}\alpha\neg A]) = 1$ or 0, $\mathcal{J}(A_\alpha[\boldsymbol{\nu}\alpha\neg A] \rightarrow (\forall\alpha)A) = 1$. By the same reasoning, instances of this axiom are valid in any non-empty domain for which $\mathcal{D}$ and $\mathcal{J}$ are such that $\mathcal{D}(\alpha) = \emptyset$. Thus, if A2 is not valid, there is a non-empty domain D, a domain assignment $\mathcal{D}$ and a value assignment $\mathcal{J}$ over D, such that $\mathcal{D}(\alpha) \neq \emptyset$ and $\mathcal{J}(A_\alpha[\boldsymbol{\nu}\alpha\neg A] \rightarrow (\forall\alpha)A) = 0$. Then it must be the case that $\mathcal{J}(A_\alpha[\boldsymbol{\nu}\alpha\neg A]) = 1$ and $\mathcal{J}((\forall\alpha)A) = 0$. Either $\mathcal{D}(\boldsymbol{\nu}\alpha\neg A) = \emptyset$ or $\mathcal{D}(\boldsymbol{\nu}\alpha\neg A) \neq \emptyset$. If $\mathcal{D}(\boldsymbol{\nu}\alpha\neg A) = \emptyset$, and thus $\{\mathsf{d} \text{ in } \mathcal{D}(\alpha)\colon\ \mathcal{J}^{\alpha}_{\mathsf{d}}(\neg A) = 1\}$ is empty, it follows that there is no individual d in $\mathcal{D}(\alpha)$ for which $\mathcal{J}^{\alpha}_{\mathsf{d}}(\neg A) = 1$ and,

therefore, for every d in $\mathfrak{D}(\alpha)$, $\mathfrak{I}_d^\alpha(\neg A) = 0$ or, for every d in $\mathfrak{D}(\alpha)$, $\mathfrak{I}_d^\alpha(A) = 1$. Then $\mathfrak{I}((\forall\alpha)A) = 1$, contrary to assumption. Thus $\mathfrak{D}(\boldsymbol{\nu}\alpha\neg A) \neq \emptyset$, so $\mathfrak{I}(\boldsymbol{\nu}\alpha\neg A)$ is defined and belongs to $\mathfrak{D}(\boldsymbol{\nu}\alpha\neg A)$. Now, $\mathfrak{D}(\boldsymbol{\nu}\alpha\neg A) = \{\text{d in } \mathfrak{D}(\alpha): \mathfrak{I}_d^\alpha(\neg A) = 1\}$, and since $\mathfrak{I}(\boldsymbol{\nu}\alpha\neg A)$ is in $\mathfrak{D}(\boldsymbol{\nu}\alpha\neg A)$, $\mathfrak{I}_{\mathfrak{I}(\boldsymbol{\nu}\alpha\neg A)}^{\alpha}(\neg A) = 1$ and thus $\mathfrak{I}_{\mathfrak{I}(\boldsymbol{\nu}\alpha\neg A)}^{\alpha}(A) = 0$. Then by **3.1**, $\mathfrak{I}(A_\alpha[\boldsymbol{\nu}\alpha\neg A]) = 0$, contrary to assumption. Therefore, $\mathfrak{I}(A_\alpha[\boldsymbol{\nu}\alpha\neg A] \rightarrow (\forall\alpha)A) \neq 0$, and A2 is valid.

3. Again, instances of $(\forall\alpha)(A \rightarrow B) \rightarrow ((\forall\alpha)A \rightarrow (\forall\alpha)B)$ are valid in $\emptyset$, and they are also valid in any non-empty domain D over which $\mathfrak{D}$ and $\mathfrak{I}$ are defined so that $\mathfrak{D}(\alpha) = \emptyset$. Thus, if A3 is not valid, there is a non-empty domain D and assignments $\mathfrak{D}$ and $\mathfrak{I}$ over D such that $\mathfrak{D}(\alpha) \neq \emptyset$, $\mathfrak{I}((\forall\alpha)(A \rightarrow B)) = 1$, and $\mathfrak{I}((\forall\alpha)A \rightarrow (\forall\alpha)B) = 0$. $\mathfrak{I}((\forall\alpha)A \rightarrow (\forall\alpha)B) = 0$ iff $\mathfrak{I}((\forall\alpha)A) = 1$ and $\mathfrak{I}((\forall\alpha)B) = 0$, and $\mathfrak{I}((\forall\alpha)B) = 0$ iff for some d in $\mathfrak{D}(\alpha)$, $\mathfrak{I}_d^\alpha(B) = 0$. But $\mathfrak{I}((\forall\alpha)(A \rightarrow B)) = 1$, so for this d, $\mathfrak{I}_d^\alpha(A \rightarrow B) = 1$; and since $\mathfrak{I}((\forall\alpha)A) = 1$, $\mathfrak{I}_d^\alpha(A) = 1$. But if $\mathfrak{I}_d^\alpha(A) = 1$ and $\mathfrak{I}_d^\alpha(A \rightarrow B) = 1$, $\mathfrak{I}_d^\alpha(B)$ must be 1, contrary to the above. Thus the assumption that A3 is not valid leads to a contradiction, so A3 is valid.

4. If $\mathsf{D} = \emptyset$, then for every $\mathfrak{I}$, $\mathfrak{I}((\forall\beta)f) = 1$ and $\mathfrak{I}((\forall\alpha)\neg A) = 1$, so $\mathfrak{I}(\neg(\forall\beta)f) = 0$ and $\mathfrak{I}(\neg(\forall\alpha)\neg A) = 0$. So $\mathfrak{I}(\neg(\forall\beta)f \rightarrow B_\alpha[\beta]) = 1$ and $\mathfrak{I}(\neg(\forall\alpha)\neg A \rightarrow B_\alpha[\boldsymbol{\nu}\alpha A]) = 1$, therefore,

$$\mathfrak{I}((\forall\alpha)B \rightarrow (\neg(\forall\beta)f \rightarrow B_\alpha[\beta])) = 1$$

and

$$\mathfrak{I}((\forall\alpha)B \rightarrow (\neg(\forall\alpha)\neg A \rightarrow B_\alpha[\boldsymbol{\nu}\alpha A])) = 1,$$

thus A4a and A4b are valid in $\emptyset$. If $\mathsf{D} \neq \emptyset$, but $\mathfrak{D}$ and $\mathfrak{I}$ are defined so that $\mathfrak{D}(\alpha) = \emptyset$, the same argument applies, because $\mathfrak{D}(\beta) = \mathfrak{D}(\alpha)$ and thus $\mathfrak{I}((\forall\beta)f) = 1$. So, if A4 is not valid, there is a non-empty domain D and assignments $\mathfrak{D}$ and $\mathfrak{I}$ over D such that $\mathfrak{I}((\forall\alpha)B) = 1$ and either (a) $\mathfrak{I}(\neg(\forall\beta)f) = 1$ and $\mathfrak{I}(B_\alpha[\beta]) = 0$, or (b) $\mathfrak{I}(\neg(\forall\alpha)\neg A) = 1$ and $\mathfrak{I}(B_\alpha[\boldsymbol{\nu}\alpha A]) = 0$. If $\mathfrak{D}(\alpha) \neq \emptyset$ then $\mathfrak{D}(\beta) \neq \emptyset$ and $\mathfrak{I}(\beta)$ is in $\mathfrak{D}(\alpha)$, because α and β are alphabetic variants. Since $\mathfrak{I}((\forall\alpha)B) = 1$, $\mathfrak{I}_d^\alpha(B) = 1$ for every individual d in $\mathfrak{D}(\alpha)$; in particular $\mathfrak{I}_{\mathfrak{I}(\beta)}^{\alpha}(B) = 1$. But then by **3.1**, $\mathfrak{I}(B_\alpha[\beta]) = 1$, contrary to assumption (a). If $\mathfrak{I}(\neg(\forall\alpha)\neg A) = 1$ then $\mathfrak{I}((\forall\alpha)\neg A) = 0$, so for some d in $\mathfrak{D}(\alpha)$, $\mathfrak{I}_d^\alpha(\neg A) = 0$, hence for some d in $\mathfrak{D}(\alpha)$, $\mathfrak{I}_d^\alpha(A) = 1$. Thus $\mathfrak{D}(\boldsymbol{\nu}\alpha A) \neq \emptyset$, so $\mathfrak{I}(\boldsymbol{\nu}\alpha A)$ is defined and belongs to $\mathfrak{D}(\boldsymbol{\nu}\alpha A) \subseteq \mathfrak{D}(\alpha)$, and since $\mathfrak{I}((\forall\alpha)B) = 1$, $\mathfrak{I}_{\mathfrak{I}(\boldsymbol{\nu}\alpha A)}^{\alpha}(B) = 1$, and thus $\mathfrak{I}(B_\alpha[\boldsymbol{\nu}\alpha A]) = 1$ contrary to assumption (b). Therefore, A4a and A4b must be valid.

5. Axioms of the form $(\forall\boldsymbol{\nu}\alpha A)\neg(\forall\alpha)\neg A$ are true whenever $\mathfrak{D}(\boldsymbol{\nu}\alpha A) = \emptyset$, so let D be a non-empty domain and $\mathfrak{D}$ and $\mathfrak{I}$ assignments defined over D such that $\mathfrak{D}(\boldsymbol{\nu}\alpha A)$ is not empty, and $\mathfrak{I}((\forall\boldsymbol{\nu}\alpha A)\neg(\forall\alpha)\neg A) = 0$. Then for some d in $\mathfrak{D}(\boldsymbol{\nu}\alpha A)$, $\mathfrak{I}_d^{\boldsymbol{\nu}\alpha A}(\neg(\forall\alpha)\neg A) = 0$, so for this d $\mathfrak{I}_d^{\boldsymbol{\nu}\alpha A}((\forall\alpha)\neg A) = 1$, therefore, $\mathfrak{I}_{d\,d'}^{\boldsymbol{\nu}\alpha A\,\alpha}(\neg A) = 1$ for every d′ in $\mathfrak{D}(\alpha)$. Thus there is a d in $\mathfrak{D}(\boldsymbol{\nu}\alpha A)$ such that for every d′ in $\mathfrak{D}(\alpha)$, $\mathfrak{I}_{d\,d'}^{\boldsymbol{\nu}\alpha A\,\alpha}(A) = 0$. But $\mathfrak{I}(\boldsymbol{\nu}\alpha A)$ belongs to $\mathfrak{D}(\alpha)$, so it follows that $\mathfrak{I}_{d\;\mathfrak{I}(\boldsymbol{\nu}\alpha A)}^{\boldsymbol{\nu}\alpha A\;\alpha}(A) = 0$ and, therefore, that $\mathfrak{I}_d^{\boldsymbol{\nu}\alpha A}(A_\alpha[\boldsymbol{\nu}\alpha A]) = 0$. But this is impossible by **3.4**. Therefore, A5 is valid.

6. In any domain over which $\mathfrak{D}$ and $\mathfrak{I}$ are defined so that $\mathfrak{D}(\beta) = \emptyset$,

$$\mathcal{I}((\forall \boldsymbol{\nu}\alpha A)B \rightarrow (\forall\beta)(A_\alpha[\beta] \rightarrow B_{\nu\alpha A}[\beta])) = 1.$$

So assume that $\mathbf{D}$ is not empty and $\mathcal{D}(\beta) \neq \emptyset$. If A6 is not valid in $\mathbf{D}$, then $\mathcal{I}((\forall \boldsymbol{\nu}\alpha A)B) = 1$ and

$$\mathcal{I}((\forall\beta)(A_\alpha[\beta] \rightarrow B_{\nu\alpha A}[\beta])) = 0,$$

so there is some $\mathbf{d}$ in $\mathcal{D}(\beta)$ such that $\mathcal{I}^\beta_{\mathbf{d}}(A_\alpha[\beta]) = 1$ and $\mathcal{I}^\beta_{\mathbf{d}}(B_{\nu\alpha A}[\beta]) = 0$. If $\mathcal{I}((\forall \boldsymbol{\nu}\alpha A)B) = 1$, then either (i) $\mathcal{D}(\boldsymbol{\nu}\alpha A) = \emptyset$, or (ii) $\mathcal{D}(\boldsymbol{\nu}\alpha A) \neq \emptyset$ and for every $\mathbf{d}'$ belonging to $\mathcal{D}(\boldsymbol{\nu}\alpha A)$ $\mathcal{I}^{\nu\alpha A}_{\mathbf{d}'}(B) = 1$. But in case (i), there is no individual $\mathbf{d}$ in $\mathcal{D}(\alpha)$ such that $\mathcal{I}^\alpha_{\mathbf{d}}(A) = 1$, hence no $\mathbf{d}$ in $\mathcal{D}(\beta) = \mathcal{D}(\alpha)$ such that $\mathcal{I}^\beta_{\mathbf{d}}(A_\alpha[\beta]) = 1$; therefore, $\mathcal{I}((\forall\beta)(A_\alpha[\beta] \rightarrow B_{\nu\alpha A}[\beta])) = 1$. In case (ii), $\mathcal{I}^{\nu\alpha A}_{\mathbf{d}'}(B) = 1$ for every $\mathbf{d}'$ in $\mathcal{D}(\boldsymbol{\nu}\alpha A)$. But $\mathcal{I}(\beta)$ belongs to $\mathcal{D}(\boldsymbol{\nu}\alpha A)$ since $\mathcal{I}(\beta)$ is in $\mathcal{D}(\alpha)$ and $\mathcal{I}^\alpha_{\mathcal{I}(\beta)}(A) = 1$ because $\mathcal{I}^\beta_{\mathbf{d}}(A_\alpha[\beta]) = 1$ and $\mathcal{I}^{\beta\ \alpha}_{\mathbf{d}\,\mathcal{I}(\beta)}(A) = \mathcal{I}^\alpha_{\mathcal{I}(\beta)}(A)$. Therefore, $\mathcal{I}^{\nu\alpha A}_{\mathcal{I}(\beta)}(B) = 1$ so $\mathcal{I}(B_{\nu\alpha A}[\beta]) = 1$ which contradicts the assumption that

$$\mathcal{I}((\forall\beta)(A_\alpha[\beta] \rightarrow B_{\nu\alpha A}[\beta])) = 0.$$

Therefore, A6 is valid.

7. A7 is actually a rule for generating axioms from formulas which are specified as axioms by A1 through A6. The proof that it is valid is by induction on the number of uses of the rule. Suppose that $(\forall\alpha)A$ is an axiom which has not been generated by a use of A7. Then $(\forall\alpha)A$ is an axiom by reason of schemas A1 through A6, so it is valid. Now, assume that if $(\forall\beta)B$ is an axiom by virtue of fewer than n uses of A7, then it is valid. Let $(\forall\alpha)A$ be an axiom generated by n uses of A7. If $(\forall\alpha)A$ is not valid, then there is a non-empty domain $\mathbf{D}$ and assignments $\mathcal{D}$ and $\mathcal{I}$ over $\mathbf{D}$ such that $\mathcal{D}(\alpha) \neq \emptyset$ and $\mathcal{I}((\forall\alpha)A) = 0$. Then for some $\mathbf{d}$ in $\mathcal{D}(\alpha)$, $\mathcal{I}^\alpha_{\mathbf{d}}(A) = 0$. But A must be an axiom generated by fewer than n uses of A7, so A is valid in $\mathbf{D}$, and since $\mathcal{I}^\alpha_{\mathbf{d}}$ is a value assignment over $\mathbf{D}$, $\mathcal{I}^\alpha_{\mathbf{d}}(A)$ must be 1, contrary to the above. Therefore, A7 is valid.

Finally, I need to show that the rule **MP** preserves validity, that is, if A and $A \rightarrow B$ are valid, then B is valid. If B were not valid, there would be some (possibly empty) domain $\mathbf{D}$ and a value assignment $\mathcal{I}$ over $\mathbf{D}$, such that $\mathcal{I}(B) = 0$. Since A is valid, $\mathcal{I}(A) = 1$ and, therefore, $\mathcal{I}(A \rightarrow B) = 0$. But $A \rightarrow B$ was assumed to be valid. Thus B must be valid.

That $\mathcal{QR}_\nu$ is sound is now clear. For if A is a theorem of $\mathcal{QR}_\nu$ there is a proof of it, say $B_1, \ldots, B_{n-1}, A$. If A is an axiom then the proof of A consists of the single formula A, and since A is an axiom it is valid. Suppose that $B_1, \ldots, B_{n-1}$, are valid and that A follows from B_i and B_j by **MP**. Then since **MP** preserves validity, A is valid.

The main result of this section is that every consistent set of formulas of $\mathcal{QR}_\nu$ is verifiable. The theorem that every valid formula is provable, and thus that $\mathcal{QR}_\nu$ is semantically complete, follows from this with the aid of some easily established lemmas. The proof that a consistent set S of formulas is verifiable is carried out by constructing a model of $\mathcal{QR}_\nu$ from syntactic materials; the domain of individuals consists of all those expressions which actually refer to individuals. In $\mathcal{QR}_\nu$ these expressions are among the variables, but since variables are used for quantification as well as to refer to specific individuals, some confusion can result unless the variables used for these two purposes are distinguished somehow. This is accomplished by extending the language by the addition of constant-like expressions which will be used only in free and adherent occurrences in formulas, while the variables are reserved for use in quantifiers and as $\boldsymbol{\nu}$-operator variables. The axioms will be extended to include those instances with constant terms in places occupied by free variables, and the definition of a model adapted to take the constants into account. It is from among these constant terms that the individuals of the model I construct will be obtained.

The language of $\mathcal{QR}_\nu$ is enlarged as follows, by the addition of special constants called $\boldsymbol{\sigma}$-terms, in such a way that to each variable a unique special constant is associated:

(i) For each proper variable x, x^σ is a $\boldsymbol{\sigma}$-term.
(ii) For each restricted variable, $(\boldsymbol{\nu}\alpha A)^\sigma$ is the $\boldsymbol{\sigma}$-term gotten by replacing every free or adherent variable by its corresponding $\boldsymbol{\sigma}$-term, and replacing each $\boldsymbol{\nu}$ not immediately preceded by $\forall$ or $\boldsymbol{\nu}$ by $\boldsymbol{\sigma}$.

A $\boldsymbol{\sigma}$-term has no free or adherent occurrences of a variable; any individual expression occurring in a $\boldsymbol{\sigma}$-term is either a $\boldsymbol{\sigma}$-term or a variable bound by $\forall$, $\boldsymbol{\nu}$, or $\boldsymbol{\sigma}$.

Next, the axioms are extended to include formulas with $\boldsymbol{\sigma}$-terms, which are instances of A1 through A6. In particular, the formulas

$$A_\alpha[\boldsymbol{\sigma}\alpha\neg A] \rightarrow (\forall\alpha)A,$$
$$(\forall\alpha)B \rightarrow (\neg(\forall\beta)f \rightarrow B_\alpha[\beta^\sigma]),$$
$$(\forall\alpha)B \rightarrow (\neg(\forall\alpha)\neg A \rightarrow B_\alpha[\boldsymbol{\sigma}\alpha A])$$

are axioms. The extended theory is called $\mathcal{QR}_\nu^\sigma$, and the symbol $\vdash_{\overline{\sigma}}$ means provable in $\mathcal{QR}_\nu^\sigma$.

$\mathcal{QR}_\nu^\sigma$ is a conservative extension of $\mathcal{QR}_\nu$, in the sense that if A is a formula of $\mathcal{QR}_\nu$ which is a theorem of $\mathcal{QR}_\nu^\sigma$, then A is a theorem of $\mathcal{QR}_\nu$. For if $\vdash_{\overline{\sigma}} A$, then there is a sequence of formulas of $\mathcal{QR}_\nu^\sigma$ which is a proof of A. In each formula of this sequence replace each $\boldsymbol{\sigma}$ by $\boldsymbol{\nu}$, and replace each x^σ by x, for any proper variable x. If the original formula was an axiom of $\mathcal{QR}_\nu^\sigma$, the resulting formula is an axiom of $\mathcal{QR}_\nu$, and the transformed sequence is a proof of A in $\mathcal{QR}_\nu$.

The definition of a model must be modified to take $\boldsymbol{\sigma}$-terms into account. Clearly, the level of any $\boldsymbol{\sigma}$-term ought to be 0. Since a $\boldsymbol{\sigma}$-term replaces its corresponding variable, I require that for any $\boldsymbol{\sigma}$-term α^σ, $\mathcal{I}(\alpha^\sigma)$ belongs to $\mathcal{D}(\alpha)$ whenever $\mathcal{D}(\alpha) \neq \emptyset$. If $\mathcal{D}(\alpha) = \emptyset$ then α^σ is a non-designating constant, and $\mathcal{I}(\alpha^\sigma)$ is not defined.

The following theorem can now be proved:

Theorem **4.1** *If S is a consistent set of formulas, then S is verifiable.*

Proof: Let S be a consistent set of formulas of $\mathcal{QR}_\nu$. Let S^σ be the set of formulas obtained from S by replacing each non-bound occurrence of a variable in a formula of S by its corresponding σ-term. Let $A_1, A_2, A_3, \ldots$ be a list of all the closed formulas of $\mathcal{QR}_\nu^\sigma$, that is, A_i has no non-bound occurrences of variables, although it may have free or adherent occurrences of σ-terms. Define a sequence of sets of closed formulas of $\mathcal{QR}_\nu^\sigma$ as follows:

$$S = S^\sigma,$$

$$S_{i+1} = \begin{cases} S_i \cup \{A_{i+1}\}, \text{ if } S_i \cup \{A_{i+1}\} \text{ is consistent,} \\ S_i \cup \{\neg A_{i+1}\}, \text{ if } S_i \cup \{A_{i+1}\} \text{ is inconsistent.} \end{cases}$$

Let $S_\infty = \bigcup_{i=0}^{\infty} S_i$.

Theorem **4.2** *S_i is consistent for $i = 0, 1, 2, \ldots$.*

Proof: S_o is S^σ. If S^σ is not consistent, then $S \vdash_{\overline{\sigma}} f$. But f is a formula of $\mathcal{QR}_\nu$, so the derivation of f from S^σ can be transformed into a derivation of f from S by replacing σ-terms in the formulas of the derivation by the corresponding variables. But then $S \vdash f$, which is impossible since S is consistent by hypothesis. Therefore, S^σ is consistent.

Assume that for some k greater than or equal to 1, S_{k-1} is consistent. S_k is either $S_{k-1} \cup \{A_k\}$ or $S_{k-1} \cup \{\neg A_k\}$, depending on whether or not $S_{k-1} \cup \{A_k\}$ is consistent. If S_k is $S_{k-1} \cup \{A_k\}$, then S_k is consistent. So suppose S_k is $S_{k-1} \cup \{\neg A_k\}$. In this case $S_{k-1} \cup \{A_k\}$ is inconsistent, so by definition $S_{k-1} \cup \{A_k\} \vdash_{\overline{\sigma}} f$. By the deduction theorem $S_{k-1} \vdash_{\overline{\sigma}} A_k \to f$. $(A_k \to f) \to (t \to \neg A_k)$ is a tautology, so $S_{k-1} \vdash_{\overline{\sigma}} (A_k \to f) \to (t \to \neg A_k)$ and thus, by **MP**, $S_{k-1} \vdash_{\overline{\sigma}} t \to \neg A_k$. But $S_{k-1} \vdash_{\overline{\sigma}} t$, so $S_{k-1} \vdash_{\overline{\sigma}} \neg A_k$. If S_k were inconsistent, that is, if $S_{k-1} \cup \{\neg A_k\}$ were inconsistent, it would follow by **2.1**, that $S_{k-1} \vdash_{\overline{\sigma}} A_k$. But then $S_{k-1} \vdash_{\overline{\sigma}} \neg A_k$ and $S_{k-1} \vdash_{\overline{\sigma}} A_k$, so by **2.2**, S_{k-1} is inconsistent, contrary to assumption. Thus S_k is consistent if S_{k-1} is consistent, so S_i is consistent for every i.

Theorem **4.3** *S_∞ is consistent.*

Proof: If S_∞ is not consistent, then $S_\infty \vdash_{\overline{\sigma}} f$, and there is a sequence $B_1, B_2, \ldots, B_p$ of formulas such that B_p is f, and each B_i is either a formula in S_∞, an axiom of $\mathcal{QR}_\nu^\sigma$, or the result of **MP** applied to formulas B_j and $B_j \to B_i$ preceding B_i. If B_i is an element of S_∞, then either B_i is in S^σ or B is A_{k_i} for some k_i. Let $K = 0$ if no B_i is an A_{k_i}, otherwise, let K be the largest of all the integers k_i such that B_i is A_{k_i}. Then, each B_i is an axiom, a member of S_K, or the result of **MP**, so $B_1, B_2, \ldots, B_p$ is a derivation of f from S_K, and $S_K \vdash_{\overline{\sigma}} f$. But this is impossible since S_K is consistent. Therefore, S_∞ is consistent.

Theorem **4.4** *Let A be a closed formula of $\mathcal{QR}_\nu^\sigma$. If A is not in S_∞ then $S_\infty \vdash_{\overline{\sigma}} \neg A$.*

Proof: Since A is a closed formula of $\mathcal{QR}_\nu^\sigma$, it occurs in the list $A_1, A_2, A_3, \ldots$. Suppose it is the k'th formula A_k. If A_k does not belong to S_∞, then A_k does not belong to $\bigcup_{i=0}^{\infty} S_i$, hence A_k is not a member of any S_i. In particular, A_k is not a member of S_k. But since S_k is either $S_{k-1} \cup \{A_k\}$ or $S_{k-1} \cup \{\neg A_k\}$, and A_k is not in S_k, S_k must be $S_{k-1} \cup \{\neg A_k\}$. Then $\neg A_k$ belongs to S_k, so $\neg A_k$ belongs to S_∞, and, therefore, $S_\infty \vdash_{\bar{\sigma}} \neg A_k$, or $S_\infty \vdash_{\bar{\sigma}} \neg A$.

Theorem **4.4** shows that S_∞ is a maximal consistent set of formulas.

Theorem **4.5** *If* $\neg(\forall\alpha)A$ *is a closed negated quantification such that* $S_\infty \vdash_{\bar{\sigma}} \neg(\forall\alpha)A$, *then there is a constant* s *such that* $S_\infty \vdash_{\bar{\sigma}} \neg A_\alpha[s]$.

Proof: $A_\alpha[\boldsymbol{\sigma}\alpha\neg A] \rightarrow (\forall\alpha)A$ is an axiom so $S_\infty \vdash_{\bar{\sigma}} A_\alpha[\boldsymbol{\sigma}\alpha\neg A] \rightarrow (\forall\alpha)A$. Then $S_\infty \vdash_{\bar{\sigma}} \neg(\forall\alpha)A \rightarrow \neg A_\alpha[\boldsymbol{\sigma}\alpha\neg A]$, so if $S_\infty \vdash_{\bar{\sigma}} \neg(\forall\alpha)A$, then by **MP**, $S_\infty \vdash_{\bar{\sigma}} \neg A_\alpha[\boldsymbol{\sigma}\alpha\neg A]$. $\boldsymbol{\sigma}\alpha\neg A$ is the desired constant s.

Now a model for $\mathcal{QR}_\nu^\sigma$ can be constructed. The domain of the model has as its individuals those closed expressions of $\mathcal{QR}_\nu^\sigma$ which actually designate specific individuals. These are the $\boldsymbol{\sigma}$-terms whose corresponding variables have non-empty domains; the non-emptiness of $\mathfrak{D}(\boldsymbol{\nu}\alpha A)$ is expressed by the formula $(\exists\alpha)A$, and the non-emptiness of $\mathfrak{D}(x)$ by $(\exists x)t$. Thus $\mathbf{D}$ is defined as follows:

$$\mathbf{D} = \begin{cases} x^\sigma \text{ is in } \mathbf{D} \text{ if and only if } S_\infty \vdash_{\bar{\sigma}} \neg(\forall x)f, \\ \boldsymbol{\sigma}\alpha A \text{ is in } \mathbf{D} \text{ if and only if } S_\infty \vdash_{\bar{\sigma}} \neg(\forall\alpha)\neg A. \end{cases}$$

The truth value of an atomic formula depends on whether or not the formula is derivable from S_∞. Thus the mapping Π from predicates to functions from $\mathbf{D}^m$ to $\{0, 1\}$ is defined as follows:

$$\Pi(P)(s_1, \ldots, s_m) = 1 \text{ if and only if } S_\infty \vdash_{\bar{\sigma}} P(s_1, \ldots, s_m).$$

Then $\mathfrak{D}$ and $\mathfrak{I}$ are defined inductively and simultaneously as before, except that I require $\mathfrak{I}(s) = s$ for every designating $\boldsymbol{\sigma}$-term s. If the domain of the variable corresponding to s is empty, then $\mathfrak{I}(s)$ is not defined. When $P(s_1, \ldots, s_n)$ is a closed formula which contains non-designating constants, $(s_1, \ldots, s_n)$ is not in $\mathbf{D}^n$ so $\Pi(P)$ is not defined for this n-tuple. In this case, $\mathfrak{I}(P(s_1, \ldots, s_n)) = 1$ iff $S_\infty \vdash_{\bar{\sigma}} P(s_1, \ldots, s_n)$.

If A is any closed atomic formula of $\mathcal{QR}_\nu^\sigma$, it is evident that $\mathfrak{I}(A) = 1$ if and only if $S_\infty \vdash_{\bar{\sigma}} A$. If A is $P(s_1, \ldots, s_n)$ and some s_i is non-designating, then $\mathfrak{I}(P(s_1, \ldots, s_n)) = 1$ iff $S_\infty \vdash_{\bar{\sigma}} P(s_1, \ldots, s_n)$ by definition, while if every s_i is a designating constant, then $(s_1, \ldots, s_n)$ belongs to $\mathbf{D}^n$, and

$$\begin{aligned} \mathfrak{I}(P(s_1, \ldots, s_n)) &= \Pi(P)(\mathfrak{I}(s_1), \ldots, \mathfrak{I}(s_n)) \\ &= \Pi(P)(s_1, \ldots, s_n) \\ &= 1 \text{ iff } S_\infty \vdash_{\bar{\sigma}} P(s_1, \ldots, s_n). \end{aligned}$$

In order to prove that $\mathfrak{I}(A) = 1$ iff $S_\infty \vdash_{\bar{\sigma}} A$, for any closed formula A of $\mathcal{QR}_\nu^\sigma$, the following theorems are necessary:

Theorem **4.6** *For every closed negation* $\neg A$ *of* $\mathcal{QR}_\nu^\sigma$, $S_\infty \vdash_{\bar{\sigma}} \neg A$ *if and only if it is not the case that* $S_\infty \vdash_{\bar{\sigma}} A$.

Proof: If A is not derivable from S_∞, then A does not belong to S_∞, because if it did, it would follow that $S_\infty \vdash_{\bar{\sigma}} A$. But if A is not a member of S_∞, $S_\infty \vdash_{\bar{\sigma}} \neg A$ by **4.4**. If $S_\infty \vdash_{\bar{\sigma}} \neg A$ and $S_\infty \vdash_{\bar{\sigma}} A$, then S_∞ would be inconsistent, contrary to **4.3**. Therefore, if $S_\infty \vdash_{\bar{\sigma}} \neg A$, it is not the case that $S_\infty \vdash_{\bar{\sigma}} A$.

Theorem **4.7** *For every closed implication $A \rightarrow B$ of $\mathcal{QR}_\nu^\sigma$, $S_\infty \vdash_{\bar{\sigma}} A \rightarrow B$ if and only if either $S_\infty \vdash_{\bar{\sigma}} B$ or not $S_\infty \vdash_{\bar{\sigma}} A$.*

Proof: If $S_\infty \vdash_{\bar{\sigma}} A \rightarrow B$, then either $S_\infty \vdash_{\bar{\sigma}} A$ or not $S_\infty \vdash_{\bar{\sigma}} A$, and if $S_\infty \vdash_{\bar{\sigma}} A$ then $S_\infty \vdash_{\bar{\sigma}} B$ by **MP**. Therefore, either $S_\infty \vdash_{\bar{\sigma}} B$ or not $S_\infty \vdash_{\bar{\sigma}} A$. Conversely, if not $S_\infty \vdash_{\bar{\sigma}} A$, then by **4.6**, $S_\infty \vdash_{\bar{\sigma}} \neg A$. $S_\infty \vdash_{\bar{\sigma}} \neg A \rightarrow (\neg B \rightarrow \neg A)$, so by **MP** $S_\infty \vdash_{\bar{\sigma}} \neg B \rightarrow \neg A$, and since $S_\infty \vdash_{\bar{\sigma}} (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$, $S_\infty \vdash_{\bar{\sigma}} A \rightarrow B$. If $S_\infty \vdash_{\bar{\sigma}} B$, then $S_\infty \vdash_{\bar{\sigma}} A \rightarrow B$ because $S_\infty \vdash_{\bar{\sigma}} B \rightarrow (A \rightarrow B)$.

Theorem **4.8** *If $S_\infty \vdash_{\bar{\sigma}} (\forall x)A$, then for every $\boldsymbol{\sigma}$-term α^σ such that $S_\infty \vdash_{\bar{\sigma}} \neg(\forall\alpha)f$, $S_\infty \vdash_{\bar{\sigma}} A_x[\alpha^\sigma]$.*

Proof: Suppose α^σ is y^σ. By A4a,

$$S_\infty \vdash_{\bar{\sigma}} (\forall x)A \rightarrow (\neg(\forall y)f \rightarrow A_x[y^\sigma]),$$

so if $S_\infty \vdash_{\bar{\sigma}} (\forall x)A$, then $S_\infty \vdash_{\bar{\sigma}} \neg(\forall y)f \rightarrow A_x[y^\sigma]$. But if y^σ is such that $S_\infty \vdash_{\bar{\sigma}} \neg(\forall y)f$, then $S_\infty \vdash_{\bar{\sigma}} A_x[y^\sigma]$. Suppose α^σ is $\boldsymbol{\sigma}\beta B$. By A4b,

$$S_\infty \vdash_{\bar{\sigma}} (\forall\beta)A \rightarrow (\neg(\forall\beta)\neg B \rightarrow A_\beta[\boldsymbol{\sigma}\beta B]),$$

and by **2.14**, $S_\infty \vdash_{\bar{\sigma}} (\forall x)A \rightarrow (\forall\beta)A_x[\beta]$. So if $S_\infty \vdash_{\bar{\sigma}} (\forall x)A$, then $S_\infty \vdash_{\bar{\sigma}} (\forall\beta)A_x[\beta]$, and thus $S_\infty \vdash_{\bar{\sigma}} \neg(\forall\beta)\neg B \rightarrow A_\beta[\boldsymbol{\sigma}\beta B]$. If $\boldsymbol{\sigma}\beta B$ is such that $S_\infty \vdash_{\bar{\sigma}} \neg(\forall\boldsymbol{\nu}\beta B)f$, it follows from **2.15** that $S_\infty \vdash_{\bar{\sigma}} \neg(\forall\beta)\neg B$ and, therefore, that $S_\infty \vdash_{\bar{\sigma}} A_\beta[\boldsymbol{\sigma}\beta B]$.

Theorem **4.9** *If $\boldsymbol{\nu}\alpha A$ is nested in γ and $S_\infty \vdash \neg(\forall\gamma)\neg C$, then $S_\infty \vdash_{\bar{\sigma}} A_\alpha[\boldsymbol{\sigma}\gamma C]$.*

Proof: Let γ be $\boldsymbol{\nu}^n\boldsymbol{\nu}\alpha A\{B_n\}$. In the course of the proof of **2.12**, it was shown that $\vdash(\forall\boldsymbol{\nu}\alpha A)A_\alpha[\boldsymbol{\nu}\alpha A]$. By **2.13**, $S_\infty \vdash_{\bar{\sigma}} (\forall\boldsymbol{\nu}\alpha A)A_\alpha[\boldsymbol{\nu}\alpha A] \rightarrow (\forall\gamma)A_\alpha[\gamma]$ since $\boldsymbol{\nu}\alpha A$ is nested in γ and, therefore, $S_\infty \vdash_{\bar{\sigma}} (\forall\gamma)A_\alpha[\gamma]$. By A4b,

$$S_\infty \vdash_{\bar{\sigma}} (\forall\gamma)A_\alpha[\gamma] \rightarrow (\neg(\forall\gamma)\neg C \rightarrow A_\alpha[\boldsymbol{\sigma}\gamma C]),$$

thus $S_\infty \vdash_{\bar{\sigma}} \neg(\forall\gamma)\neg C \rightarrow A_\alpha[\boldsymbol{\sigma}\gamma C]$. But if $S_\infty \vdash_{\bar{\sigma}} \neg(\forall\gamma)\neg C$ then $S_\infty \vdash_{\bar{\sigma}} A_\alpha[\boldsymbol{\sigma}\gamma C]$.

Theorem **4.10** *If A is a closed formula of $\mathcal{QR}_\nu^\sigma$, then $S_\infty \vdash_{\bar{\sigma}} A$ if and only if $\mathcal{I}(A) = 1$.*

Proof: The proof is by induction on the number of occurrences of $\neg$, $\rightarrow$, and $\forall$ in A. The theorem has already been proved for the case $n = 0$; that is, for closed atomic formulas of the form $P(s_1, \ldots, s_n)$, $S_\infty \vdash_{\bar{\sigma}} P(s_1, \ldots, s_n)$ iff $\mathcal{I}(P(s_1, \ldots, s_n)) = 1$. Assume that the theorem holds for formulas with fewer than n occurrences of $\neg$, $\rightarrow$, and $\forall$. Let A be a formula with n occurrences of $\neg$, $\rightarrow$, and $\forall$.

(i) Suppose A is $\neg B$. $\mathcal{I}(\neg B) = 1$ iff $\mathcal{I}(B) = 0$ iff it is not the case that $S_\infty \vdash_{\bar{\sigma}} B$ iff $S_\infty \vdash_{\bar{\sigma}} \neg B$, by **4.6**.
(ii) Suppose A is $B \rightarrow C$. $\mathcal{I}(B \rightarrow C) = 0$ iff $\mathcal{I}(B) = 1$ and $\mathcal{I}(C) = 0$ iff $S_\infty \vdash_{\bar{\sigma}} B$ and not $S_\infty \vdash_{\bar{\sigma}} C$ iff it is not the case that $S_\infty \vdash_{\bar{\sigma}} B \rightarrow C$, by **4.7**.

(iii) Suppose A is $(\forall\beta)B$. The proof that $S_\infty \vdash_{\overline{\sigma}} (\forall\beta)B$ iff $\mathcal{I}((\forall\beta)B) = 1$ is by induction on the structure of β. Suppose β is a proper variable x. If $S_\infty \vdash_{\overline{\sigma}} (\forall x)B$, then for every $\boldsymbol{\sigma}$-term α^σ such that $S_\infty \vdash_{\overline{\sigma}} \neg(\forall\alpha)f$, $S_\infty \vdash_{\overline{\sigma}} B_x[\alpha^\sigma]$, by **4.8**, and thus for every such $\boldsymbol{\sigma}$-term, $\mathcal{I}(B_x[\alpha^\sigma]) = 1$ by the induction hypothesis. Then by **3.1**, $\mathcal{I}^{x}_{\alpha^\sigma}(B) = 1$ because $\mathcal{I}(\alpha^\sigma) = \alpha^\sigma$. But the $\boldsymbol{\sigma}$-terms for which $S_\infty \vdash_{\overline{\sigma}} \neg(\forall\alpha)f$, or equivalently, $S_\infty \vdash_{\overline{\sigma}} \neg(\forall\gamma)\neg C$, are precisely the $\boldsymbol{\sigma}$-terms which are members of D, and since $\mathcal{D}(x) = \mathsf{D}$, $\mathcal{I}^{x}_{\alpha^\sigma}(B) = 1$ for every $\boldsymbol{\sigma}$-term α^σ in $\mathcal{D}(x)$. Therefore, $\mathcal{I}((\forall x)B) = 1$.

If it is not the case that $S_\infty \vdash_{\overline{\sigma}} (\forall x)B$, then $S_\infty \vdash_{\overline{\sigma}} \neg(\forall x)B$, and from **4.5**, $S_\infty \vdash_{\overline{\sigma}} \neg B_x[\boldsymbol{\sigma}x\neg B]$. Therefore, it is not true that $S_\infty \vdash_{\overline{\sigma}} B_x[\boldsymbol{\sigma}x\neg B]$, so by the induction hypothesis $\mathcal{I}(B_x[\boldsymbol{\sigma}x\neg B]) = 0$. But $\boldsymbol{\sigma}x\neg B$ belongs to D, which is $\mathcal{D}(x)$, because $S_\infty \vdash_{\overline{\sigma}} \neg(\forall x)B$, and $\mathcal{I}^{x}_{\boldsymbol{\sigma}x\neg B}(B) = 0$ by **3.1**; thus, there is a $\boldsymbol{\sigma}$-term $\boldsymbol{\sigma}x\neg B$ in $\mathcal{D}(x)$ such that $\mathcal{I}^{x}_{\boldsymbol{\sigma}x\neg B}(B) = 0$ and, therefore, $\mathcal{I}((\forall x)B) = 0$.

Assume as induction hypothesis for this part of the proof, that for every variable γ in which x is nested to depth less than n, iff $S_\infty \vdash_{\overline{\sigma}} (\forall\gamma)D$ then $\mathcal{I}((\forall\gamma)D) = 1$. Let A be $(\forall\boldsymbol{\nu}\gamma C)B$. If $S_\infty \vdash_{\overline{\sigma}} (\forall\boldsymbol{\nu}\gamma C)B$ then, using **MP** and A6, $S_\infty \vdash_{\overline{\sigma}} (\forall\gamma')(C_\gamma[\gamma'] \rightarrow B_{\boldsymbol{\nu}\gamma C}[\gamma'])$ for an alphabetic variant γ' of γ. By the induction hypothesis,

$$\mathcal{I}((\forall\gamma')(C_\gamma[\gamma'] \rightarrow B_{\boldsymbol{\nu}\gamma C}[\gamma'])) = 1,$$

and, therefore, for every s in $\mathcal{D}(\gamma')$,

$$\mathcal{I}^{\gamma'}_{s}(C_\gamma[\gamma'] \rightarrow B_{\boldsymbol{\nu}\gamma C}[\gamma']) = 1.$$

Then by **3.1**, for every s in $\mathcal{D}(\gamma)$,

$$\mathcal{I}(C_\gamma[s] \rightarrow B_{\boldsymbol{\nu}\gamma C}[s]) = 1.$$

Now, for s' in D, s' belongs to $\mathcal{D}(\boldsymbol{\nu}\gamma C)$ iff s' is in $\mathcal{D}(\gamma)$ and $\mathcal{I}^{\gamma}_{s'}(C) = 1$; therefore, for every s' in $\mathcal{D}(\boldsymbol{\nu}\gamma C)$,

$$\mathcal{I}(C_\gamma[s'] \rightarrow B_{\boldsymbol{\nu}\gamma C}[s']) = 1.$$

Also, since $\mathcal{I}^{\gamma}_{s'}(C) = 1$, $\mathcal{I}(C_\gamma[s']) = 1$, so for every s' in $\mathcal{D}(\boldsymbol{\nu}\gamma C)$, $\mathcal{I}(B_{\boldsymbol{\nu}\gamma C}[s']) = 1$, and thus $\mathcal{I}^{\boldsymbol{\nu}\gamma C}_{s'}(B) = 1$. Therefore, $\mathcal{I}((\forall\boldsymbol{\nu}\gamma C)B) = 1$.

If it is not the case that $S_\infty \vdash_{\overline{\sigma}} (\forall\boldsymbol{\nu}\gamma C)B$, then $S_\infty \vdash_{\overline{\sigma}} \neg(\forall\boldsymbol{\nu}\gamma C)B$, hence $S_\infty \vdash_{\overline{\sigma}} \neg B_{\boldsymbol{\nu}\gamma C}[\boldsymbol{\sigma}(\boldsymbol{\nu}\gamma C)\neg B]$, therefore, it is not the case that $S_\infty \vdash_{\overline{\sigma}} B_{\boldsymbol{\nu}\gamma C}[\boldsymbol{\sigma}(\boldsymbol{\nu}\gamma C)\neg B]$.

At this point in the proof I wish to separate from it the following theorem:

Theorem **4.11** *For every variable α which is nested in a $\boldsymbol{\sigma}$-term s in* D, *s belongs to $\mathcal{D}(\alpha)$.*

Proof: If α is a proper variable x, and s is $\boldsymbol{\sigma}xA$, then s belongs to $\mathcal{D}(x)$. Suppose α is the restricted variable $\boldsymbol{\nu}yB$ and s is $\boldsymbol{\sigma}(\boldsymbol{\nu}yB)A$. s belongs to $\mathcal{D}(y)$ and, by **4.9**, $S_\infty \vdash_{\overline{\sigma}} B_y[\boldsymbol{\sigma}(\boldsymbol{\nu}yB)A]$, therefore, $\mathcal{I}(B_y[s]) = 1$, thus $\mathcal{I}^{y}_{s}(B) = 1$ and, therefore, $\boldsymbol{\sigma}(\boldsymbol{\nu}yB)A$ belongs to $\mathcal{D}(\boldsymbol{\nu}yB)$. If α is $\boldsymbol{\nu}\gamma C$, then γ has a proper variable nested in it to a depth less than the depth y is nested in α, so by the induction assumption $\boldsymbol{\sigma}(\boldsymbol{\nu}\gamma C)A$ belongs to $\mathcal{D}(\gamma)$. Again, by **4.9**, $S_\infty \vdash_{\overline{\sigma}} C_\gamma[\boldsymbol{\sigma}(\boldsymbol{\nu}\gamma C)A]$, so letting s be $\boldsymbol{\sigma}(\boldsymbol{\nu}\gamma C)A$, $\mathcal{I}(C_\gamma[s]) = 1$ and, therefore, $\mathcal{I}^{\gamma}_{s}(C) = 1$. Thus s belongs to $\mathcal{D}(\boldsymbol{\nu}\gamma C)$.

Continuing with the proof of **4.10**, by **4.11**, $\boldsymbol{\sigma}(\boldsymbol{\nu}\gamma C)\neg B$ belongs to $\mathfrak{D}(\boldsymbol{\nu}\gamma C)$, and $\mathcal{J}(B_{\boldsymbol{\nu}\gamma C}[\boldsymbol{\sigma}(\boldsymbol{\nu}\gamma C)\neg B]) = \mathbf{0}$, therefore, $\mathcal{J}^{\boldsymbol{\nu}\gamma C}_{\boldsymbol{\sigma}(\boldsymbol{\nu}\gamma C)\neg B}(B) = \mathbf{0}$. That is, there is a $\boldsymbol{\sigma}$-term $s = \boldsymbol{\sigma}(\boldsymbol{\nu}\gamma C)\neg B$ in $\mathfrak{D}(\boldsymbol{\nu}\gamma C)$ such that $\mathcal{J}^{\boldsymbol{\nu}\gamma C}_{s}(B) = \mathbf{0}$; therefore, $\mathcal{J}((\forall\boldsymbol{\nu}\gamma C)B) = \mathbf{0}$.

It is now possible to complete the proof of **4.1** and show that S is verifiable. First, S_∞ is verifiable, because if A belongs to S_∞ then $S_\infty \vdash_{\overline{\sigma}} A$, so $\mathcal{J}(A) = \mathbf{1}$. Clearly S^σ is verifiable, since it is a subset of S_∞. Let A be a formula of S. It differs (possibly) from its corresponding formula A^σ in S^σ in having free or adherent variables in places where A^σ has $\boldsymbol{\sigma}$-terms. In the model just constructed for $\mathcal{QR}^\sigma_\nu$ the assignments for variables were not specified; the only requirement was that $\mathcal{J}(\alpha)$ be a member of $\mathfrak{D}(\alpha)$ whenever $\mathfrak{D}(\alpha) \neq \emptyset$. So the formulas of S which are not in S^σ will be verifiable if, for every α occurring free or adherent in a formula of S, $\mathcal{J}$ is defined so that $\mathcal{J}(\alpha) = \alpha^\sigma$. In this case $\mathcal{J}(A) = \mathcal{J}(A^\sigma)$ because $\mathcal{J}(\alpha) = \mathcal{J}(\alpha^\sigma)$.

Theorem **4.12** *$\mathcal{QR}_\nu$ is semantically complete.*

Proof: Let A be a valid formula of $\mathcal{QR}_\nu$. For every domain $\mathbf{D}$ and for every $\mathcal{J}$ over $\mathbf{D}$, $\mathcal{J}(A) = \mathbf{1}$, therefore, $\mathcal{J}(\neg A) = \mathbf{0}$ and $\{\neg A\}$ is not verifiable. So by the contrapositive of **4.1**, $\{\neg A\}$ is not consistent, and thus $\{\neg A\} \vdash f$. By the deduction theorem, $\vdash \neg A \rightarrow f$, and since $\vdash (\neg A \rightarrow f) \rightarrow (t \rightarrow A)$, by **MP** $\vdash t \rightarrow A$. But t is a tautology, hence $\vdash t$, and therefore $\vdash A$.

The final theorem of this section shows, as would be expected, that restricted quantifiers can be eliminated. This result is accomplished by associating with each closed formula A of $\mathcal{QR}_\nu$ a formula A^* of ordinary quantification theory, called the $\boldsymbol{\nu}$-less transform of A, and showing that $\vdash A$ if and only if $\vdash A^*$. Hailperin proved this result in [3] for his less general variables. The present proof makes use of the completeness of $\mathcal{QR}_\nu$, which has just been demonstrated, without reference to the provable equivalence of A and A^*.

Let A be a closed formula of $\mathcal{QR}_\nu$. (A thus has no free or adherent occurrences of restricted variables.) The $\boldsymbol{\nu}$-less transform of A is constructed as follows:

Assume that the proper variables have been arranged in some fixed order. Let $(\forall\alpha_1)$ be the left-most restricted quantifier occurring in A, and let B be its scope. Since α_1 is restricted it has the form $\boldsymbol{\nu}^n x\{C_n\}$ for some proper variable x and formulas $C_1, \ldots, C_n$. Replace $(\forall\boldsymbol{\nu}^n x\{C_n\})B$ in A by

$$(\forall y)(C_{1x}[y] \rightarrow C_{2\boldsymbol{\nu}xC_1}[y] \rightarrow \ldots \rightarrow C_{n\boldsymbol{\nu}^{n-1}x\{C_{n-1}\}}[y] \rightarrow B_\alpha[y]),$$

where y is the first proper variable not occurring in $(\forall\alpha_1)B$. The resulting formula has one less restricted quantifier than A. If A has only one restricted quantifier then this resulting formula is A^*. If A has more than one restricted quantifier then repeat the process on the resulting formula, moving from left to right. Assuming that A has n restricted quantifiers, at the end of n replacements a formula without restricted quantifiers is obtained; this formula is A^*, the $\boldsymbol{\nu}$-less transform of A. Assuming a fixed ordering of proper variables, this A^* is uniquely determined. As an immediate consequence of the construction of A^*, the following hold:

$(A^*)^*$ is A^*, and if A is $\boldsymbol{\nu}$-less then A^* is A,
$(A \to B)^*$ is $A^* \to B^*$,
$(\neg A)^*$ is $\neg A^*$,
$((\forall x)B)^*$ is $(\forall x)B^*$,
$((\forall \alpha)B)^*$ is $(\forall y)(A^*_{1x}[y] \to \ldots \to A^*_{n\boldsymbol{\nu}^{n-1}x\{A_n\}}[y] \to B^*_\alpha[y])$, where α is $\boldsymbol{\nu}^n x\{A_n\}$.

By reason of the completeness theorem, $\vdash A$ if and only if A is valid, and $\vdash A^*$ if and only if A^* is valid; thus, in order to prove $\vdash A$ iff $\vdash A^*$, it is sufficient to show that A is valid iff A^* is valid. If A were valid and A^* were not valid, there would be a domain $\mathbf{D}$ and an interpretation $\mathcal{J}$ over $\mathbf{D}$ such that $\mathcal{J}(A) = 1$ and $\mathcal{J}(A^*) = 0$; thus I shall show that for every $\mathbf{D}$ and every $\mathcal{J}$ over $\mathbf{D}$, $\mathcal{J}(A) = \mathcal{J}(A^*)$.

Let A be a closed formula of $\mathcal{QR}_\nu$, $\mathbf{D}$ a domain of individuals, and $\mathcal{J}$ a value assignment and $\mathcal{D}$ a domain assignment over $\mathbf{D}$. It is clear from the facts stated above about the $\boldsymbol{\nu}$-less transform of formulas built up from the logical symbols $\neg$, $\to$, and $\forall$, that it is sufficient to consider only formulas whose initial symbol is a restricted quantifier. The proof proceeds by induction on the number of restricted quantifiers in A. If A has no restricted quantifier then A^* is A, so $\mathcal{J}(A) = \mathcal{J}(A^*)$. Assuming that the theorem holds for formulas with fewer than n restricted quantifiers, let A be a formula with n restricted quantifiers. Suppose A is $(\forall\alpha)B$, where B has fewer than n quantifiers and α is $\boldsymbol{\nu}^n x\{C_n\}$. From the above, $((\forall\alpha)B)^*$ is

$$(\forall y)(C^*_{1x}[y] \to C^*_{2\boldsymbol{\nu}xC_1}[y] \to \ldots \to C^*_{n\boldsymbol{\nu}^{n-1}x\{C_{n-1}\}}[y] \to B^*_\alpha[y]).$$

$\mathcal{J}((\forall\alpha)B) = 0$ iff for some individual $\mathbf{d}$ in $\mathcal{D}(\alpha)$, $\mathcal{J}^\alpha_{\mathbf{d}}(B) = 0$. $\mathcal{D}(\alpha) = \mathcal{D}(\boldsymbol{\nu}^n x\{C_n\})$, and

$$\mathcal{D}(\alpha) \subseteq \mathcal{D}(\boldsymbol{\nu}^{n-1}x\{C_{n-1}\}) \subseteq \ldots \subseteq \mathcal{D}(\boldsymbol{\nu}xC_1) \subseteq \mathcal{D}(x),$$

so if $\mathbf{d}$ is in $\mathcal{D}(\alpha)$, then $\mathbf{d}$ belongs to $\mathcal{D}(\boldsymbol{\nu}^i x\{C_i\})$ for $i = 1, 2, \ldots, n - 1$, and $\mathbf{d}$ belongs to $\mathcal{D}(x)$. Therefore, $\mathcal{J}^x_{\mathbf{d}}(C_1) = 1$ and $\mathcal{J}^{\boldsymbol{\nu}^i x\{C_i\}}_{\mathbf{d}}(C_{i+1}) = 1$ for $i = 1, 2, \ldots, n - 1$, so $\mathcal{J}^y_{\mathbf{d}}(C_{1x}[y]) = 1$ and $\mathcal{J}^y_{\mathbf{d}}(C_{i+1\boldsymbol{\nu}^i x\{C_i\}}[y]) = 1$ for $i = 1, 2, \ldots, n - 1$. But each C_i, as well as B, has fewer than n restricted quantifiers and, therefore, $\mathcal{J}^y_{\mathbf{d}}(B^*_\alpha[y]) = 0$, $\mathcal{J}^y_{\mathbf{d}}(C^*_{1x}[y]) = 1$ and $\mathcal{J}^y_{\mathbf{d}}(C^*_{i+1\boldsymbol{\nu}^i x\{C_i\}}[y]) = 1$ for $i = 1, 2, \ldots, n - 1$. Therefore,

$$\mathcal{J}^y_{\mathbf{d}}(C^*_{1x}[y] \to \ldots \to C^*_{n\boldsymbol{\nu}^{n-1}x\{C_{n-1}\}}[y] \to B^*_\alpha[y]) = 0$$

for some $\mathbf{d}$ in $\mathcal{D}(y)$, so

$$\mathcal{J}((\forall y)(C^*_{1x}[y] \to \ldots \to C^*_{n\boldsymbol{\nu}^{n-1}x\{C_{n-1}\}}[y] \to B^*_\alpha[y])) = 0,$$

and thus $\mathcal{J}(((\forall\alpha)B)^*) = 0$.

A similar argument shows that $\mathcal{J}((\forall\alpha)B) = 0$ if $\mathcal{J}(((\forall\alpha)B)^*) = 0$, and thus the following theorem is proven:

Theorem **4.13** *If A is a closed formula of $\mathcal{QR}_\nu$ and A^* is its $\boldsymbol{\nu}$-less transform, then $\vdash A$ if and only if $\vdash A^*$.*

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Lehigh University
Bethlehem, Pennsylvania

and

Indiana University at South Bend
South Bend, Indiana