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LEŚNIEWSKI'S ANALYSIS OF RUSSELL'S ANTINOMY

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According to Sobociński, Leśniewski's analysis of Russell's antinomy ". . . fût point de départ pour la construction du système des fondements des mathématiques de Leśniewski . . .," and it ". . . déterminait, pour ainsi dire, le caractère des théories deductives comprises dans son système."[1] Leśniewski's theory of collective classes, which came to be called "Mereology," is a direct result of his first analysis of the antinomy. In 1914 he published this analysis in his paper "Czy klasa klas, niepodporządkowanych sobie, jest podporządkowana sobie?" (Is the class of classes which are not subordinate to themselves subordinate to itself?) in the Polish journal *Przegląd Filozoficzny* (Philosophical Review), XVII, pp. 63-75, and a year later he created his first axiomatization of Mereology which formalized the fundamental concepts he used in his earlier analysis of the antinomy.[2] During the period 1913-1914 he arrived at another analysis of the antinomy, and some of the results of this analysis were published in [27]. In [49-50] Sobociński recounted Leśniewski's third (unpublished) analysis of the antinomy.

Speaking of his 1914 paper, "Czy klasa klas . . ., " Leśniewski said in 1927: "In this poor paper I expressed my views on Russell's antinomy. Not yet having my own axiomatic theory of classes, I there appealed from case to case to various theses of this discipline in which I believed and which were necessary for my analyses. My procedure was in this respect completely similar to the procedure of all those 'set theorists' who do not construct their work on clear, axiomatic foundations."[3] Despite the disclaimer, this paper is a seminal work: Leśniewski's use of "is" (Polish "jest") foreshadows his use of "ε" in singular propositions of Ontology, and for the first time he introduces and uses the nuclear concept of Mereology, the concept of a collective class, a class literally constituted by its members.[4]

As mentioned above, Leśniewski's third analysis of Russell's antinomy has been published by Sobociński in [49-50]. Luschei in [62] summarized Leśniewski's second analysis, which was published in [27]. However, the historically important first analysis of 1914, the analysis which determined

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the character of Leśniewski's later logical theories, is not accessible to those who do not read Polish.[5] This analysis should be of interest not only to those concerned with Leśniewski's Ontology and Mereology but also to those concerned with the antinomy. My purpose here is to state and explain the main points of Leśniewski's 1914 paper "Czy klasa klas, niepodporządkowanych sobie, jest podporządkowana sobie?" in order to reveal some of the factors which determined the form of Mereology, and to help stimulate interest in his work.

In [27] Leśniewski said that he became acquainted with Russell's antinomy in 1911 when he read Łukasiewicz's formulation of it in the latter's *O zasadzie sprzeczności u Arystotelesa* (The Principle of Contradiction in Aristotle), published in 1910. For more than eleven years the problems connected with the antinomies became the most persistent subject of his reflections. He began to reflect on examples of situations in which in practice he considered or did not consider an object to be a class or set of objects, and it was from this perspective that he began his analysis of the assumptions of the antinomy. He reported in [27] that "The problem of 'empty classes' did not constitute a subject of my investigations on this occasion, since I treated the concept of 'empty classes' from the very first time when I encountered it as a 'mythological' concept, taking without any hesitation the position that if some object is the class of objects a, then some object is a." Leśniewski began his paper "Czy klasa klas . . .," which hereafter will also be referred to as "[14]", saying:

> To the question, is the class of classes which are not subordinate to themselves subordinate to itself? it may be that the affirmative answer "it is subordinate to itself" is true, or that the negative answer "it is not subordinate to itself" is true—but only if some object is the class of classes which are not subordinate to themselves; if no object is the class of classes which are not subordinate to themselves, then the expression "the class of classes which are not subordinate to themselves" does not denote any object, and consequently every sentence in which this expression occurs as the subject is a false sentence.

At this point in the discussion Leśniewski refers to a part of his earlier paper "Krytyka logicznej zasady wyłączonego środka" (Critique of the logical principle of excluded middle), published in 1913, where he introduces some general statements which, he says, may be considered as a formulation of two "formal" conditions for the truth of any sentence: 1) a true sentence always has a subject which denotes something, 2) a true sentence always has a connoting predicate. On the basis of these conditions he asserted: every sentence whose subject does not denote anything is a false sentence. He gives the following as examples: "every centaur has a tail," "some centaur does not have a tail," "every square circle is a circle," "some square circle is not a circle." He says that the subject of the first two sentences is "centaur," while of the last two it is "square circle."[6] Thus it would seem that when Leśniewski says in [14] that if the expression "the class of classes which are not subordinate to themselves" does not denote any object, then every sentence in which this expression occurs as subject is a false sentence, he is applying the principle, stated in [13], that every sentence whose subject does not denote anything is a false sentence. Those familiar with Leśniewski's Ontology, which was not formulated until 1920, will recall that in Ontology a singular sentence of the form "$A \varepsilon b$" is false if the term substituted for "A" is not a uniquely referring expression, i.e., every sentence of the form "$A \varepsilon b$" whose substituend for "A" does not denote uniquely anything is a false sentence. However, from the contexts cited above it is not entirely clear whether Leśniewski in [14] was anticipating his later analysis of singular sentences of Ontology. (It should be noted that in [13] and [14] Leśniewski did not relativize his arguments to any formalized language, and that both are written in colloquial Polish with a minimum of symbolism.)

If the expression "the class of classes which are not subordinate to themselves" does not denote any object, then both "the class of classes which are not subordinate to themselves is subordinate to itself" as well as "the class of classes which are not subordinate to themselves is not subordinate to itself" will be false sentences, and if both are false, then Russell's antinomy cannot be generated. For Leśniewski, to answer the question, is the class of classes which are not subordinate to themselves subordinate to itself?, it must first be determined whether any object is the class of classes which are not subordinate to themselves, and an answer to the question, is any object the class of classes which are not subordinate to themselves?, depends upon an answer to the prior question, is any object a class which is not subordinate to itself? According to Leśniewski, [14], "If no object were a class which is not subordinate to itself, then, of course, no object would be, likewise, some set of classes which are not subordinate to themselves, for example, a set of all classes which are not subordinate to themselves, i.e., precisely a class of classes which are not subordinate to themselves. (By a class of objects a I mean here the set of all objects which are a; the class of classes which are not subordinate to themselves would be, according to this conception, just the set of all classes which are not subordinate to themselves.)"[7]

At this point Leśniewski's strategy is to prove that no object is a class which is not subordinate to itself, and since a proof of this fact is also a proof that no object is the class of classes which are not subordinate to themselves, he will have shown thereby that the assumptions leading to Russell's antinomy are false. According to Leśniewski [14], p. 64:

> I call any object P an object subordinate to a class K if in some sense of the expression "a" the following two conditions hold: 1) K is a class (of objects) a,[8] 2) P is an a.[9] Examples: A) any man C is subordinate to the class of men because if the expression "a" is used in the sense of the expression "man," then both of the conditions indicated above hold, i.e., 1) the class of men is a class (of objects) a, 2) the man C is an a; B) any half P of the sphere Q is subordinate to the class of halves of the sphere Q, since if the expression "a" is used in the sense of the expression "half of the sphere Q," then 1) the class of halves of the sphere Q is a class (of

> objects) a, 2) the half P of the sphere Q is an a. Since the class of halves of the sphere Q is just the sphere Q itself, the latter is not only the class of all of its halves, but also the class of all of its quarters, i.e., the class of halves of the sphere Q is also the class of quarters of the sphere Q, and conversely; thus, C) any half P of the sphere Q is subordinate to the class of quarters of the sphere Q because if the expression "a" is used in the sense of the expression "half of the sphere Q," then 1) the class of quarters of the sphere Q is a class (of objects) a, 2) the half P of the sphere Q is an a.

Given the above sense of the expression "object subordinate to the class K," an elephant S is not subordinate to the class of men because in no sense of the expression "a" do the following two conditions hold: 1) the class of men is a class (of objects) a, 2) an elephant S is an a. Nor is a man C subordinate to the class of human heads since in no sense of the expression "a" do the following two conditions hold: 1) the class of human heads is a class (of objects) a, 2) a man C is an a. According to Leśniewski [14], pp. 65-66:

> . . . one might think, by analogy to the sense of the expression "object subordinate to the class K," [specified above] that in order for some object P' not to be subordinate to the class K it suffices that the conditions mentioned [1) K is a class (of objects) a, 2) P' is an a] not hold in *some* sense of the expression "a." However, such a position would imply a contradiction because it would follow from it that one and the same object P could be subordinate to the class K and at the same time not subordinate to the class K. Example: a half P of the sphere Q is, as we know . . . , subordinate to the class of quarters of the sphere Q because if the expression "a" is used in the sense of the expression "half of the sphere Q," then both conditions hold: 1) the class of quarters of the sphere Q is a class (of objects) a, 2) the half P of the sphere Q is an a. However, the very same half P of the sphere Q is not subordinate to the class of quarters of the sphere Q, since in *some* sense of the expression "a," namely when the expression is used in the sense of the expression "quarter of the sphere Q," the first condition [the class of quarters of the sphere Q is a class (of objects) a] indeed holds; however, the half P of the sphere Q is not an a, i.e., the second condition does not hold. In this case the half P of the sphere Q would be both subordinate and not subordinate to the class of quarters of the sphere Q.

Leśniewski noted that some readers would find it theoretically shocking that by assuming the above sense of the expression "object subordinate to the class K" some half of the sphere Q may be subordinate to the class of quarters of the sphere Q, despite the fact that no half of the sphere Q is a quarter of the sphere Q. It might be objected more generally: it does not follow that some object which is not an a should nevertheless be subordinate to the class of objects a. And it might be thought that this shock would be avoided if the expression "object subordinate to the class K" were to denote some object P not when in *some* sense of the expression "a" the two conditions above stated hold, but rather when these conditions hold in *every* sense of the expression "a." Leśniewski observed that with this modification:

> . . . a half *P* of the sphere *Q* would be not subordinate to the class of quarters of the sphere *Q*, since it is not the case that in *every* sense of the expression "*a*"—1) the class of quarters of the sphere *Q* is a class (of objects) *a*, and, at the same time, 2) the half *P* of the sphere *Q* is an *a*. In *some* sense of the expression "*a*," as, for example, when the expression "*a*" is used in the sense of the expression "quarter of the sphere *Q*," the first condition holds (the class of quarters of the sphere *Q* is a class of quarters of the sphere *Q*), but the second condition does not hold, since the half *P* of the sphere *Q* is not an *a*. ([14], pp. 66-67)

However, as Leśniewski pointed out, this modification in the sense of the expression "object subordinate to the class *K*" entails that no object is subordinate to any class since any object *P* is with respect to any class *K* in a relation such that in *some* sense of the expression "*a*" both conditions do not hold: 1) *K* is a class (of objects) *a*, 2) *P* is an *a*. That is, no object *P* is with respect to any class *K* in a relation such that in every sense of the expression "*a*" both of the conditions mentioned hold. Leśniewski's proof of this assertion is as follows:

> Let us assume that some object *P* is subordinate to the class *K*. Hence, it follows that in *every* sense of the expression "*a*" these conditions hold: 1) *K* is a class (of objects) *a*, 2) *P* is an *a*. Let us use the expression "*a*" in the sense of the expression "square circle"; of course conditions 1) and 2) hold, and in just this sense of the expression "*a*," since they hold in *every* sense of this expression. Therefore, 1) *K* is a class of square circles, 2) *P* is a square circle. The last sentence is a contradictory sentence, and as a contradictory sentence it is a false sentence. So the assumption which leads to this false consequence must also be false, that is, the assumption that some object *P* is subordinate to some class *K*; and since the assumption that an object is subordinate to some class is a fallacious assumption, no object is subordinate to any class. ([14], p. 67)

Consequently, the proposed modification in the sense of the expression "object subordinate to the class *K*" must be rejected. It would seem, however, that a slight change in the proposal just rejected would eliminate the fallacious consequences derived above, and at the same time not imply the "paradoxical" consequences of Leśniewski's sense of the expression "object subordinate to the class *K*." Consider the proposal that the expression "object subordinate to the class *K*" will be used to denote some object *P* when in every sense of the expression "*a*" both conditions 1) and 2) hold provided one of these conditions holds. However, this proposal also leads to the conclusion that no object is subordinate to any class. Leśniewski's proof of this is as follows:

> I assume for the proof that some object *P* is subordinate to a class *K*. On the basis of the principle of simplification I can affirm that *P* is either *P* or not *P*; thus, if I use the expression "*a*" in the sense of the expression "either *P* or not *P*," then *P* is *a*. Thus, one of the two conditions—1) *K* is a class (of objects) *a*, 2) *P* is *a* (in this sense of the expression "*a*")—holds, since the object *P* is subordinate to the class *K*. Therefore, if one of the conditions holds, then both hold. Thus, in this sense of the expression "*a*"

condition 1)—K is a class (of objects) a—also holds. Substituting for the expression "a" its meaning, "either P or not P," we obtain: K is a class (of objects) which are either P or not P. On the basis of the ontological principle of excluded middle, every object is either P or not P, the class (of objects) which are either P or not P is, thus, the set of all objects, i.e., a universe. Consequently, K is a universe. Since (in every sense of the expression "a") the expressions "class (of objects) a" and "class of classes (of objects) a" are two different symbols for one and the same object, *viz.*, an object which is the set of all a's, it follows that the expression "class of classes (of objects) which are either P or not P" is a symbol for the same object that the expression "class (of objects) which are either P or not P" is. Therefore, K, being a class (of objects) which are either P or not P, is identical with the class of classes (of objects) which are either P or not P. If, then, we use the expression "a" in the sense of the expression "class (of objects) which are either P or not P," then K is a class (of objects) a. Thus, in this sense of the expression "a" condition 1) holds, and since this is the case, the second condition also holds, that is, P is a. Substituting for the expression "a" its present sense, that is, the expression "class (of objects) which are either P or not P," we obtain: P is a class (of objects) which are either P or not P. However, we already know that the class (of objects) which are either P or not P is a universe, and consequently the object P, which is the class (of objects) which are either P or not P, is a universe (I). Let us consider the class of objects which are not universes. This is the set of *all* objects, each of which is not a universe. Of course, such a set is a universe. We can, then, consider a universe as a class of objects which are not universes. Thus, if we use the expression "a" in the sense of the expression "object which is not a universe," then a universe is a class (of objects) a. We already know that K is a universe; consequently, it follows that also in this new sense of the expression "a" K is a class (of objects) a. So then in this sense of the expression "a" condition 1) of conditions 1) and 2) also holds. Hence, the second condition also holds, that is, P is an a. Substituting for the expression "a" its new sense, that is, the expression "object which is not a universe," we obtain: P is an object which is not a universe, that is, P is not a universe (II). Comparing theses (I) and (II), we have, on the basis of the principle of composition: P is a universe and not a universe. The last sentence is a contradictory sentence, and as such is false. Hence the assumption which leads to this false consequence, that is, the assumption that some object P is subordinate to some class K, must also be false. And since the assumption that some object is subordinate to some class is a fallacious assumption, no object is subordinate to any class. ([14], pp. 68-70)

The preceding argument shows that the proposed modification (i.e., the proposal that the expression "object subordinate to the class K" be used to denote some object P when in every sense of the expression "a" both conditions 1) and 2) hold provided one of these conditions holds) must be rejected since it too entails that no object is subordinate to any class.

In the immediately preceding long passage from [14] it will be noted that Leśniewski asserts without argument that ". . . the expressions 'class (of objects) a' and 'class of classes (of objects) a' are two different symbols for one and the same object, *viz.*, an object which is the set of all *a*'s . . ." If so, it would seem to follow that for Leśniewski the class (of objects) *a* is identical with the class of classes (of objects) *a*. Thus, in [14], before his axiomatization of Mereology, Leśniewski was appealing to what came to be one of the most characteristic properties of collective classes. In the later, [29], axiomatization of Mereology the following two theorems jointly express this characteristic property:

Theorem LXXII: *If P is the class of objects a, then P is the class of classes of objects a.*

Theorem XCVII: *If P is the class of classes of objects a, then P is the class of objects a.*

In connection with these theorems the following four theorems should also be noted. Leśniewski's [28], a recapitulation of his [16], contains the following two theorems:

Theorem XXIV: *If P is the class of sets of objects a, then P is the class of objects a.*

Theorem XXV: *If P is the class of objects a, then P is the class of sets of objects a.*

("Set" is used here according to Leśniewski's Definition III: *P* is the set of objects *a* if and only if the following conditions are satisfied: α) *P* is an object; β) for all *Q*—if *Q* is an ingredient of the object *P*, then some ingredient of the object *Q* is an ingredient of some *a*, which is an ingredient of the object *P*.) Lesniewski's [29] contains the following two theorems:

Theorem LXXIX: *If P is the set of sets of objects a, then P is the set of objects a.*

Theorem IC: *If P is the set of classes of objects a, then P is the class of objects a.*

In Sobociński's [49-50], vol. II, p. 243, fourteen theses establishing the most characteristic and elementary properties of the terms "class" and "element" (in the collective sense of "class") are given; the sixth thesis is "$[A\ a] : A \varepsilon \mathsf{Kl}(a)\ .\!\equiv\!.\ A \varepsilon \mathsf{Kl}(\mathsf{Kl}(a))$," wherein "ε" is the epsilon of Leśniewski's Ontology.

Having disposed of the two proposed modifications of the sense of the expression "object subordinate to the class *K*," Leśniewski draws two consequences "of prime importance for the theory of classes" from his definition of the expression "object subordinate to the class *K*." The first is that every object *n* is subordinate to the class (of objects) *n*, while the second is that not every object which is subordinate to the class (of objects) *n* is an *n*. He offers a proof of the first assertion:

> Let us assume that some object *P* is an *n*. Let us use the expression "*a*" in the sense of the expression "*n*." In this sense of the expression "*a*"— 1) the class (of objects) *n* is the class (of objects) *a*, 2) *P* is an *a*. Since the two conditions just now specified hold in *some* sense of the expression "*a*," the object *P* is subordinate to the class (of objects) *n*.

His proof of the second assertion is as follows:

> If every object subordinate to the class (of objects) *n* were an *n*, then every object subordinate to the class of quarters of the sphere *Q* would be a quarter of the sphere *Q*. As we know . . . any half *P* of the sphere *Q* is subordinate to the class of quarters of the sphere *Q*; from whence it would follow, therefore, that a half *P* of the sphere *Q* is a quarter of the sphere *Q*, which is obviously false. And consequently the initial hypothesis that every object subordinate to the class (of objects) *n* is an *n* must be false. Consequently, *not every object subordinate to the class (of objects) n is an n.* ([14], p. 70)

Leśniewski maintained that keeping in mind these two simple consequences of his definition of "object subordinate to the class *K*" ". . . is exceedingly important as a protective device against certain widespread methods of false inference in the theory of classes." These two fundamental concepts came to be expressed as theorems in [28]:

Theorem XX: *If P is the class of objects a, then every a is an element of the object P.*

Theorem XXII: *If P is part of the object Q, then not (for all R and a—if R is an element of some set of objects a, then R is an a).*

In a footnote to Theorem XXII, Leśniewski refers to his [16], Theorem XXVII, saying that this theorem states: "The theorem 'if *P* is an element of the set of objects *m*, then *P* is an *m*' is false." He adds:

> Today, instead of this clumsy sentence I would say in a similar situation: "not (for all *P* and *m*—if *P* is an element of some set of objects *m*, then *P* is an *m*)." In the proof of Theorem XXVII I relied on the assumption, about which I had no doubts (although I did not prove it in my "general theory of sets" [i.e., [16]]), that some object is a part of some object. In the present exposition I do not reiterate this error any longer, and I express my present theorem XXII in the form of a conditional sentence, which is equivalent to the sentence "if some object is a part of some object, then not (for all *P* and *m*—if *P* is an element of some set of objects *m*, then *P* is an *m*)." Further on we find that by assuming there are at least two different objects, the thesis asserting that some object is a part of some object is easily proved.

In [14] Leśniewski said that he used the expression "class subordinate to itself" to denote "any class *K* which is subordinate to the class *K*. Thus considering the sense of the expression 'object subordinate to the class *K*,' specified [above], I can say that I call any class *K* a class subordinate to itself if in some sense of the expression "*a*" the following two conditions hold: 1) *K* is a class (of objects) *a*, 2) *K* is an *a*." Leśniewski gives the following as examples. The class of objects which are now in my room is a class subordinate to itself because if we use the expression "*a*" in the sense of the expression "object which is now in my room," then 1) the class of objects which are now in my room is a class (of objects) *a*, 2) the class of objects which are now in my room is an *a* (this class is itself an object which is now in my room). As another example he gives the following. The class of classes is a class subordinate to itself because if we use the expression "a" in the sense of the expression "class," then 1) the class of classes is a class (of objects) a, 2) the class of classes is an a.

Leśniewski uses the expression "class not subordinate to itself" to denote any class K' which is not subordinate to the class K'. On the basis of his analysis of the expression "object not subordinate to the class K," he can say: ". . . I call any class K' a class which is not subordinate to itself if in no sense of the expression 'a' do the following two conditions hold: 1) K' is a class (of objects) a, 2) K' is an a." In Sobociński [49-50], vol. I, p. 101, the expression "A is a class which is not subordinate to itself" is defined as follows:

$$[A] \therefore A \, \varepsilon \, * \, .\equiv: A \, \varepsilon A \; : [a] : A \, \varepsilon \, \mathsf{Kl}(a) \; .\supset. \sim(A \; \varepsilon \; a),$$

where "ε" functions as the epsilon of Leśniewski's Ontology. It should be noted that nothing like the preceding *formula* occurs in Leśniewski's [14]; the discussion in [14] is not couched in terms of any "formalized" language. As indicated earlier, Leśniewski's [14] is written in Polish with a minimum of symbols. Nevertheless, as we have seen on numerous occasions above, many of Leśniewski's assertions in [14] are easily translated and incorporated into the "formalized" languages he subsequently formulated. (Those familiar with Leśniewski's logical systems recognize the need for "$A \, \varepsilon A$" in the above formula; it guarantees that A is an object, and precludes the derivation of contradictions.)

While Leśniewski was able to give examples of classes which *are* subordinate to themselves, he was not able to give examples of classes which are *not* subordinate to themselves because his analysis revealed that there are no such classes. According to Leśniewski, "*every class is subordinate to itself.*" This assertion from [14] may be compared to Theorem XIX of [28], which states "If P is an object, then P is an element of the object P." As Leśniewski pointed out, this theorem is weaker than Theorem XIV of his [16], which states "Every object is its own element." Theorem XIX of [28] may be stated more formally (using the epsilon of Ontology) as "$[P] : P \, \varepsilon \, P \; .\supset. \; P \, \varepsilon \, \mathsf{el}(P)$."

Leśniewski's proof of the assertion that every class is subordinate to itself is as follows ([14], pp. 71-72):

> . . . let us assume that some class K is not subordinate to itself. This means . . . that
>
> (I) in no sense of the expression "a" do both conditions—1) K is a class (of objects) a, 2) K is an a—hold.
>
> The class K is necessarily a class of some objects n. Let us denote the class of objects n by the sign "Σn." On the basis of the law of tautology
>
> (II) $\Sigma n = \Sigma n + \Sigma n$.
>
> As I have already remarked . . . the expression "class of objects n" is a

symbol for the same object that the expression "class of classes of objects n" is; in other words, the expression "Σn" is a symbol for the same object that the expression "$\Sigma\Sigma n$" is. Substituting the expression "$\Sigma\Sigma n$" into the formula

(II) for one of the expressions "Σn" we obtain

(III) $\Sigma n = \Sigma\Sigma n + \Sigma n$.

It is known that the logical sum of two classes, one of which is the class (of objects) a, and the other the class (of objects) b, is the class (of objects) a or b. Consequently, $\Sigma\Sigma n + \Sigma n$, that is, the logical sum of the two classes $\Sigma\Sigma n$ and Σn, one of which is the class of objects Σn, and the other the class of objects n, is the class (of objects) Σn or n. In other words, (IV) $\Sigma\Sigma n + \Sigma n = \Sigma(\Sigma n$ or $n)$.

Since $\Sigma n = \Sigma\Sigma n + \Sigma n$ (III), and $\Sigma\Sigma n + \Sigma n = \Sigma(\Sigma n$ or $n)$, (IV), then

(V) $\Sigma n = \Sigma(\Sigma n$ or $n)$.

And since K is Σn,

(VI) K is $\Sigma(\Sigma n$ or $n)$.

Since K is Σn, therefore, on the basis of the principles of simplification and syllogism

(VII) K is Σn or n.

Let us use the expression "a" in the sense of the expression "Σn or n." Substituting the expression "a" for the expression "Σn or n" in (VI) and (VII) we obtain

(VIII) K is Σa,

and

(IX) K is a.

Therefore,

(X) In some sense of the expression "a," *viz.*, when the expression "a" is used in the sense of the expression "Σn or n," both conditions— 1) K is a class (of objects) a, 2) K is an a—hold.

Comparing statements (I) and (X) we note that these statements are contradictory sentences. Consequently, from the assumption admitted at the beginning of the present paragraph, that some class K is not subordinate to itself, it follows that two contradictory sentences are true. Since the logical principle of contradiction prevents us from accepting such a conclusion, the assumption (that some class K is not subordinate to itself) leading to it must be false. And in view of this, *every class is subordinate to itself.* (Consequently, those classes, as, for example, the class of men, which hitherto were considered as being not subordinate to themselves, are in fact subordinate to themselves. If we use the expression "a" in the sense of the expression "man or class of men," then 1) the class of men is a class (of objects) a, 2) the class of men is a.)

Since every class is subordinate to itself, *no object is a class which is not subordinate to itself.* It will be recalled that according to Leśniewski, if no object is a class which is not subordinate to itself, then no object is the class of classes which are not subordinate to themselves. Having shown that no object is a class which is not subordinate to itself, Leśniewski concludes that *no object is the class of classes which are not subordinate to themselves.* Hence, the expression "the class of classes which are not subordinate to themselves" does not denote any object, and both "the class of classes which are not subordinate to themselves is subordinate to itself" as well as "the class of classes which are not subordinate to themselves is not subordinate to itself" are false sentences. Thus, Russell's antinomy cannot be generated.

In his second analysis of Russell's antinomy (arrived at during the period 1913-14, and published in [27]) Leśniewski derived the conclusion "no object is the class of classes which are not subordinate to themselves" *via* a different argument. Speaking of this conclusion he said ([27], p. 188):

> Having the strongest belief in [this conclusion], I did not feel the slightest shadow of an "antinomy" in the fact that both the assumption that the class of classes which are not subordinate to themselves is subordinate to itself, as well as the assumption that the class of classes which are not subordinate to themselves is not subordinate to itself, lead to a contradiction, just as, seeing that no object is a round square I did not feel an "antinomy" in the fact that both the assumption that a round square is a circle, as well as the assumption that a round square is not a circle, lead to a contradiction. Then I ceased to see an "antinomy" in Russell's construction, ceasing to believe in the existence of the class of classes which are not subordinate to themselves, and thus rejecting one of the fundamental positions of the aforementioned construction.[10]

In short, the solution to the problem expressed in the title of Leśniewski's [14] is simply: the question "is the class of classes which are not subordinate to themselves subordinate to itself?" does not allow either an affirmative or negative true answer.

Up to this point in [14] Leśniewski has been concerned with answering the question: is the class of classes which are not subordinate to themselves subordinate to itself? His concern with the problem associated with this question was generated by Russell's paradox. In the concluding part of [14] Leśniewski states the paradox (in colloquial language), and proposes two solutions. The first solution consists in pointing out two different errors in the construction of the paradox. The second solution consists in showing that the paradox could be solved if the expressions "object subordinate to the class K" and "object not subordinate to the class K" were used in the senses specified by Leśniewski in the early part of [14].

According to Leśniewski ([14], pp. 73-74), Russell's paradox

> . . . is based on the fact that both answers to the question, is the class of classes which are not subordinate to themselves subordinate to itself? apparently lead to a contradiction. 1) If the class of classes which are not subordinate to themselves is subordinate to itself, then it is subordinate to the class of classes which are not subordinate to themselves, and if it is

> subordinate to the class of classes which are not subordinate to themselves, then it is a class which is not subordinate to itself, which contradicts the assumption. 2) If the class of classes which are not subordinate to themselves is not subordinate to itself, then it is not subordinate to the class of classes which are not subordinate to themselves, and if it is not subordinate to the class of classes which are not subordinate to themselves, then it is not a class which is not subordinate to itself, that is, it is a class subordinate to itself, which contradicts the second assumption. However, since hypothesis 1), that the class of classes which are not subordinate to themselves is subordinate to itself, is false, then it is true that *this class is not subordinate to itself* (I). However, since on the other hand hypothesis 2), that the class of classes which are not subordinate to themselves is not subordinate to itself, is false, then it is true that *this class is subordinate to itself* (II). Comparing theses (I) and (II) it follows that the class of classes which are not subordinate to themselves both is subordinate to itself and is not subordinate to itself. And consequently, a "paradox."

Leśniewski maintains that the paradox could be solved by showing the errors in its formulation. Hypothesis 1) does not lead to the contradiction since

> . . . from the fact that the class of classes which are not subordinate to themselves is subordinate to the class of classes which are not subordinate to themselves, it is not possible to infer that the class of classes which are not subordinate to themselves is a class which is not subordinate to itself, since [as Leśniewski has already shown] *not every* object subordinate to the class of objects n is an n. ([14], p. 74)

This is, according to Leśniewski, the first error in the construction of the paradox. The second error arises in the following way. From the falsity of hypothesis 1) it is not possible to derive (I) above, i.e., "this class is not subordinate to itself," since

> . . . from the falsity of one of two contradictory sentences it is possible to infer that the other is true only in those cases in which these sentences have subjects which denote something. But in our case the subject of the sentences considered *viz.*, the expression "the class of classes which are not subordinate to themselves," does not denote anything [as Leśniewski has shown above]. ([14], p. 74)

Leśniewski concluded his [14] by pointing out that Russell's paradox also could have been solved if the expression "object subordinate to the class K" is used to denote an object P provided that in *every* sense of the expression "a" the following two conditions hold: 1) K is a class (of objects) a, 2) P is an a; or if the expression is used to denote an object P provided that in *every* sense of the expression "a" both conditions 1) and 2) hold whenever one of these conditions holds. However, (as shown earlier) to use "object subordinate to the class K" in either of these two senses entails that no object is subordinate to any class. Thus, no class is subordinate to itself.

Therefore, this can also be said of the class of classes which are not subordinate to themselves. This latter class, like every other class, is a class which is not subordinate to itself. This does not lead to any contradiction, since from the fact that the class of classes which are not subordinate to themselves is not subordinate to the class of classes which are not subordinate to themselves, it does not now follow that this class is subordinate to itself (as was the case from hypothesis 2) [above]. After all, we cannot here appeal to the principle that if some object is not subordinate to the class of objects *n*, then it is not an *n*, since such a principle is, in the new senses of the expression, a false principle. Proof: let us assume that this principle is true. From this it follows that every object which is not subordinate to the class of men is not a man (III); we already know that no man, as well as no object generally, is subordinate to any class, and thus to the class of men as well (IV). In view of theses (III) and (IV), it follows that no man is a man. Thus, the principle mentioned is a false principle, since it leads to a contradiction. So then, under these conditions too Russell's "paradox" is "dispatched." This "paradox" has contributed to a clarification of the foundations of the theory of classes, and this is its historical contribution. So, all hail to its memory! ([14], pp. 74-75)

NOTES

1. Sobociński [49-50], vol. 1, p. 94. References are listed chronologically; bracketed numerals associated with an item in the list indicate year of publication.

2. The first version of Mereology was published in [16]. The more important results of this system are recapitulated and reformulated in Leśniewski's [28]; the four axioms, seven definitions, and theorems I-XLVIII. In [29] Leśniewski published theorems IL-CXCVIII, and three additional definitions, and in [30] he published theorems CIC-CCIX; they were not included in [16], and were obtained in the period up to 1920 inclusively. In [30] he also published theorems CCX-CCLXIV; they too were not included in [16], and were obtained in the period 1921-1923. In [30], p. 105, Leśniewski points out that on the basis of six of the previously proved theorems Mereology could take as primitive not only "part" or "ingredient" but any one of the following: "+," "set," "class," "external," "sum," "complement." Theorems I-CCLXVI are based on the axioms of [16]. In [16] axioms III and IV are formulated using the term "class" which is introduced by definition II, which in turn is formulated using the term "ingredient" introduced by definition I. The fact that these axioms contained defined terms began to irk Leśniewski, and in 1918 he created a second set of axioms for Mereology using no defined terms in the axioms, taking "part" as sole primitive and defining "ingredient" and "class" in terms of "part." The axioms and definitions are published in Leśniewski's [30], and the resulting system shown to be equivalent to that of [16]. In 1920 Leśniewski created a third set of axioms for Mereology using no defined terms in the axioms, taking "ingredient" as sole primitive, and defining "part" and "class" in terms of "ingredient." The axioms and definitions were also published in [30], and the resulting system shown to be equivalent to that of [16]. In 1921 he established that a theory equivalent to that of [16] could be obtained using no defined terms in the axioms, taking "external" as sole primitive, and defining "class," "ingredient," and "part" in terms of "external." This was published in his [31].

3. Leśniewski [27], p. 186. (Translations of Polish texts in this paper are by V.F.S.). According to Sobociński [49-50], vol. II, p. 254: "Dans la période où [Leśniewski] s'était occupé de l'analyse de l'antinomie de Russell, il n'avait pas encore élaboré son propre système de logique. Les déductions qu'il faisait dans ces temps-là, en analysant le problème soulevé par Russell, ne se basaient donc que sur ses intuitions de la logique. Pourtant, comme il l'a souligné à plusieures reprises, elles concordaient plainement avec la conception de la logique qui a trouvé plus tard une expression précise dans son système de logique."

4. In [27], p. 190, Leśniewski said: "The starting-point of all my analyses of Russell's antinomy was the conception of a class (or set) which makes it possible to assert of any class (or set) of objects whatsoever that it 'consists of' just these objects [not necessarily in a disconnected way]. . . . In this respect my conception is, on the one hand, in so far as I have been able to observe, completely consistent with the common way of using the expressions 'class' and 'set' in the colloquial language of people who have never gone in for any 'theory of classes' nor for any 'theory of sets,' while on the other hand it is based on a strong scientific tradition represented more or less consistently by numerous past and present scientists, and specifically by Georg Cantor." In [38], p. 58, Leśniewski said that the sense of the term "class" which he used seemed to be consistent with ordinary intuition, and that he used this term in discussions regarding the "evidence" or "non-evidence" of individual theses which take a part in various "antinomies" constructed by "set-theorists." He said also that expressions of the type "class of objects a" are in his Mereology names which designate certain determinate and completely "ordinary" objects, and that they are to be distinguished from the term "class" when it is used not to name an object but as a *façon de parler,* as in *Principia Mathematica.*

5. Luschei [62], p. 20, asserts that Leśniewski repudiated his paper "Czy klasa klas . . . ," and in his bibliography of Leśniewski's works, p. 321, he lists this paper under the heading "Early writings, later repudiated." Luschei also lists Leśniewski's [16] under this heading. Unfortunately, I have not found any textual evidence to support the claim that Leśniewski repudiated these two works. In [27], pp. 182-183, Leśniewski listed four articles, two published in 1911, and two published in 1913, which he solemnly repudiated, but *his* list does not contain either the paper "Czy klasa klas . . ." or his [16]. Leśniewski said that he mentioned these four works ". . . because I wish to indicate that I am very distressed that they were published at all, and I herewith solemnly 'repudiate' these works, which I have already done from a university lectern, and assert the bankruptcy of the 'philosophico'-grammatical enterprises of the first period of my research." Furthermore, Luschei [62], p. 67, summarizes Leśniewski's second analysis of the antinomy, which appeared for the first time in Leśniewski's [27], pp. 182-189, but incorrectly he attributes the analysis to Leśniewski's "Czy klasa klas . . ." of 1914. The analyses of [27] and of "Czy klasa klas . . ." are distinct. "Czy klasa klas . . ." is not, as Luschei says, recapitulated in [27].

6. *Vide* [13], pp. 325-327.

7. P. 64. As mentioned earlier, in [27] Leśniewski reaffirmed the view expressed here, asserting "if some object is the class of objects a, then some object is a." With regard to Leśniewski's remark "By a class of objects a I mean here the set of all objects which are a," *cf.*, Sobociński [49-50], vol. I, p. 100: ". . . par

'A ε **Kl**(a)' nous comprenons la même chose que par 'A est l'ensemble de tous les objets a' ou, autrement encore 'A est un ensemble formé de tous les objets a'."

8. According to Leśniewski [14], p. 64, "I add the expression 'of objects' in parentheses to emphasize the fact that the class K itself need not be an a, but that there must be objects a whose class is the class K."

9. In [27], p. 187, Leśniewski reported that in 1914 he did not know how to use quantifiers. In the colloquial language he was then using he needed some analogue of expressions of the type "($\exists a$)fa," and this analogue was expressions of the type "in some sense of the expression 'a,' $f(a)$." In practice he treated these complex expressions *mutatis mutandis* in precisely the same way as one handles expressions of the type "($\exists a$)fa." Thus, conditions 1) and 2) might also be expressed as: "($\exists a$) (K is a class (of objects) a and P is an a)." He also indicated that he considered "P is subordinate to the class K" and "P is an element of the class K" to be equivalent expressions. It should be pointed out that in the first axiomatization of Mereology (*vide* [28], p. 272) the fourth definition reads: "P is an element of the object Q if and only if for some a, Q is the class of objects a, and P is an a." In Leśniewski's third analysis of Russell's antinomy (Sobociński [49-50], vol. I, p. 100), the statement "B is an object subordinate to a class A if in some sense of the expression 'a' the following two conditions hold: 1) A is a class (of objects) a, 2) B is an a" is expressed in terms of the Ontological epsilon and quantifiers as: "[AB] : B ε **el**(A) .≡. [$\exists a$] · A ε **Kl**(a) · B ε a." According to Sobociński, *ibid.*, "On peut correctement définir le term 'élément' par le terme 'classe'" in precisely this way. The passage from [14] is of considerable historical significance in the development of Leśniewski's logical views since it very clearly foreshadows the concept of a collective class, a concept which he first axiomatized in [16]. Sobociński [49-50], vol. II, pp. 242-243, has given an informal explication of Leśniewski's concept of a collective class, and it may be helpful to recapitulate this explication here. According to Sobociński, if an object B is a collective class constituted by some objects a, and if an object A is an element of B, then A need not necessarily be one of the objects a. Consider some books which are now on a desk. We take as books only the printed pages making them up, and we assume that at this moment there is no printed page on the desk which is not a part of one of the books. Under these circumstances, B is a collective class of all the books now on the desk, and B is also a collective class of all the printed pages now on the desk. In the collective sense of a class, for all A, a, and b, if A is a class of a's and a class of b's, then the class of a's is identical with the class of b's. However, it does not follow that the objects a are the same as the objects b. If A is a class of books now on the desk, and B is an element of A, then B need not necessarily be one of these books. B could be the fifth page of one of these books. Briefly, if B is an element of the object A, then B could be any segment of the object A.

10. *Cf.*, Sobociński [49-50], vol. I, p. 96: "Nous employons ici le mot 'antinomie' au sens qui lui a été donné par L. Nelson et qui peut être décrit de la manière suivante: l'antinomie est une contradiction que nous déduisons en partant de présupposés à la vérité desquels nous croyons et avec des méthodes dont nous reconnaissons la validité. . . . L'antinomie ne sera éliminée qu'au moment, où nous nous serons persuadés que nous avons utilisé dans la construction de l'antinomie soit des règles de raisonnement incorrectes, soit des présupposés faux."

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