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A Property Which Guarantees Termination in Weak Combinatory Logic and Subtree Replacement Systems

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1 Introduction It is well-known that recursive equations may be conveniently used for defining functions and specifying their computations. A particular syntactic system for writing such equations is Weak Combinatory Logic. Such logical formalism, even if not particularly appealing to computer scientists because of its syntactical characteristics, could be useful when we want to consider also untyped functions and would like to avoid the extra problems due to the presence of variables and their bindings, as in type-free λ-calculus [1].

Those equations may be considered as production rules for deriving "simpler" terms from more complex ones, and in that case the crucial problem of the existence of the "normal form" arises: does there exist the "simplest" derivable term? Is it unique?

In this paper we study that problem and give the definition of a property which is sufficient, under some hypotheses, for assuring the existence of such

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normal forms, extending previous known results [2]. Also included is the case in which a term may be obtained, during the rewriting process, via duplication of subterms. The result is also applicable to subtree replacement systems and therefore to all computational processes for which such systems may provide useful semantical models.

2 Initial definitions and notations In order to avoid ambiguity we first recall the definition of Weak Combinatory Logic (WCL) [2]. The alphabet is:

I, K, S	constants (or basic combinators)
$(,)$	special symbols for building terms
$>, =, \geqslant$	binary infixes for building formulas
$x, y, \ldots, x_1, x_2, \ldots$	variables.

Terms are defined as follows:

i. a constant or a variable is an atomic term
ii. an applicative combination of two terms α_1 and α_2, denoted by $(\alpha_1\alpha_2)$, is a term.

We generally assume left associativity, so that $\alpha_1\alpha_2 \ldots \alpha_n$ stands for $(\ldots(\alpha_1\alpha_2)\ldots\alpha_n)$.

Combinators are terms without variables.

The reduction axioms and inference rules of WCL are:

1. $Ix_1 > x_1$ (reduction axiom of I)
2. $Kx_1x_2 > x_1$ (reduction axiom of K)
3. $Sx_1x_2x_3 > x_1x_3(x_2x_3)$ (reduction axiom of S)
4. If $\alpha_1 > \alpha_2$ then $\alpha_1 = \alpha_2$
5. Reflexivity holds for $>$
6. $\geqslant$ is the reflexive transitive closure of $>$
7. Reflexivity, symmetry, and transitivity hold for $=$
8. a. If $\alpha_1 > \alpha_2$ then $\alpha_0\alpha_1 > \alpha_0\alpha_2$
   b. If $\alpha_1 > \alpha_2$ then $\alpha_1\alpha_0 > \alpha_2\alpha_0$.

To simplify some of the following definitions we write the reduction axioms using variables (e.g., x_i) instead of generic terms (e.g., α_i). $\equiv$ denotes syntactical identity. We write $\alpha >_n \beta$ for showing that β is obtained from α by n applications of reduction axioms.

We can introduce in WCL some other constants (or basic combinators) defining them either in terms of I, K, and S or by giving a reduction axiom. This is a consequence of the Combinatory Completeness Theorem [2]. For example we can introduce a constant B by giving either $B \equiv S(KS)K$ or $Bx_1x_2x_3 > x_1(x_2x_3)$.

Let us now introduce some other definitions. A *subbase* $\mathscr{B}$ is a nonempty (possibly infinite) set of basic combinators $\mathscr{B} = \{X_1, \ldots, X_n\}$. The *applicative closure* $\mathscr{B}^+$ *of a subbase* $\mathscr{B}$ is the set of all finite applicative combinations of basic combinators in $\mathscr{B}$. As far as notations are concerned,

$\mathscr{A}, \mathscr{B}, \ldots$	denote subbases
$X, Y, \ldots, K, S, \ldots$	denote basic combinators

$x, y, \ldots$	denote variables
$\alpha, \beta, \ldots$	denote terms
$\chi, \phi, \psi, \ldots$	denote combinators.

We also use subscripts and superscripts, if necessary. Let us suppose that we are given a subbase $\mathscr{B}$ and a term α built out of basic combinators in $\mathscr{B}$ and variables. The set S_α of *subterms* of α is defined as follows:

i. if α is a basic combinator or a variable (atomic subterm) then $S_\alpha = \{\alpha\}$
ii. if $\alpha \equiv (\alpha_1\alpha_2)$ then $S_\alpha = S_{\alpha_1} \cup S_{\alpha_2} \cup \{\alpha\}$.

The set of *proper subterms* of α is $S_\alpha - \{\alpha\}$.

The set T_α of *right applied subterms* (or *branches*) of α is defined as follows:

i. if α is a basic combinator or a variable then $T_\alpha = \{\alpha\}$
ii. if $\alpha \equiv (\alpha_1\alpha_2)$ then $T_\alpha = T_{\alpha_1} \cup \{\alpha_2\}$.

We say that the basic combinator X with reduction axiom $Xx_1 \ldots x_m > \beta$ is a *proper combinator* iff β is an applicative combination of variables in the set $\{x_1, \ldots, x_m\}$. We also say that X has *duplicative effect* iff $\exists x_i$, $1 \leqslant i \leqslant m$, such that x_i occurs in β more than once; X has *compositive effect* iff a right applied subterm of β is not a variable.

Example 1:

i. W such that $Wx_1x_2 > x_1x_2x_2$ is a proper combinator with duplicative effect.
ii. B such that $Bx_1x_2x_3 > x_1(x_2x_3)$ is a proper combinator with compositive effect.
iii. Given $\mathscr{B} = \{S, K\}$ and $\alpha \equiv S(KS)x_2$,

$$S_\alpha = \{S, KS, K, x_2, S(KS), S(KS)x_2\} \text{ and } T_\alpha = \{S, KS, x_2\}.$$

Given a term α we define the corresponding *marked term* marked (α) as follows:

$$\text{marked}\ (\alpha) = \text{marked 1}\ (\langle\alpha, 0\rangle)$$

where

$$\begin{aligned}\text{marked 1}\ (\langle\alpha, n\rangle) &= \langle\alpha, n\rangle \text{ if } \alpha \text{ is a basic combinator or a variable}\\ &= (\text{marked 1}\ (\langle\alpha_1, n+1\rangle), \text{marked 1}\ (\langle\alpha_2, n\rangle))\\ &\quad \text{if } \alpha \equiv (\alpha_1\alpha_2).\end{aligned}$$

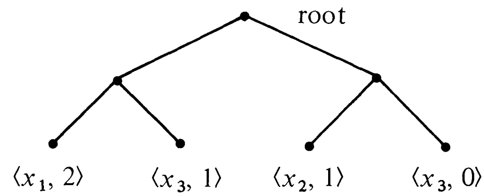
Example 2:

i. marked $(Sx_1x_2x_3) = (((\langle S, 3\rangle, \langle x_1, 2\rangle), \langle x_2, 1\rangle), \langle x_3, 0\rangle)$
ii. marked $(x_1x_3(x_2x_3)) = ((\langle x_1, 2\rangle, \langle x_3, 1\rangle), (\langle x_2, 1\rangle, \langle x_3, 0\rangle))$.

Therefore a marked term is a term in which all basic combinators and variables are associated with an integer. If we represent a term as a binary tree according to its applicative structure, then that integer is the number of "left choices" one has to make for going from the root of the tree to the considered basic combinator or variable.

Example 3: Using a binary tree representation,

marked $(x_1x_3(x_2x_3))$ =



3 The nonascending property and the basic theorem

Definition 1 We say that a proper combinator X with reduction axiom $Xx_1x_2 \ldots x_m > \beta$ has *nonascending property* (NA property) iff $\forall i$, for $1 \leqslant i \leqslant m$, if $\langle x_i, p\rangle$ occurs in marked$(Xx_1 \ldots x_m)$ and $\langle x_i, q\rangle$ occurs in marked(β) then $p \geqslant q$.

Example 4:

i. X such that $Xx_1x_2x_3 > x_1x_2x_2$ has NA property, because

marked$(Xx_1x_2x_3) = (((\langle X, 3\rangle, \langle x_1, 2\rangle), \langle x_2, 1\rangle), \langle x_3, 0\rangle)$ and
marked$(x_1x_2x_2) = ((\langle x_1, 2\rangle, \langle x_2, 1\rangle), \langle x_2, 0\rangle)$.

ii. S such that $Sx_1x_2x_3 > x_1x_3(x_2x_3)$ does not have NA property, because

$\langle x_3, 0\rangle$ occurs in marked$(Sx_1x_2x_3)$ and
$\langle x_3, 1\rangle$ occurs in marked $(x_1x_3(x_2x_3))$.

For the usual concepts of "normal form", "redex", and "reductum" in WCL, see [2].

Theorem 1 (Basic Theorem) *Given a proper combinator X with NA property and without compositive effect* $\forall \chi \in \{X\}^+$ χ *has normal form.*

In order to prove Theorem 1 we first introduce more definitions and prove some lemmas.

Definition 2 Given a term α we define the set R_α of terms *reachable from* α as follows: $R_\alpha = \{\beta | \alpha \geqslant \beta\}$.

Definition 3 Given a term α and a pair $\langle X, n\rangle$ (where X is a basic combinator) which occurs in marked(α), n is the *copy-number* (or c-number) of that occurrence of X in α. We also say that such an occurrence of X is *associated with* the c-number n or, simply, X is *with* c-number n or X *has* c-number n.

Definition 4 We say that a combinator χ *cycles* iff $\chi >_k \chi$ for some $k \geqslant 1$.

Remark 1: In proving the following lemmas and Theorem 1 we consider all reductions to be leftmost outermost. This is a safe strategy for obtaining the normal form (see [2]).

Remark 2: From now on, unless otherwise stated, we will consider that the basic proper combinator X has NA property and does not have compositive effect.

Remark 3: In what follows we say that "the maximum c-number in $\chi \in \{X\}^+$ is m" or "χ has maximum c-number m" as abbreviations for: "m is the maximum integer such that $\langle X, m \rangle$ occurs in marked(χ)".

Lemma 1 *$\forall\chi \in \{X\}^+ R_\chi$ is finite.*

Proof: Let m be the maximum c-number in χ. Since X has NA property and does not have compositive effect, $\forall\phi \in R_\chi$ such that $\chi \geqslant \phi$, ϕ must be an applicative combination of at most m subterms of χ, otherwise at least one X in ϕ would have a c-number greater than m. Since the set of all subterms of χ is finite, R_χ is finite.

Lemma 2 *If $\chi \in \{X\}^+$ does not have normal form then there exists $\bar{\chi} \in \{X\}$ such that $\bar{\chi} \equiv X\phi_2 \ldots \phi_n$, where $\forall\phi_i$, for $2 \leqslant i \leqslant n$, ϕ_i is in normal form, and $\bar{\chi}$ has no normal form.*

Proof: Let Ω be the set of all terms in $\{X\}^+$ without normal form. Suppose $\Omega \neq \phi$. We can order Ω using the "proper subterm" relation. It is a well ordering and therefore there exists $\bar{\chi} \in \Omega$ whose proper subterms have normal form.

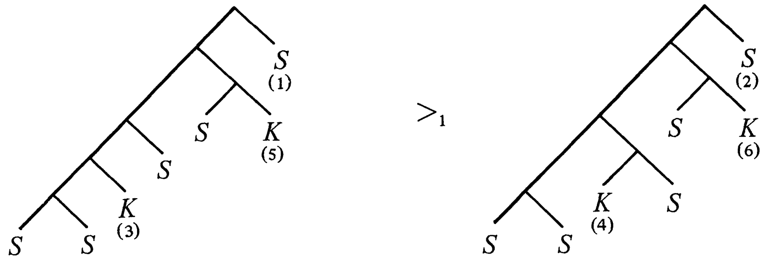
Definition 5 A reduction $\alpha >_1 \beta$ is called a *head reduction* when *either* $\alpha \equiv \rho$ *or* $\alpha \equiv \rho\alpha_2 \ldots \alpha_n$ with $n \geqslant 2$ where ρ is the reduced redex.

Lemma 3 *Given $\chi_0 \in \{X\}^+$ such that $\chi_0 \equiv X\phi_2 \ldots \phi_n$ where all ϕ_i's with $2 \leqslant i \leqslant n$ are in normal form, if $\chi_0 >_1 \chi_1 >_1 \ldots >_1 \chi_i >_1 \ldots$ then $\forall i \geqslant 0$, $\chi_i >_1 \chi_{i+1}$ is a head reduction.*

Proof: Immediate because X does not have compositive effect.

Definition 6 Given the head reduction $\alpha >_1 \beta$, that is $\rho\alpha_2 \ldots \alpha_n >_1 \rho'\alpha_2 \ldots \alpha_n$ where ρ is the redex and ρ' is the contractum, we say that a subterm σ_α of α *immediately produces* the subterm σ_β of β (or β *immediately derives* from α) iff *either* $\exists i$ $2 \leqslant i \leqslant n$ such that $\sigma_\alpha \equiv \sigma_\beta \equiv \alpha_i$ *or* σ_α and σ_β are subterms of ρ and ρ' respectively and both correspond to the same variable x_i of the reduction axiom applied in the given reduction. If $\sigma_\alpha^{(1)}\sigma_\alpha^{(2)}$ immediately produces $\sigma_\beta^{(1)}\sigma_\beta^{(2)}$ we also say that $\sigma_\alpha^{(i)}$ immediately produces $\sigma_\beta^{(i)}$ (or $\sigma_\beta^{(i)}$ immediately derives from $\sigma_\alpha^{(i)}$) for $i = 1, 2$.

Example 5: In



which is a head reduction, $\underset{(1)}{S}$ immediately produces $\underset{(2)}{S}$ (because they both correspond to α_3, writing the reduction as $\rho\alpha_2\alpha_3 >_1 \rho'\alpha_2\alpha_3$); $\underset{(4)}{K}$ immediately derives from $\underset{(3)}{K}$ (because they both correspond to the variable x_2 of the axiom

$Sx_1x_2x_3 > x_1x_3(x_2x_3)$); and $\underset{(5)}{K}$ immediately produces $\underset{(6)}{K}$ (because $S \underset{(5)}{K}$ immediately produces $S \underset{(6)}{K}$ since they both correspond to α_2).

Definition 7 Given a sequence of head reductions $\alpha \equiv \alpha_0 >_1 \ldots >_1 \alpha_n \equiv \beta$, we say that a subterm σ_α of α *produces* a subterm σ_β of β (or σ_β *derives from* σ_α) iff there exists a sequence of subterms σ_{α_0} of α_0, σ_{α_1} of α_1, . . ., σ_{α_n} of α_n such that $\forall k$, $1 \leqslant k \leqslant n$, $\sigma_{\alpha_{k-1}}$ *immediately produces* σ_{α_k}.

Example 6: Given $\alpha_0 \equiv KKK(\underset{(1)}{S}SK)KK >_1$, $\alpha_1 \equiv K(\underset{(2)}{S}SK)KK >_1$, $\alpha_2 \equiv \underset{(3)}{S}SKK >_1$, $\alpha_3 \equiv \underset{(4)}{S}K(KK)$, then $\underset{(1)}{S}$ *produces* $\underset{(4)}{S}$ because $\sigma_{\alpha_i} \equiv \underset{(i+1)}{S}$ immediately produces $\sigma_{\alpha_{i+1}} \equiv \underset{(i+2)}{S}$ for $i = 0, 1, 2$.

Remark 4: If σ_α *produces* σ_β then $\sigma_\alpha \equiv \sigma_\beta$.

Lemma 4 *Given $\chi_1, \chi_2 \in \{X\}^+$ $\chi_1 >_k \chi_2$ with $k \geqslant 1$. Let us suppose that χ_1 and χ_2 have the same maximum c-number m:* (i) *A copy of X with c-number m in χ_1 produces 0 or 1 copy of X with c-number m in χ_2.* (ii) *A copy of X with c-number m in χ_2 can derive only from a copy of X in χ_1 with c-number m.*

Proof: (i) Given $Xx_1 \ldots x_m > \beta$, since X does not have compositive effect, the branches of β are variables, i.e., elements of the set $\{x_1, \ldots, x_m\}$. Since X has NA property at most one copy of x_i, which has c-number $m - i$ in $Xx_1 \ldots x_m$, has the same c-number $m - i$ in β. (ii) Obvious, because X has NA property.

Lemma 5 *Given $\chi_0 \in \{X\}^+$ if $\chi_0 >_1 \chi_1 >_1 \ldots >_1 \chi_k \equiv \chi_0$ then $\forall i, j$, $0 \leqslant i, j \leqslant k$, the maximum c-number in χ_i is equal to the maximum c-number in χ_j.*

Proof: Since X has NA property, the maximum c-number cannot increase. If it decreases, χ cannot cycle.

Lemma 6 *Given $\chi_0 \in \{X\}^+$ if $\chi_0 >_1 \chi_1 >_1 \ldots >_1 \chi_k \equiv \chi_0$ then $\forall i, j$, $0 \leqslant i, j \leqslant k$, the number of copies of X's in χ_i with maximum c-number is equal to the number of copies of X's in χ_j with maximum c-number.*

Proof: The maximum c-number in χ_i is equal to the maximum c-number in χ_j by Lemma 5. Since X has NA property, the number of copies of X's with maximum c-number in χ_i cannot increase in χ_j for $j > i$ by Lemma 4. If it decreases then χ_0 cannot cycle.

Lemma 7 *Given $X\phi_1 \ldots \phi_k >_1 \beta$ where $\phi_i \in \{X\}^+$ with $i = 1, \ldots, k$, if ϕ_i has maximum c-number m_i in $(X\phi_1 \ldots \phi_k)$ for $i = 1, \ldots, k$ then the maximum c-number in β is at most $max(m_1, \ldots, m_k)$.*

Proof: Immediate.

Definition 8 Suppose $\alpha >_1 \beta$. We say that a particular occurrence of a subterm σ_α of α is *unbroken* in the reduction $\alpha >_1 \beta$ iff *either* σ_α is not a subterm of the reduced redex *and* the reduced redex is not a subterm of σ_α *or* σ_α is a subterm of the reduced redex *and* it corresponds to an atomic subterm of the applied reduction axiom. A subterm σ_α is *broken* if a subterm (proper or not) of it is not unbroken.

Example 7:

i. In $S(KS)x_5B >_1 KSB(x_5B)$, KS is a subterm of the reduced redex and it corresponds to the variable x_1 of the reduction axiom $Sx_1x_2x_3 >_1 x_1x_3(x_2x_3)$ we applied. Therefore KS is unbroken.
ii. In $S(KSKK) >_1 S(SK)$, KS is a subterm of the reduced redex and it does not correspond to an atomic subterm of the reduction axiom $Kx_1x_2 > x_1$ applied, because it corresponds to Kx_1. Therefore KS is broken. Also KSK, $KSKK$ and $S(KSKK)$ are broken because their subterm KS is broken.

Definition 9 Let $\chi \equiv X\phi_2 \ldots \phi_{m-1}\phi_m\phi_{m+1} \ldots \phi_n$ be an element of $\{X\}^+$ and $2 \leqslant m \leqslant n$. We say that ϕ_m is the *rightmost deepest branch* (RDB) of χ iff ϕ_m is a right applied subterm of χ containing an X with c-number which is maximum in χ and $\phi_{m+1}, \ldots, \phi_n$ do not contain an X with maximum c-number in χ.

Example 8: Let us consider $X\phi_2\phi_3\phi_4$ where $\phi_2 \equiv \phi_4 \equiv X$ and $\phi_3 \equiv XXX$. The maximum c-number is 3 and ϕ_3 is the RDB, because it contains an X with c-number 3 and ϕ_4 does not.

Lemma 8 *Let $\alpha \equiv \phi_m\psi_{m+1} \ldots \psi_l$ and $\beta \equiv X\phi_2 \ldots \phi_{m-1}\phi_m\phi_{m+1} \ldots \phi_n$. Let $\phi_i \in \{X\}^+$ for $i = 2, \ldots, n$ with $2 \leqslant m \leqslant n$ be in normal form. Let $\psi_{m+1}, \ldots, \psi_l \in \{X\}^+$ with $m \leqslant l$ be in normal form. Let the maximum c-number in α and in β be the same and let ϕ_m be the RDB of α and β. In these hypotheses:* (i) $l = n$, *and* (ii) $\alpha \not\geqslant \beta$ (i.e., $\beta \notin R_\alpha$).

Proof: (i) $l = n$ because the maximum c-number in α and β is the same. (ii) If a reduction is possible for α either $\phi_m \equiv X$ or ϕ_m is broken, because $\phi_{m+1}, \ldots, \psi_l$ are in normal form. If $\phi_m \equiv X$ the copies of X with c-number equal to the maximum c-number in β are 0 (because they are just 1 in α) and obviously in that case $\alpha \not\geqslant \beta$. If ϕ_m in α is broken we have that:

1. ϕ_m in β cannot derive from a subterm of ψ_{m+1} or ψ_{m+2} or . . . or ψ_l by Lemma 7, because in ϕ_m in β occurs an X with maximum c-number and no X with maximum c-number occurs in $\psi_{m+1}, \ldots \psi_l$ and NA property holds
2. ϕ_m in β cannot derive from ϕ_m in α because ϕ_m in α is broken and X does not have compositive effect
3. ϕ_m in β cannot be obtained by composition of terms derived from subterms of $\psi_{m+1}, \ldots, \psi_l$ or proper subterms of ϕ_m in α because X does not have compositive effect.

From 1, 2, and 3, $\alpha \not\geqslant \beta$ follows.

Definition 10 Among the right applied subterms of a term $\alpha \equiv \alpha_1\alpha_2 \ldots \alpha_n$ where α_1 is an atomic subterm, we define

1. the relation to be *at the right hand side of* (ROF) as follows:

$$\forall i, j,\ 1 \leqslant i, j \leqslant n,\ \alpha_i \text{ ROF } \alpha_j \text{ iff } i > j$$

2. the relation to be *at the left hand side of* (LOF) as follows:

$$\forall i, j\ 1 \leqslant i, j \leqslant n,\ \alpha_i \text{ LOF } \alpha_j \text{ iff } i < j.$$

Remark 5: In the case that $\alpha \equiv \alpha_1 \ldots \alpha_n$ and all α_i's are atomic, if α_i LOF α_j then α_i in mark(α) has c-number greater than that of α_j in mark(α). Analogously if α_i ROF α_j then α_i in mark(α) has a c-number less than that of α_j in mark(α).

Remark 6: If α_i ROF α_j then α_j LOF α. If α_i ROF α_j then $\forall\gamma$ subterms of α_i, $\forall\delta$ subterm of α_j, γ ROF δ. Analogously using LOF instead of ROF and ROF instead of LOF.

Lemma 9 *Consider* $\chi_1 \equiv X\phi_2^{(1)} \ldots \phi_m^{(1)} \ldots \phi_n^{(1)} \in \{X\}^+$ *with* $2 \leqslant m \leqslant n$ *and* $\chi_2 \equiv X\phi_2^{(2)} \ldots \phi_k^{(2)} \ldots \phi_t^{(2)} \in \{X\}^+$ *with* $2 \leqslant k \leqslant t$. *Suppose:*

1. $\chi_1 >_1 \chi_2$
2. *maximum c-number in* χ_1 = *maximum c-number in* χ_2
3. $\phi_m^{(1)} \equiv \phi_k^{(2)}$ *and* $\phi_m^{(1)}$ *produces* $\phi_k^{(2)}$
4. $\phi_m^{(1)}$ *and* $\phi_k^{(2)}$ *contain an X with maximum c-number in* χ_1 *and* χ_2 *respectively.*

Under these hypotheses:

(i) $n - m = t - k$

(ii) *no* $\phi_p^{(1)}$ *with* $m < p \leqslant n$ *produces any* $\phi_q^{(2)}$ *with* $2 \leqslant q \leqslant k$

(iii) *each* $\phi_p^{(1)}$ *with* $m < p \leqslant n$ *can only produce elements of the set* $\{\phi_i^{(2)} | k < i \leqslant t\}$.

Proof: (i) Immediate.

(ii) Since $\phi_m^{(1)}$ produces $\phi_k^{(2)}$ and $\chi_1 >_1 \chi_2$ by applying the axiom $Xx_1 \ldots x_m > \beta$, if $\phi_m^{(1)}$ corresponds to the variable x_i in $Xx_1 \ldots x_m$, then $\phi_k^{(2)}$ corresponds to another occurrence of the same variable x_i in β. Those two occurrences of the variable x_i have the same c-number in $Xx_1 \ldots x_m$ and β respectively, because otherwise $\phi_m^{(1)}$ and $\phi_k^{(2)}$ could not have a copy of X with maximum c-number in χ_1 and χ_2, respectively. If a $\phi_p^{(1)}$ with $m < p \leqslant n$ produces a $\phi_q^{(2)}$ for any $q = 2, \ldots, k$, that fact would imply that an x_j, with $j > i$ in $Xx_1 \ldots x_m$ such that x_j ROF x_i, where x_i in $Xx_1 \ldots x_m$ corresponds to $\phi_m^{(1)}$, would have been the same as an x_k in β such that x_k LOF x_i, where x_i in β corresponds to $\phi_k^{(2)}$.
Since the two occurrences of x_i, above mentioned, have the same c-number in $Xx_1 \ldots x_m$ and β, respectively, X could not have NA property by Remark 1.

(iii) All $\phi_p^{(1)}$'s with $m < p \leqslant n$ are unbroken because $\chi_1 >_1 \chi_2$ is a leftmost outermost reduction and $\phi_m^{(1)}$ is unbroken. From point ii and hypothesis 3, point iii follows.

Lemma 10 *Let* $\phi_i \in \{X\}^+$ *for* $i = 2, \ldots, n$ *with* $n \geqslant 2$ *be in normal form. Let* χ_0 *be* $X\phi_2 \ldots \phi_{m-1}\phi_m\phi_{m+1} \ldots \phi_n$ *where* $2 \leqslant m \leqslant n$ *and* ϕ_m *is the RDB of* χ_0. *If* $\chi_0 >_1 \chi_1 >_1 \ldots >_1 \chi_i >_1 \ldots >_1 \chi_k \equiv \chi_0$ *for* $k \geqslant 1$ *then*

(i) ϕ_m *which is the RDB of* χ_k *derives from* ϕ_m *which is the RDB of* χ_0

(ii) $\forall i$, $0 \leqslant i \leqslant k$, *a copy of* ϕ_m *occurs in* χ_i *and it is the RDB of* χ_i *and* ϕ_m *which is the RDB in* χ_i *produces the* ϕ_m *which is the RDB of* χ_{i+1}

(iii) *$\forall i$, $0 \leqslant i \leqslant k-1$, ϕ_m which is the RDB of χ_i is unbroken in the reduction $\chi_i >_1 \chi_{i+1}$.*

Proof: (i) ϕ_m, which is the RDB of χ_k, cannot be obtained as composition of subterms of $\phi_2, \ldots, \phi_n$ because X does not have compositive effect; it cannot derive from a subterm of ϕ_{m+1} or . . . or ϕ_n because in ϕ_m occurs an X with maximum c-number and no X with maximum c-number occurs in ϕ_{m+1} or . . . or ϕ_n (by Lemmas 5 and 7); and it cannot derive from a subterm of $\phi_2, \ldots, \phi_{m-1}$ because such a subterm could not have an X with maximum c-number in χ_k, say d, unless it had in χ_0 (which is identical to χ_k) an X with a c-number strictly greater, than d (by Lemma 5). Therefore the copy of ϕ_m which is RDB of χ_k derives from the copy of ϕ_m which is RDB in χ_0.

(ii) The derivation of ϕ_m which is the RDB of χ_k from ϕ_m which is the RDB of χ_0, since X does not have compositive effect, is obtained by a sequence of ϕ_m's such that $\forall i$, $0 \leqslant i \leqslant k-1$, a copy of ϕ_m in χ_{i+1} derives from a copy of ϕ_m in χ_i and each copy is a right applied subterm of the corresponding combinator. In order to prove point ii we have to show that each ϕ_m of the above-mentioned sequence of ϕ_m's is the RDB of the corresponding χ_i. Since we know (by point i) that ϕ_m in χ_k is the RDB of χ_k, we have only to prove that: $\forall i$, $1 \leqslant i \leqslant k$, if ϕ_m which is the RDB of χ_i derives from a copy of ϕ_m in χ_{i-1} then that copy of ϕ_m in χ_{i-1} is the RDB of χ_{i-1}. In fact:

(1) an X with maximum c-number must occur in that copy of ϕ_m in χ_{i-1} because otherwise that copy of ϕ_m would not produce a ϕ_m which is the RDB of χ_i (by Lemma 4); and

(2) no other right applied subterms of χ_{i-1} containing an X with maximum c-number occur in χ_{i-1} on the right hand side of the copy of ϕ_m which produces the RDB of χ_i, because if any would occur, it would produce in χ_i subterms containing copies of X without maximum c-number by Lemma 9 (point iii) and therefore, by Lemma 6, χ_0 could not cycle.

(iii) If ϕ_m, which is the RDB of χ_i for some i such that $0 \leqslant i \leqslant k-1$, is broken in the reduction $\chi_i >_1 \chi_{i+1}$ then $\chi_i \equiv \phi_m \psi_{m+1} \ldots \psi_l$ by Lemma 3 and no k exists such that $\chi_i >_k \chi_0$ by Lemma 8.

Lemma 11 *Let $\phi_i \in \{X\}^+$ for $i = 2, \ldots, n$ with $n \geqslant 2$ be in normal form. Let χ_0 be $X\phi_2 \ldots \phi_{m-1}\phi_m\phi_{m+1} \ldots \phi_n$ where $m \leqslant n$ and ϕ_m is the RDB of χ_0. Let $\chi_0 >_1 \chi_1 >_1 \ldots >_1 \chi_i >_1 \ldots >_1 \chi_k \equiv \chi_0$ for $k \geqslant 1$. On these hypotheses:*

(i) *$\psi_m, \psi_{m+1}, \ldots, \psi_n \in \{X\}^+$ exist such that $\chi' \equiv X\phi_2 \ldots \phi_{m-1}\psi_m\psi_{m+1} \ldots \psi_n$ cycles, and*

(ii) *$\psi_m, \psi_{m+1}, \ldots, \psi_n$ do not have X's with maximum c-number in χ'.*

(Note that χ differs from χ' also for the subterm ϕ_m which is transformed into ψ_m).

Proof: (i) Let us denote by $\phi_j^{(i)}$ for $2 \leqslant j \leqslant n$ the right applied subterms of χ_i for $i = 0, 1, \ldots, k$. By Lemma 10 $\phi_m^{(k)}$ derives from $\phi_m^{(0)}$. Now let us prove the following:

Assertion 1 *Each $\phi_{m+1}^{(k)}, \ldots, \phi_n^{(k)}$ may be derived either from $\phi_m^{(0)}$ or $\phi_{m+1}^{(0)}$ or . . . or $\phi_n^{(0)}$ as a copy of one of them or from a subterm of $\phi_2^{(0)}$ or . . . or $\phi_{m-1}^{(0)}$.*

Proof of Assertion 1: Each $\phi_j^{(k)}$ with $m + 1 \leqslant j \leqslant n$:

1. cannot derive from a *proper* subterm of $\phi_{m+1}^{(0)}$ or . . . or $\phi_n^{(0)}$ because during the reduction from χ_0 to $\chi_k \equiv \chi_0$, ϕ_m is unbroken (by Lemma 10) and since we make only leftmost outermost reductions, also $\phi_{m+1}^{(0)}, \ldots, \phi_n^{(0)}$ are unbroken

2. cannot derive from a *proper* subterm of $\phi_m^{(0)}$ because it would imply that $\exists i$, $0 \leqslant i \leqslant k - 1$, such that the copy of ϕ_m, which is the RDB of χ_i and derives from $\phi_m^{(0)}$ by Lemma 10, is broken. In fact in each χ_i, $0 \leqslant i \leqslant k$, no other copies of ϕ_m are on the left hand side of the copy which is the RDB of χ_i, because otherwise the copy of ϕ_m which is the RDB of χ_i could not contain a copy of X with maximum c-number in χ_i. Therefore since all reductions from χ_0 to χ_k are head reductions, subterms of $\phi_m^{(0)}$ can only be derived by breaking in χ_i, for some i such that $0 \leqslant i \leqslant k - 1$, a copy of ϕ_m which is the RDB of χ_i. But in that case, by Lemma 10, point iii, χ_0 could not cycle.

3. cannot be obtained by composition of subterms because X does not have compositive effect.

From 1, 2, and 3, Assertion 1 follows.

To continue the proof of Lemma 11(i), now let us choose:

1. $\psi_m \equiv X$
2. $\forall i, j$, $m + 1 \leqslant i \leqslant n$ and $m \leqslant j \leqslant n$, if $\phi_i^{(k)}$ derives from $\phi_j^{(0)}$ (as a copy of it) then $\psi_i \equiv \psi_j \equiv X$
3. $\forall i$, $m + 1 \leqslant i \leqslant n$, if $\phi_i^{(k)}$ derives from a subterm of $\phi_2^{(0)}$ or . . . or $\phi_{m-1}^{(0)}$ then $\psi_i \equiv \phi_i$.

With the above choices $\chi' \equiv X\phi_2 \ldots \phi_{m-1}\psi_m\psi_{m+1} \ldots \psi_n$ cycles because χ_0 cycles and the subterm structure of χ' preserves the same derivation relationships among subterms as they are in χ_0.

(ii) Obviously $\psi_m, \psi_{m+1}, \ldots, \psi_n$ do not have X's with maximum c-number in χ'.

Lemma 12 *No combinator* $\bar{\chi} \equiv X\phi_2 \ldots \phi_n$ *exists in* $\{X\}^+$ *such that* $\phi_2, \ldots, \phi_n$ *are in normal form and* $\bar{\chi}$ *does not have normal form.*

Proof: Since by Lemma 1 $R_{\bar{\chi}}$ is a finite set, we have to prove that $\bar{\chi}$ cannot cycle. The proof is by structural induction on $\bar{\chi}$.

Basis: $\bar{\chi} \equiv X$ does not cycle.

Induction step: $\bar{\chi} \equiv \phi_1\phi_2$. Suppose ϕ_1 and ϕ_2 are in normal form.

By iteratively applying Lemma 11, if $(\phi_1\phi_2)$ cycles then also so does $\chi \equiv X\psi_2 \ldots \psi_k$ where $k \geqslant 2$ and $\forall i$, $2 \leqslant i \leqslant k$, ψ_i is in normal form and the leftmost X in χ has maximum c-number in χ. But χ cannot cycle by Lemma 6 because at the first leftmost outermost reduction (if a reduction is possible at all) an X with maximum c-number is deleted. Therefore $\phi_1\phi_2$ cannot cycle.

Proof of Theorem 1: Immediate from Lemmas 2 and 12.

Example 9: Given $\mathscr{B} = \{X\}$ such that $Xx_1x_2x_3x_4x_5 > x_1x_2x_2x_4x_4$ $\forall\chi \in \{X\}^+$ χ has normal form. In fact, X has NA property and it does not have compositive effect.

Remark 7: If in Definition 1 we had $p > q$ instead of $p \geqslant q$ then the proof of Theorem 1 would have been immediate, using a "well order" argument (cf. [3]). In fact, at each contraction step the c-number of each copy of X diminishes.

4 Some consequences of the basic theorem Given a combinator χ, in general more than one redex occurs in it and therefore more than one combinator χ' exists such that $\chi >_1 \chi'$.

Definition 11 A *reduction strategy* is a rule which chooses the redex (or the redexes) to be reduced in a term with more than one redex when a reduction step must be performed.

Theorem 2 *Given a proper combinator X with NA property and without compositive effect, $\forall\chi \in \{X\}^+$ every reduction strategy applied to χ leads to its normal form.*

Proof: By Theorem 1, $\forall\chi \in \{X\}^+$ all subterms of χ, which are themselves elements of $\{X\}^+$, have normal form.

Theorem 2 can also be extended to subbases with more than one basic combinator.

Theorem 3 *Given a subbase $\mathscr{B} = \{X_1, \ldots, X_n\}$ such that $\forall X_i \in \mathscr{B}$ X_i has NA property and doesn't have compositive effect, $\forall\chi \in \mathscr{B}^+$, every reduction strategy applied to χ leads to its normal form.*

Proof: In the proofs of lemmas necessary for proving Theorem 1 we use only the facts that NA property holds and combinators don't have compositive effect.

Example 10: Given $\mathscr{B} = \{X_1, X_2\}$ where $X_1x_1x_2x_3 > x_1x_2x_2$ and $X_2x_1x_2x_3x_4 > x_1x_2x_2x_3$, $\forall\chi \in \mathscr{B}^+$ χ has normal form. In fact, X_1 and X_2 have NA property and no compositive effect. For instance:

$$\begin{aligned}\chi \equiv X_1(X_1X_2)(X_1X_1X_2)X_2X_2X_1 &>_1 X_1X_2(X_1X_1X_2)(X_1X_1X_2)X_2X_1\\ &>_1 X_2(X_1X_1X_2)(X_1X_1X_2)X_2X_1\\ &>_1 X_1X_1X_2(X_1X_1X_2)(X_1X_1X_2)X_2\\ &>_1 X_1X_2X_2(X_1X_1X_2)X_2\\ &>_1 X_2X_2X_2X_2.\end{aligned}$$

Theorem 3 improves the result in [2], page 180, where termination of a combinator $\chi \in \mathscr{B}^+$ is guaranteed if there are no basic combinators in $\mathscr{B}$ with duplicative effect. According to Curry and Feys [2] the results about termination are as follows:

1. If all combinators in $\mathscr{B}$ are without duplicative effect then $\forall\chi \in \mathscr{B}^+$ χ has normal form.
2. If some combinators in $\mathscr{B}$ have duplicative effect then it could be the case that:

i. $\forall \chi \in \mathscr{B}^+$, χ has normal form (see Example 11.1)
ii. some χ's in $\mathscr{B}^+$ do not have normal form (see Example 11.2).

Example 11:

1. Let us consider $\mathscr{B} = \{X\}$ such that $Xx_1x_2 > x_1x_1$.

   All $\chi \in \mathscr{B}^+$ have normal form because eventually the leftmost outermost redex will be of the form $XX\phi$ and from that reduction onward the number of right applied subterms of the subterm where the reduction occurred diminishes.

2. Let us consider $\mathscr{B} = \{W\}$ such that $Wx_1x_2 > x_1x_2x_2$.

   WWW has no normal form because it cycles.

   As a consequence of our result Case 2 is split into two subcases:

   2.1 If all combinators in $\mathscr{B}$ have NA property and no compositive effect then $\forall \chi \in \mathscr{B}^+$ χ has normal form.
   2.2 If some combinators in $\mathscr{B}$ do not have NA property or they have compositive effect then it could be the case that:

   i. $\forall \chi \in \mathscr{B}^+$ χ has normal form (see Example 12.1)
   ii. some χ's in $\mathscr{B}^+$ do not have normal form (see Example 12.2).

Example 12:

1. Let us consider any $\mathscr{B} = \{X\}$ where X doesn't have NA property and has no duplicative effect (e.g., $X \equiv C$ where $Cx_1x_2x_3 > x_1x_3x_2$). By Curry's result, any $\chi \in \{X\}^+$ has normal form.

2. We consider three cases:

   2.1 $\mathscr{B} = \{S\}$ where S does not have NA property and has compositive effect. If we denote SSS by $\mathit{3}$ then $S\mathit{33}(S\mathit{33})$ does not have normal form. In fact, $\mathit{3}x_1x_2 \geqslant x_1x_2(Sx_1x_2)$ and $s_0 \equiv S\mathit{33}(S\mathit{33}) \geqslant s_1 \equiv S\mathit{33}(\mathit{3}(S\mathit{33})) \ldots \geqslant s_2 \equiv S\mathit{33}(\mathit{3}(\mathit{3}(S\mathit{33})))$ and so on.
   2.2 $\mathscr{B} = \{W\}$ where W does not have NA property and no compositive effect. WWW doesn't have normal form.
   2.3 $\mathscr{B} = \{X\}$ where X is any combinator which has NA property and compositive effect.
   In general, the problem of determining whether or not termination is guaranteed for all $\chi \in \{X\}^+$ when NA property is the only condition on X is open. In some particular cases that problem has a simple solution. For example, if $Xx_1x_2x_3 > x_1(x_2x_2)$, then termination is guaranteed for all $\chi \in \{X\}^+$ (this is a consequence of Example 10).

5 ***Conclusions and related work*** We introduced the NA property for proving termination of combinators in Weak Combinatory Logic (WCL). The result is applicable also to term-rewriting systems.

The proof of the main theorem is based on the finiteness of the set of terms that can be obtained by reduction from a given combinator and on the absence of cycling reductions (cf. [6]).

Some other methods have been used by other authors for similar proofs and we would like to refer to [3] and [4], in which "well ordering" and "value preserving" techniques were used for proving termination of "context free" tree rewriting systems. With regard to those methods we notice that: (i) NA property does not allow the direct application of a "well-ordering" technique because it defines a nonstrict order between the structure of the redex and the structure of the contractum; (ii) our termination result is obtained in WCL whose power of computation is the same as a Turing machine (i.e., a "type 0" grammar and not a "context free" one). We would like also to relate our study with Sanchis' result on termination of terms of typed λ-calculus [8]. Obviously WCL terms can be translated into λ-calculus, but, in the case of combinators with NA property, self application is possible and, in general, typed λ-calculus would not be sufficient for such a translation. Therefore our result is outside the scope of Sanchis'.

In general, for type 0 grammars the termination problem is undecidable, and our study is an effort in the direction of defining properties for extending the class of terms in which termination is decidable. It could also have useful applications in equational logic and in the theory of recursive equations, LISP and Lucid (as shown in [5]) as well as in type-free languages for tree manipulation (cf. [7]).

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