Notre Dame Journal of Formal Logic
Volume 22, Number 1, January 1981

The Completeness of Intuitionistic Propositional Calculus for its Intended Interpretation

JOHN P. BURGESS

If certain plausible though not absolutely compelling assumptions about choice sequences are admitted as postulates of intuitionistic analysis, then the usual formal system of intuitionistic propositional calculus can be proved complete for its intended interpretation: Any formula of the system which is intuitively correct no matter what propositions of intuitionistic mathematics are substituted for its variables can be formally deduced as a thesis of the system. This result has been known for some twenty years now, Kreisel's [5] being the first fully worked-out version. But thus far a streamlined and self-contained account of Kreisel's completeness theorem has been lacking in the literature. The aim of the present notes is to supply that lack.

We work with a well-known equivalent, presented in Section 1, of Heyting's 'classic' axiomatization [2]. The first step in a proof of completeness of such an axiomatization for its intended interpretation is always to prove completeness for some useful though artificial *un*intended interpretation, e.g., the topological models of McKinsey and Tarski, or the tree models of Beth.

We prefer to work with the relational models of Kripke, presenting in Section 2 a proof of Kripke's completeness theorem which, like the original proof [8] (so far as the latter pertains to propositional calculus), is finitistic. For a finitistic proof, ours is relatively quick and painless.

The Outlaw Schema, the choice-sequence assumption on which our work depends, is expounded in Section 3. Its statement requires only symbols for the basic operations of logic and arithmetic, and quantification over natural numbers and infinite sequences thereof. Our work requires (besides the Outlaw Schema) only noncontroversial axioms of intuitionistic logic and arithmetic.

Received May 25, 1979; revised November 16, 1979

Formally, the Outlaw Schema can be written in the austere language of Kleene [4], and our work can be formalized in that language using only Kleene's basic Postulates A-D.

Though the Outlaw Schema can be formalized 'extensionally', it can only be justified 'intensionally', through appeal to the notion of *lawless* sequences developed in the work of Kreisel, Troelstra, and others (see, e.g., [10]). It must be confessed that intuitionists are not uniformly enthusiastic about this notion. Brouwer himself expressed doubts in a mysterious footnote, and today the Nijmegen school has its reservations (see the historical appendix to [10]). We leave it to the reader to judge the plausibility of the Outlaw Schema, confining ourself here to two remarks: First, while certain classical tautologies, e.g., $\forall p(\neg p \vee \neg\neg p)$, can be refuted intuitionistically using only Continuity Principles, no *general* refutation of all intuitionistic nontheses is known which does not involve lawlessness. Second, the Outlaw Schema is at least *consistent* with such better-known postulates as Dependent Choice, Bar Induction, Creative Subject, and $\forall\exists!$-Continuity. For as J. R. Moschovakis [9] has shown, all these hold in Scott's topological model of intuitionistic analysis; and we indicate in Section 4 that the Outlaw Schema holds there, too.

At last in Section 5 we are ready to prove the completeness of intuitionistic propositional calculus for its intended interpretation as the logic of intuitionistic mathematics. Can this result be extended? Work of Veldman and de Swart of Nijmegen has been interpreted by Dummett as providing a completeness proof for the negation-free part of intuitionistic predicate calculus; work of Gödel and Kreisel is usually interpreted as ruling out any completeness proof for full predicate calculus. Dummett ([1], Ch. 5) has a full account of these matters.

1 ***Rudiments*** As mentioned in the introduction, we will work with a variant of the usual Heyting axiomatization. As primitives for our system I of intuitionistic propositional calculus we take conjunction &, disjunction $\vee$, implication $\rightarrow$, and absurdity f. As sole rule of inference for I we take modus ponens (*MP*): from ϕ and $\phi \rightarrow \psi$ to infer ψ. As axioms for I we take all substitution instances of the following:

Axioms

(a) $p \rightarrow (q \rightarrow p)$
(b) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
(c) $f \rightarrow p$
(d) $(p \,\&\, q) \rightarrow p$
(e) $(p \,\&\, q) \rightarrow q$
(f) $p \rightarrow (q \rightarrow (p \,\&\, q))$
(g) $p \rightarrow (p \vee q)$
(h) $q \rightarrow (p \vee q)$
(i) $(p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \vee q) \rightarrow r))$.

Negation can be introduced by definition: $\neg\phi$ abbreviates $\phi \rightarrow f$. (For an equivalent system with negation as primitive, replace Axiom c by $(p \rightarrow \neg q) \rightarrow (q \rightarrow \neg p)$ and $\neg p \rightarrow (p \rightarrow q)$.)

Intuitionism upholds a conceptualist or psychologistic philosophy of mathematics, according to which every mathematical proposition calls for a thought-construction. The classical mathematician trying to understand does not go too far wrong in thinking of the simplest, such as $7 + 5 = 12$, as calling for the verification of a mental calculation. The interpretation of compounds runs along the following lines:

Hermeneutics

(a) $p_0 \& p_1$ calls for a p_0-construction followed by a p_1-construction
(b) $p_0 \vee p_1$ calls for a choice of $i \in \{0, 1\}$ followed by a p_i-construction
(c) $p_0 \rightarrow p_1$ calls for the construction of a method together with a proof that it will convert any p_0-construction into a p_1-construction
(d) f calls for the impossible, a verification that $0 = 1$
(e) $\forall x P(x)$ calls for the construction of a method together with a proof that applied to any ξ in the range of x it will produce a $P(\xi)$-construction
(f) $\exists x P(x)$ calls for the construction of some ξ in the range of x followed by a $P(\xi)$-construction.

"The only positive contention" the intuitionist ([2], p. 11) opposes "is that classical mathematics has a clear sense". The classicist returns the compliment, calling the explanations above circular, impredicative, a case of the obscure explained through the more obscure. Yet the candid classicist must admit that these explanations make intelligible the notorious rejection of the excluded middle. Read intuitionistically, $\forall p(p \vee \neg p)$ amounts to the preposterous claim to have a general method for settling all mathematical questions. And one can see this from the above explanations however imperfect one's grasp of 'construction', 'method', or 'proof' may be.

One can equally well see why $\forall p \forall q(p \rightarrow (q \rightarrow p))$ is accepted. For on the above reading of $\rightarrow$, we can assert this if we possess a method M which demonstrably converts any p-construction C into a $(q \rightarrow p)$-construction, that is, into a method M' for converting any q-construction C' into a p-construction C''. Plainly, one such M is that which, given C, produces this M': Throw away C', and take $C'' = C$. It is an easy and pleasant task to 'talk through' Axioms b-i in this way, and prove:

Theorem 1 (Soundness) *Every thesis of I is intuitively correct no matter what propositions of intuitionistic mathematics are substituted for its variables.*

2 Models Intuitionistically, a proposition is *decided* if either it or its negation holds, and a set is *decidable* if it is decided for every relevant mathematical construct whether it belongs to the set. For present purposes, relational 'semantics' will be introduced thus:

Definition 1 A *model* is a triple (W, R, S) where:

(a) W is a finite decidable set of elements called *worlds.*
(b) R is a decidable partial order on W called *accessibility.*
(c) S is a constructive function assigning each world a finite decidable set of propositional variables of I in a *cumulative* way: if uRv, then $S(u) \subseteq S(v)$.

Definition 2 The relation $\models$ of a formula of I being *realized* at a world w in a model (W, R, S) is defined inductively:

(a) $w \models p$, a variable iff $p \in S(w)$
(b) not $w \models f$
(c) $w \models \phi \,\&\, \psi$ iff $w \models \phi$ and $w \models \psi$
(d) $w \models \phi \vee \psi$ iff $w \models \phi$ or $w \models \psi$
(e) $w \models \phi \rightarrow \psi$ iff whenever wRv and $v \models \phi$, then $v \models \psi$.

A little thought shows that $\models$ is decidable and cumulative: if uRv and $u \models \phi$, then $v \models \phi$. A formula is *realized throughout* a model if it is realized at each of its worlds, and is *universally realized* if it is realized throughout every model. It is a tedious but routine exercise in unpacking definitions to check that each of Axioms a-i is universally realized, and that *MP* preserves this property, thus showing:

Theorem 2 (Pseudo-Soundness) *Every thesis of I is universally realized.*

We wish to prove the converse *intuitionistically,* indeed finitistically.

A *deduction* of a formula ϕ from a set of formulas Φ is a finite list of formulas ending with ϕ, each of which is either an element of Φ, or an axiom of I, or follows from earlier formulas on the list by *MP*. If such exists we say ϕ is *deducible* from Φ and write $\Phi \vdash \phi$. Thus ϕ is a thesis iff it is deducible from the empty set Λ.

Lemma 1

(a) $\phi \in \Phi$ *and* $(\phi \rightarrow \psi) \in \Phi$ *imply* $\Phi \vdash \psi$
(b) $\theta = \phi \rightarrow \psi$ *and* $\psi \in \Phi$ *imply* $\Phi \vdash \theta$
(c) $f \in \Phi$ *implies* $\Phi \vdash \phi$
(d) $(\phi \,\&\, \psi) \in \Phi$ *implies* $\Phi \vdash \phi$ *and* $\Phi \vdash \psi$
(e) $\theta = \phi \,\&\, \psi$ *and* $\phi \in \Phi$ *and* $\psi \in \Phi$ *imply* $\Phi \vdash \theta$
(f) $\theta = \phi \vee \psi$ *and* $\phi \in \Phi$ *or* $\psi \in \Phi$ *imply* $\Phi \vdash \theta$.

Proof: (a) requires only *MP*; (b) uses also Axiom a; (c) uses c; (d) uses d and e; (e) uses f; (f) uses g and h.

Lemma 2 (Deduction Theorem) *If* $\theta = \phi \rightarrow \psi$ *and* $\Phi \cup \{\phi\} \vdash \psi$, *then* $\Phi \vdash \theta$.

Proof: This can be proved constructively for any system containing *MP*, Axioms a,b, and hence their consequence $p \rightarrow p$. The proofs in most classical logic texts, e.g., [3], are adequate for our purposes.

Lemma 3 *If* $\theta = \phi_0 \vee \phi_1$ *and* $\Phi \cup \{\phi_0\} \vdash \psi$ *and* $\Phi \cup \{\phi_1\} \vdash \psi$, *then* $\Phi \cup \{\theta\} \vdash \psi$.

Proof: Under these hypotheses the Deduction Theorem provides deductions of $\phi_0 \rightarrow \psi$ and of $\phi_1 \rightarrow \psi$ from (Φ and *a fortiori* from) $\Phi \cup \{\theta\}$. Adding to these a few more steps using Axiom i and *MP*, we easily get a deduction of ψ.

We now suppose that a suitable Gödel numbering of all finite lists of formulas of I has been introduced. Let χ be an arbitrary but fixed formula. We reserve ϕ and ψ and θ to range over its subformulas, and Φ and Ψ and Θ to range over *decidable* sets of such subformulas. There are only finitely many of these.

Now in Lemma 1 the antecedent of any of (a)-(f) is always decided. It follows that there exists a fixed number n_0 such that whenever one of these antecedents is fulfilled for some given subformulas of χ and some given set of subformulas, then there is a deduction of the sort called for by the corresponding consequent which has Gödel number $< n_0$.

A given Φ will be called *n-closed* if whenever $\Phi \vdash \phi$ by a deduction of Gödel number $< \max(n_0, n)$, then $\phi \in \Phi$. Note that an n-closed set is closed under simple operations corresponding to the clauses of Lemma 1; for instance, (1a) gives closure under *MP*; also such a set is m-closed for any $m < n$.

Starting from a given Φ one can, by a constructive process whose steps involve adding to Φ certain formulas deducible from it, arrive after a finite number of steps at a decidable set with the property of being the smallest n-closed set containing Φ. This we call the *n-closure* $K(n, \Phi)$ of Φ.

The proofs of Lemmas 2 and 3 being constructive, they give us constructive functions γ and α such that:

whenever $\theta = \phi \to \psi$ and $\psi \in K(n, \Phi \cup \{\phi\})$, then $\theta \in K(\gamma(n), \Phi)$
whenever $\theta = \phi_0 \vee \phi_1$ and $\psi \in K(n, \Phi \cup \{\phi_0\}) \cap K(n, \Phi \cup \{\phi_1\})$, then $\psi \in K(\alpha(n), \Phi \cup \{\theta\})$.

We write α^i for the i^{th} iterate of α, so $\alpha^0(n) = n$, $\alpha^1(n) = \alpha(n)$, $\alpha^2(n) = \alpha(\alpha(n))$, etc.

For any given Φ, let $C(\Phi)$ be the set of all subformulas θ of χ of form $\phi \to \psi$ with neither ϕ nor θ in Φ. Let $A(\Phi)$ be the set of all $\theta = \phi_0 \vee \phi_1$ with neither ϕ_i in Φ. Note that if $\Phi \subseteq \Psi$, then $C(\Psi) \subseteq C(\Phi)$ and $A(\Psi) \subseteq A(\Phi)$. We bring this tedious series of definitions to a close by calling Φ *bisective* if whenever it contains a disjunction, then it contains at least one of the disjuncts, i.e., if $A(\Phi) \cap \Phi = \Lambda$.

Lemma 4 *If $\psi \notin K(\alpha^{\operatorname{card} A(\Phi)}(n), \Phi)$, then there exists an n-closed, bisective Ψ with $\Phi \subseteq \Psi$ and $\psi \notin \Psi$.*

Proof: Let $a = \operatorname{card} A(\Phi)$. We will construct inductively Ψ_m for $m \leqslant a$ so that:

(a) Ψ_m is $\alpha^{a-m}(n)$-closed, and hence n-closed
(b) $\operatorname{card} A(\Psi_m) = \max(0, \operatorname{card} A(\Psi_{m-1}) - 1)$, if Ψ_{m-1} is not bisective, and hence $\leqslant a-m$ in this case
(c) $\Phi \subseteq \Psi_0 \subseteq \Psi_1 \subseteq \ldots \subseteq \Psi_a$
(d) $\psi \notin \Psi_m$.

Indeed, let $\Psi_0 = K(\alpha^a(n), \Phi)$. Then suppose $m < a$ and we have Ψ_m. If it is already bisective, set $\Psi_{m+1} = \Psi_m$. Otherwise, pick an element $\phi_0 \vee \phi_1$ of $A(\Psi_m) \cap \Psi_m$. By the defining property of α, one (at least) of the sets $K(\alpha^{a-m-1}(n), \Psi_m \cup \{\phi_i\})$ for $i \in \{0, 1\}$ does not contain ψ. Let Ψ_{m+1} be this set. It is easy to verify (a)-(d) inductively. To complete the proof, take $\Psi = \Psi_a$.

Now let $\mu(n) = \gamma(\alpha^{\operatorname{card} A(\Lambda)}(n))$, and $\nu(\Phi) = \mu^{\operatorname{card} C(\Phi)}(0)$. Call Φ *replete* if it is $\nu(\Phi)$-closed, bisective, and does not contain f.

Lemma 5 *If $\theta = \phi \rightarrow \psi$ and Φ is replete and $\theta \notin \Phi$, then there exists a replete Ψ with $\Phi \cup \{\phi\} \subseteq \Psi$ and $\psi \notin \Psi$.*

Proof: *Case 1.* $\phi \in \Phi$. By Lemma 1b, $\theta \notin \Phi$ implies $\psi \notin \Phi$. So take $\Psi = \Phi$. *Case 2.* $\phi \notin \Phi$, hence $\theta \in C(\Phi)$. Let $c = \text{card } C(\Phi)$, $a = \text{card } A(\Lambda)$. (Note card $A(\Phi) \leqslant a$.) Now $\theta \notin \Phi = K(n, \Phi)$ for $n = \nu(\Phi) = \mu^c(0) = \gamma(\alpha^a(\mu^{c-1}(0)))$. By the defining property of γ, $\psi \notin K(\alpha^a(\mu^{c-1}(0)), \Phi \cup \{\phi\})$—call this set Θ. By Lemma 4, there is a $\mu^{c-1}(0)$-closed, bisective Ψ containing Θ with $\psi \notin \Psi$ (and hence by Lemma 1c with $f \notin \Psi$). Since $\theta \in C(\Phi)$, card $C(\Psi) < c$, and $\nu(\Psi) \leqslant \mu^{c-1}(0)$, so Ψ is replete.

Definition 3 The *master model* $M(\chi)$ for χ is the model (W, R, S) where:

(a) W is the set of all replete Φ
(b) R is the inclusion relation $\Phi \subseteq \Psi$
(c) S assigns to each Φ precisely the set of all propositional variables in Φ.

Lemma 6 *For all subformulas θ of χ and all replete Φ we have:*

$$\Phi \vDash \theta \textit{ in } M(\chi) \textit{ iff } \theta \in \Phi \textit{ as a set of formulas.}$$

Proof: By induction on the complexity of ϕ, unpacking all the definitions and using the simple closure properties of Lemma 1. We do the case $\theta = \phi \rightarrow \chi$, leaving the rest for the reader. Assume as Induction Hypothesis that the Lemma holds for ϕ and for ψ. Either $\theta \in \Phi$ or $\theta \notin \Phi$.

In the former case, things are easy. For any Ψ with $\Phi R \Psi$ and $\Psi \vDash \phi$, we have $(\phi \rightarrow \psi) = \theta \in \Phi \subseteq \Psi$, and by Induction Hypothesis, $\psi \in \Psi$. Replete sets are closed under *MP* by Lemma 1a, so $\psi \in \Psi$ and by Induction Hypothesis $\Psi \vDash \psi$. Thus the Definition 2e of $\Psi \vDash \theta$ is fulfilled.

In the opposite case, we need to invoke Lemma 5 to get a replete Ψ containing $\Phi \cup \{\phi\}$ with $\psi \notin \Psi$. Plainly $\Phi R \Psi$ and by Induction Hypothesis $\Psi \vDash \phi$ and not $\Psi \vDash \psi$. So the definition of $\Psi \vDash \theta$ is not fulfilled.

Lemma 7 *Either χ is a thesis of I or else it does not hold throughout $M(\chi)$.*

Proof: It is decided whether $\chi \in K(\alpha^{\text{card} A(\Lambda)}(\nu(\Lambda)), \Lambda)$. If so, it is deducible from Λ and a thesis. If not, Lemma 4 provides a replete Ψ with $\chi \notin \Psi$. By Lemma 6, χ is not realized at Ψ in $M(\chi)$.

We thus have:

Theorem 3 (Pseudo-Completeness, cf. [8]) *If a formula is universally realized, then it is a thesis of I.*

Theorem 4 *The set of theses of I is decidable.*

It remains to establish a connection between being universally realized and being intuitively correct.

3 Lawlessness For intuitionism, mathematics is 'all in the mind', and an actual completed infinity won't fit there. *Potentially* infinite or infinitely *proceeding* sequences of natural numbers form the basis of intutionistic analysis. By any given time, the construction of no more than a finite number of terms of such a sequence will actually have been completed, but there is in principle no bound or limit to the sequence's eventual growth. We reserve $\alpha, \beta, \gamma, \delta$ for such sequences.

The simplest sequences are those, like the famous one whose k^{th} term is the k^{th} digit in the decimal expansion of π, for which a law of construction can be given, determining in advance what each term of the sequence is to be. Such are called *constructive* or *lawful* sequences.

Intuitionists, in contrast to finitists, also admit 'choice' sequences generated by the creative will of the mathematical subject. (Some also admit empirical sequences, arrived at through passive experience.) The simplest of these would seem to be the so-called *absolutely free* or *lawless* sequences, in which terms are obtained by *independent* acts of free choice subject to *no advance restrictions* on what number may be chosen.

We may call *projective* those sequences obtainable by a constructive law from a finite number (possibly zero) of lawless sequences. This takes in as degenerate cases the lawful and lawless sequences. We also get for any lawless α such sequences as the following:

$$(2\alpha(0), 2\alpha(1), 2\alpha(2), 2\alpha(3), \ldots)$$
$$(\alpha(0), 0, \alpha(1), 0, \alpha(2), 0, \ldots).$$

A rich analysis can be built up for such sequences.

Prominent among the projective sequences are what we will call *outlaw* sequences: An outlaw sequence β is introduced by stipulating what a certain finite number of its terms are to be, and by stipulating that the rest of its terms are to be copied from some previously introduced lawless sequence α. We aim to derive with 'informal rigor' (cf., [6]) some key properties of outlaw sequences. Our work will be, we hope, in the spirit of the 'classic' treatments of lawlessness (e.g., [7]). First a couple of definitions.

If I say today "let α be the sequence $\alpha(k) = 1 + 3 + 5 + \ldots + (2k + 1)$" and say tomorrow "let β be the sequence $\beta(k) = (k + 1)^2$", there is a sense in which α and β are different, and a sense in which they are the same. They are *given* by different stipulations, and we say they are not (intensionally) *identical.* They have, however, the same *course of values,* and we say they are (extensionally) *equal.* Note that by virtue of the meaning (2e) of intuitionistic $\forall$, equality $\forall k(\alpha(k) = \beta(k))$ never holds 'by accident'; it requires a proof.

A property of sequences (or a relation between numbers and sequences) will be called *strictly mathematical* if it can be expressed by a formula *without parameters* of the austere language of [4] mentioned in the introduction. This includes most properties commonly met with in mathematics, but not 'being lawful' or 'being lawless' or 'being equal to α' (unless α happens to be definable). If P is strictly mathematical, then whenever $P(\alpha)$ holds and α and β are equal, then $P(\beta)$ holds.

Now we are ready to state the fundamental insight that will be taken for granted throughout this section:

Intuition For purposes of proving that a given outlaw sequence β has some strictly mathematical property P, the only information about β available at any given time will be:

(a) That a certain finite number of terms of β are what they are.
(b) That the remaining terms of β will result from independent unrestricted choices.

To restate this insight: Let β be an outlaw sequence; so β was introduced by stipulating its k_1st, k_2nd, . . ., k_rth terms, and by stipulating that its remaining terms should be copied from those of some lawless sequence α. At any given time after its introduction, only finitely many terms of β have actually been determined, as *per* (a) above; namely, the k_1st through k_rth, plus the kth term for all those (finitely many) other k for which $\alpha(k)$ has already been chosen by the time in question. The remaining terms of β are ultimately to result from independent unrestricted choices, as *per* (b); for they are copied from the terms of α, which directly result from such choices. Information about β beyond (a) and (b) includes such facts as that it is α, and not some other lawless sequence, from which all but finitely many terms of β are being copied. Such information, according to our fundamental insight, is irrelevant to establishing *strictly mathematical* properties of β. If this is granted, two important principles follow:

Proposition 1 $\forall$ *outlaw* $\beta \forall i \neg \forall n \exists k > n\ \beta(k) \neq i$.

Proposition 2 *For any strictly mathematical P we have:* $\forall$ *outlaw* $\beta[P(\beta) \rightarrow \exists m \forall \gamma(\forall k < m\ \gamma(k) = \beta(k)\ \&\ \exists n \forall k > n\ \gamma(k) = \beta(k) \rightarrow P(\gamma))]$.

Justification: Information of types (a) and (b) can clearly never suffice to prove anything substantive about what *infinitely many* terms of β may or may not be. In particular, no proof based on such information can force us to choose $\beta(k) \neq i$ for infinitely many k. This gives us Proposition 1.

Now suppose we have a proof of $P(\beta)$ for some strictly mathematical P and some outlaw sequence β. If m is taken sufficiently large, all information of type (a) used in the proof is subsumed under the fact that the restriction $\beta|m$ of β to its first m terms is the particular finite string s that it is. The information (b) is equally true of any outlaw sequence. Thus the proof of $P(\beta)$ constitutes a proof of $P(\delta)$ for any outlaw sequence δ with $\delta|m = s$. Indeed, $P(\gamma)$ is true for any γ with $\gamma|m = s$ which is even *equal to* an outlaw sequence. Now the condition $\exists n \forall k > n\ \gamma(k) = \beta(k)$ guarantees that γ is in fact equal to an outlaw sequence, and indeed one for which all but finitely many terms are copied from the same lawless sequence α from which all but finitely many terms of β are copied. Namely, it guarantees that γ is equal to the outlaw sequence δ obtained by suitably stipulating $\delta(0), \delta(1), \ldots, \delta(n)$, and by letting the remaining terms of δ be copied from α. This noted, a little thought gives us Proposition 2.

We are at last in a position to deduce something not mentioning outlawry:

Proposition 3 (Outlaw Schema) *For any strictly mathematical Q we have:*

$$\exists \beta \forall i \{\neg \forall n \exists k > n\ \beta(k) \neq i\ \&\ [Q(i,\beta) \rightarrow \exists m \forall \gamma(\forall k < m\ \gamma(k) = \beta(k)\ \&\ \exists n \forall k > n\ \gamma(k) = \beta(k) \rightarrow Q(i,\gamma))]\}.$$

Proof: Let β be an outlaw sequence and apply Propositions 1 and 2.

We will make heavy use of the Outlaw Schema in Section 5, but will make no further direct reference to the notions of lawlessness and outlawry.

4 Consistency We digress a moment to mention a fact, already alluded to in the introduction, which demonstrates (at least classically) the consistency of the Outlaw Schema with many more popular intuitionistic postulates. But first, some technical apparatus that will be useful in this section and the next.

For every n and every string $\tau = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ of permutations of $\{0, 1, \ldots, n-1\}$, let π_τ be the permutation of the set of all infinite number-sequences given by defining $\pi_\tau(\beta)$ thus:

$$(\pi_\tau(\beta))(k) = \begin{cases} \sigma_k(\beta(k)) \text{ if } k < n \text{ and } \beta(k) < n \\ \beta(k) \text{ otherwise} \end{cases}$$

Let Π be the set of all π_τ (for all relevant n and τ); and let Π_m for each m be the subset consisting of those π_τ coming from strings $\tau = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ in which σ_i is the identity for all $i < \min(m, n)$.

Lemma 8

(a) $\forall\beta\forall n\forall s$ (s a string of length $n \to \exists\pi \in \Pi\ (\forall k < n\ (\pi(\beta))(k) = s(k)\ \&$ $\forall k \geq n\ (\pi(\beta))(k) = \beta(k))$

(b) $\forall\beta\forall\gamma(\exists n\forall k > n\ \gamma(k) = \beta(k) \longleftrightarrow \exists\pi \in \Pi\ \forall k\ \gamma(k) = (\pi(\beta))(k))$

(c) $\forall\beta\forall m\forall\gamma[(\forall k < m\ \gamma(k) = \beta(k)\ \&\ \exists n\forall k > n\ \gamma(k) = \beta(k)) \longleftrightarrow \exists\pi \in \Pi_m\forall k$ $\gamma(k) = (\pi(\beta))(k)]$.

Proof: (a) Given β and s, let $n' > n$ be so large that $s(k) < n'$ and $\beta(k) < n'$ for all $k < n$. Let $\tau = (\sigma_0, \sigma_1, \ldots, \sigma_{n'-1})$ where for $k < n$, σ_k is the permutation of $\{0, 1, \ldots, n' - 1\}$ switching $s(k)$ and $\beta(k)$, while for $n \leq k < n'$, σ_k is the identity. Then $\pi = \pi_\tau$ satisfies the requirements of (a).

(b) To go from left to right, assuming $\forall k > n\ \gamma(k) = \beta(k)$, simply apply (a) to β and $s = \gamma|n$ to obtain the required π. To go from right to left is immediate. (c) is similar to (b).

We can now state:

Proposition 4 *The Outlaw Schema is verified in the topological model of intuitionistic 2nd order arithmetic* (of [9]).

Sketch of a Classical Proof: Recall that the *Baire space* U is the set of all infinite number-sequences equipped with the topology generated by the sets $U_s = \{\alpha\colon \alpha \text{ extends } s\}$, where s ranges over all finite strings. In the topological model, the *truth-value* $\|P\|$ of a proposition P is an open subset of U, and P is considered *verified* by the model if $\|P\| = U$. In the inductive definition of $\|P\|$, sequence variables α, β, γ are instantiated by continuous functions from U to U.

In verifying the Outlaw Schema, it proves convenient to invoke (21c) and replace the schema as originally formulated by the following equivalent:

$$(*)\quad \exists\beta\forall i\{\neg\forall n\exists k > n\ \beta(k) \neq i\ \&\ [Q(i,\beta) \to \exists m\forall\pi \in \Pi_m Q(i, \pi(\beta))]\}.$$

Instantiate the variable β by the identity function η on U. For each i, the fact that $\{\alpha\colon \exists n\forall k > n\ \alpha(k) = i\}$ is a dense subset of U implies, by techniques used again and again in [9], that the first conjunct of (*) is verified. Those same techniques reduce the verification of the second conjunct to proving this: If some basic open set U_s is contained in $\|Q(i,\eta)\|$, then there exists an m such that for all $\pi \in \Pi_m$, U_s is contained in $\|Q(i,\pi)\|$. To prove this, we invoke [9], Lemma 7, which says:

$$U_s \subseteq \|Q(i,\pi)\| \longleftrightarrow \pi^{-1}[U_s] \subseteq \|Q(i,\eta)\|.$$

It will clearly suffice to choose as m the length of the string s, for then $\pi^{-1}[U_s] = U_s$ for all $\pi \in \Pi_m$.

To keep the length of this digression in bounds, it has been necessary to suppress many details. However, the reader acquainted with [9] should have no serious difficulty in reconstructing a fully rigorous proof. It will, however, (like many of the techniques of [9], including Lemma 7 of [9] in particular) only be classically valid.

*5 **Completeness*** Returning now to our main task, let χ be, as in Section 2, an arbitrary but fixed formula of I, and reserve the letters ϕ, ψ, θ for its subformulas. Let $(W,R\ S)$ be an isomorph of the master-model $M(\chi)$ with $W = \{0, 1, \ldots, r\}$ for some r.

To each infinite number-sequence α we associate an infinite sequence α^* of possible worlds, each accessible from the one before:

$$\alpha^*(n) = \begin{cases} \min(r, \alpha(0)) & \text{if } n = 0 \\ \alpha(n) & \text{if } n > 0 \text{ and } \alpha(n) \leqslant r \text{ and } \alpha^*(n-1)R\alpha(n) \\ \alpha^*(n-1) & \text{otherwise.} \end{cases}$$

Note that if $\alpha|n = \beta|n$, then $\alpha^*|n = \beta^*|n$.

Lemma 9

(a) $\exists n(\alpha^*(n) \vDash \phi \rightarrow \psi)\ \&\ \exists n(\alpha^*(n) \vDash \phi) \rightarrow \exists n(\alpha^*(n) \vDash \psi)$
(b) $\neg\exists n(\alpha^*(n) \vDash f)$
(c) $\exists n(\alpha^*(n) \vDash \phi\ \&\ \psi) \rightarrow \exists n(\alpha^*(n) \vDash \phi)\ \&\ \exists n(\alpha^*(n) \vDash \psi)$
(d) $\exists n(\alpha^*(n) \vDash \phi)\ \&\ \exists n(\alpha^*(n) \vDash \psi) \rightarrow \exists n(\alpha^*(n) \vDash \phi\ \&\ \psi)$
(e) $\exists n(\alpha^*(n) \vDash \phi \vee \psi) \rightarrow \exists n(\alpha^*(n) \vDash \phi) \vee \exists n(\alpha^*(n) \vDash \psi)$
(f) $\exists n(\alpha^*(n) \vDash \phi) \vee \exists n(\alpha^*(n) \vDash \psi) \rightarrow \exists n(\alpha^*(n) \vDash \phi \vee \psi)$.

Proof: (a) Suppose $\alpha^*(n_0) \vDash \phi \rightarrow \psi$ and $\alpha^*(n_1) \vDash \phi$, and let $n = \max(n_0, n_1)$. Since $\vDash$ is cumulative, both $\phi \rightarrow \psi$ and ϕ are realized at $\alpha^*(n)$. From Definition 2e it follows $\alpha^*(n) \vDash \psi$, proving (a).

(b)-(f) are equally straightforward, unpacking the definitions.

It is a tedious but routine exercise in Gödel numbering to produce a strictly mathematical relation Q between numbers and sequences such that as i runs through all natural numbers, $Q(i, \alpha)$ runs through all properties of the form:

$$\exists n((\pi(\alpha))^*(n) \vDash \phi) \rightarrow \exists n((\pi(\alpha))^*(n) \vDash \psi)$$

for $\pi \in \Pi$ and for ϕ and ψ subformulas of χ. Apply the Outlaw Schema (or its equivalent (*) of the last section) to this particular Q to obtain a sequence β, and set $B = \{\pi(\beta): \pi \in \Pi\}$.

Lemma 10

(a) $\forall\alpha(\exists i \exists n \forall k > n\ \alpha(k) = i \rightarrow \neg\forall\gamma \in B \exists k\ \gamma(k) \neq \alpha(k))$
(b) $\forall\psi, \psi\forall\gamma \in B\{[\exists n(\gamma^*(n) \vDash \phi) \rightarrow \exists n(\gamma^*(n) \vDash \psi)] \rightarrow$
$\exists m \forall\delta \in B[\forall k < m\ \delta(k) = \gamma(k)\ \&\ \exists n(\delta^*(n) \vDash \phi) \rightarrow \exists n(\delta^*(n) \vDash \psi)]\}$.

Proof: (a) Suppose for contradiction that $\forall k > n_0\, \alpha(k) = i$ but that $\forall\pi \in \Pi$ $\exists k(\pi(\beta))(k) \neq \alpha(k)$. Given any $n \geqslant n_0$, apply Lemma 8a to β and $s = \alpha | n$ to obtain a certain π. By our supposition $\exists k(\pi(\beta))(k) \neq \alpha(k))$. Necessarily $k > n \geqslant n_0$, so $\beta(k) = (\pi(\beta))(k)$ and $\alpha(k) = i$. Thus we have shown $\forall n \exists k > n\ \beta(k) \neq i$, contrary to the first property claimed for β in the Outlaw Schema.

(b) is an equally routine, and equally tedious, matter of unpacking definitions, and is left as an exercise.

Lemma 11 $\forall\phi, \psi\forall\gamma \in B\{[\exists n(\gamma^*(n) \vDash \phi) \rightarrow \exists n(\gamma^*(n) \vDash \psi)] \rightarrow \exists n(\gamma^*(n) \vDash \phi \rightarrow \psi)\}$.

Proof: Indeed, suppose that $\gamma \in B$ satisfies the antecedent. Apply Lemma 10b, and let m be the number given by that lemma. We claim $\gamma^*(m-1) \vDash \phi \rightarrow \psi$.

Note that it is decided whether this is so or not, so it will suffice to derive a contradiction from the assumption that it is not. In that case, $\exists w(\gamma^*(m-1)\, Rw\ \&\ w \vDash \phi\ \&\ \neg w \vDash \psi)$. Let α be the sequence agreeing with γ in its first m terms and taking the constant value w thereafter.

For any $\delta \in B$, either $\exists k \leqslant m\ \delta(k) \neq \alpha(k)$, or else $\delta | m = \gamma | m$ and $\delta(m) = w$. Consider the latter case. Going back to the definitions we see that $\delta^*(m-1) = \gamma^*(m-1)$ and that $\delta^*(m) = w \vDash \phi$. Then by choice of m we must conclude $\exists n\, \delta^*(n) \vDash \psi$. But for such an n, $\delta^*(n) \neq w = \alpha^*(n)$, so we must have $\delta(k) \neq \alpha(k)$ for some $k \leqslant n$. Thus in either case, $\forall\delta \in B \exists k\ \delta(k) \neq \alpha(k)$, contradicting Lemma 10a.

We next consider the result of substituting propositions of form $\exists n(\gamma^*(n) \vDash p_i)$ for the variables p_i of subformulas of χ.

Lemma 12 *For any $\gamma \in B$ and any ϕ involving variables $p_1, \ldots, p_s$ we have:* $\phi(\exists n(\gamma^*(n) \vDash p_1), \ldots, \exists n(\gamma^*(n) \vDash p_s)) \longleftrightarrow \exists n(\gamma^*(n) \vDash \phi(p_1, \ldots, p_s))$.

Proof: A straightforward induction using Lemmas 9 and 11.

Lemma 13 *If χ is intuitively correct no matter what propositions of form $\exists n(\alpha(n) = 0)$ are substituted for its variables, then it holds throughout the master model $M(\chi)$.*

Proof: The proposition $\exists n(\gamma^*(n) \vDash p_i)$ is equivalent to $\exists n(\alpha(n) = 0)$, where α is so defined that $\alpha(n) = 0$ if $p_i \in S(\gamma^*(n))$ and $=1$ otherwise. Thus the antecedent of the lemma implies that the result of substituting $\exists n(\gamma^*(n) \vDash p_i)$ for the variable p_i in χ is intuitively correct for any γ. Lemma 12 then tells us $\forall\gamma \in B \exists n(\gamma^*(n) \vDash \chi)$.

Now, it is decided whether χ holds throughout $M(\chi)$. Suppose for contradiction that it is not, and hence is also not realized at some w in the isomorphic model (W, R, S). Let α be the constant sequence $\alpha(k) = w$. Then we have $\forall\gamma \in B \exists k\ \gamma(k) \neq \alpha(k)$, contrary to Lemma 10a.

Putting everything together:

Completeness Theorem (cf. [5]) *If a formula is intuitively correct no matter what propositions of analysis of form* $\exists n(\alpha(n) = 0)$ *are substituted for its variables, then it is a thesis of I.*

REFERENCES

[1] Dummett, M. A. E., *Elements of Intuitionism,* Clarendon, Oxford, 1977.

[2] Heyting, A., *Intuitionism: An Introduction,* North-Holland, Amsterdam, 1956.

[3] Kleene, S. C., *Introduction to Metamathematics,* Van Nostrand, Princeton, 1952.

[4] Kleene, S. C. and R. E. Vesley, *The Foundations of Intuitionistic Mathematics,* North-Holland, Amsterdam, 1965.

[5] Kreisel, G., "A remark on free choice sequences and the topological completeness proof," *The Journal of Symbolic Logic,* vol. 23 (1958), pp. 369-388.

[6] Kreisel, G., "Informal rigor and completeness proofs," pp. 138-186 in *Problems in the Philosophy of Mathematics,* ed., I. Lakatos, North Holland, Amsterdam, 1967.

[7] Kreisel, G., "Lawless sequences of natural numbers," *Compositio Mathematica,* vol. 20 (1968), pp. 222-248.

[8] Kripke, S. A., "Semantical analysis of intuitionistic logic I," pp. 93-130 in *Formal Systems and Recursive Functions,* eds., J. N. Crossley and M. A. E. Dummett, North Holland, Amsterdam, 1965.

[9] Moschovakis, J. R., "A topological interpretation of second-order intuitionistic arithmetic," *Compositio Mathematica,* vol. 26 (1973), pp. 261-275.

[10] Troelstra, A. S., *Choice Sequences: A Chapter in Intuitionistic Mathematics,* Clarendon Press, Oxford, 1976.

Department of Philosophy
Princeton University
Princeton, New Jersey 08544