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A THEORY OF CLASSES AND INDIVIDUALS BASED ON A 3-VALUED SIGNIFICANCE LOGIC

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1 *Introduction*

1.1 *The need for the theory* A problem that arises when one tries to introduce individuals into a theory of sets or classes is that of distinguishing the null class from an individual, since both have no members.* The class theory will contain an axiom of extensionality which will identify two classes or individuals if they have exactly the same members. The null class and an arbitrary individual will have no members and hence, by the axiom, be identical.

The difficulty is discussed by Quine [9], pp. 29-32. One way out is to use separate variables for individuals and for classes or to introduce the primitive predicate 'is an individual' into the system. Quine dismisses these as "unwelcome sacrifices of elegance" and says that happily these can be avoided. Quine instead suggests regarding $x \epsilon y$, where y is an individual, as $x = y$. This avoids the problem with the axiom of extensionality because if y and z are individuals $(\mathbf{A}x)(x \epsilon y \equiv x \epsilon z)$ is equivalent to $(\mathbf{A}x)(x = y \equiv x = z)$, i.e., $y = z$. Quine also shows that this implies that an individual is equal to its unit class and says that this does not affect the development of class theory as required for mathematics. But if one takes a material object and forms its unit class, then, according to Quine, this material object would be equal to its unit class, an abstract entity, and this is unsatisfactory.

By taking $x \epsilon y$ as nonsignificant when y is an individual and using a 3-valued significance logic,[1] one can avoid all the problems that have arisen in connection with distinguishing the null class from individuals. The predicate 'is an individual' can be defined in terms of the logic, i.e., $I(x) =_{df} \sim(\mathbf{S}y)S(y \epsilon x)$, i.e., $y \epsilon x$ is nonsignificant for all y, where the variables x and y range over classes and individuals.

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The advantages of this over Quine's are obvious. No longer can it be said that there is an "unwelcome sacrifice of elegance". One can distinguish between individuals and their unit classes and avoid the identity of a material object with an abstract entity. The axiom of extensionality has to be restricted to classes using the predicate 'is a class' to restrict the general variables to class variables. The identity of individuals would have to be established separately within the theory of individuals.

Just as ϵ is taken as a paradigm predicate used for generating classes by an abstraction axiom, membership of an individual can be taken as a paradigm case of nonsignificance for generating significance ranges. The rest of the classes can be obtained by adding arbitrary predicates, which does not affect the consistency nor the general features of the theory. Similarly with significance ranges, by the addition of arbitrary predicates, the significance ranges of these predicates can be formed.

A theory of classes and individuals based on a 3-valued significance logic is the next theory to be developed after completing the development of some 3-valued significance predicate logics as in [1]. The theory is also needed for the development of significance range theory because significance ranges are classes which can be generated from their own form of abstraction axiom, which would have to be added to such a theory of classes.

1.2 *The choice of the formal theory* I use the functionally-complete significance logic in [1], because it is necessary to be able to restrict general variables so that they range over individuals, over classes, and over sets.

I use an axiomatic theory of individuals to ensure the existence of individuals and to widen the scope of the formal theory so that fusions[2] of individuals as well as classes of individuals can be formed. Suppes points out about the set theory **ZF**, "However, our axioms do not actually postulate the existence of any individuals, and they are thus consistent with the view that there are only sets in the domain of discourse" ([12], p. 20). The addition of an axiomatic theory of individuals to such a set theory would overcome this problem. The theory of individuals I use is due to Goodman [6], but some additions are made in order to adapt it for inclusion into a theory of classes and individuals. Also the primitive '$\circ$' (read 'overlaps') needs to be interpreted in such a way that it is significant for any two individuals to overlap.

I use the class theory **NBG** as it is stronger than **ZF** and I follow Mendelson's treatment in Chapter 4 of [8] except for certain modifications due to the presence of individuals or due to the use of the 3-valued significance logic.

2 *The formal theory*

1. Primitives

$U', V', W', X', Y', Z', \ldots$ (general variables over classes and individuals)

$\circ$ (overlaps), ϵ (is a member of)
$\sim$, $\supset$, T_n (connectives of the 3-valued significance logic)
A, **S** (quantifiers of the 3-valued significance logic).[3]

Formation Rules

1. If X' and Y' are variables then $X' \circ Y'$ and $X' \epsilon Y'$ are atomic wffs.

2. If B and C are wffs then $\sim B$, $(B \supset C)$, $T_n B$ are wffs.

3. If B is a wff and X' is a variable then $(\mathbf{A}X')B$ and $(\mathbf{S}X')B$ are wffs.

Definitions

$Cl(X') =_{df} (\mathbf{S}Y')S(Y' \epsilon X')$. ($X'$ is a class)
$I(X') =_{df} \sim Cl(X')$. (X' is an individual)

Let us assume for the moment that there is at least one set (and hence at least one class) and at least one individual.

$(\mathbf{A}k)\phi(k) =_{df} (\mathbf{A}X')(I(X') \supset \phi(X'))$.
$(\mathbf{S}k)\phi(k) =_{df} (\mathbf{S}X')(T_n I(X') \mathbin{\&} \phi(X'))$.

Let $k, l, m, n, \ldots$ be individual variables.

$(\mathbf{A}X)\phi(X) =_{df} (\mathbf{A}X')(Cl(X') \supset \phi(X'))$.
$(\mathbf{S}X)\phi(X) =_{df} (\mathbf{S}X')(T_n Cl(X') \mathbin{\&} \phi(X'))$.

Let $U, V, W, X, Y, Z, \ldots$ be class variables.

$M(X) =_{df} (\mathbf{S}Y)(X \epsilon Y)$. ($X$ is a set)
$(\mathbf{A}x)\phi(x) =_{df} (\mathbf{A}X)(M(X) \supset \phi(X))$.
$(\mathbf{S}x)\phi(x) =_{df} (\mathbf{S}X)(T_n(M(X) \vee I(X)) \mathbin{\&} \phi(X))$.

Let $u, v, w, x, v, z, \ldots$ be set variables.

$(\mathbf{A}x')\phi(x') =_{df} (\mathbf{A}X')(M(X') \vee I(X') \supset \phi(X'))$.
$(\mathbf{S}x')\phi(x') =_{df} (\mathbf{S}X')(T_n(M(X') \vee I(X')) \mathbin{\&} \phi(X'))$.

Let $u', v', w', x', y', z', \ldots$ be variables over sets and individuals.

$X' = Y' =_{df} (\mathbf{A}k)(k \circ X' \doteqdot k \circ Y') \vee (\mathbf{A}Z')(Z' \epsilon X' \doteqdot Z' \epsilon Y')$.
(X' is identical with Y')

Notice, in this definition, that if X' and Y' are classes then $X' = Y' \simeq (\mathbf{A}Z')(Z' \epsilon X' \equiv Z' \epsilon Y')$, if X' and Y' are individuals then $X' = Y' \simeq (\mathbf{A}k)(k \circ X' \equiv k \circ Y')$, and if X' is an individual and Y' is a class, or vice versa, then $X' = Y'$ is nonsignificant. Notice also how the disjunction $\vee$ is used to define identity over a range containing two different types of things. The definition could have been made without the use of $\vee$ by taking each case in turn but it seems easier and more natural to use $\vee$.

I now give definitions restricted to individuals. In using these definitions, one cannot substitute one side of the definition for the other unless the variables X', Y', etc., are restricted to individuals or sets (as the case may be).

$k \leqslant l =_{df} (\mathbf{A}m)(m \circ k \supset m \circ l)$. ($k$ is part of l)
$k = l =_{df} (\mathbf{A}m)(m \circ k \equiv m \circ l)$. ($k$ is identical with l)

[This is derived from the definition of $X' = Y'$.]

$k < l =_{df} k \leqslant l \mathbin{\&} \sim(k = l)$. ($k$ is a proper part of l)
$k \not\circ l =_{df} (k \circ l)$. ($k$ is discrete from l)
$kFux =_{df} (\mathbf{A}m)(m \not\circ k \equiv (\mathbf{A}l)(l \in x \supset m \not\circ l))$. ($k$ is the fusion of x)
$kNux =_{df} (\mathbf{A}m)(m \leqslant k \equiv (\mathbf{A}l)(l \in x \supset m \leqslant l))$. ($k$ is the nucleus of x)

I now give definitions restricted to classes. Again, the definitions cannot be used unless the appropriate variables restrictions are made.

$X \subseteq Y =_{df} (\mathbf{A}X')(X' \in X \supset X' \in Y)$. ($X$ is a subclass of Y)
$X = Y =_{df} (\mathbf{A}X')(X' \in X \equiv X' \in Y)$. ($X$ is identical with Y)

[This is derived from the definition of $X = Y$.]

$X \subset Y =_{df} X \subseteq Y \mathbin{\&} \sim(X = Y)$. ($X$ is a proper subclass of Y)

If the variable restrictions are violated a nonsignificant wff may result. For example, $k \subseteq l$, $k \subset g$, $X \leqslant Y$, and $X \leqslant k$ are all nonsignificant. Except in the case of $X' \circ Y'$ and $X' \in Y'$, these nonsignificant wffs are avoided because it is simpler to use restricted variables and also the only purpose they could serve would be to form significance ranges but they do not introduce any new significance ranges which are not already obtained from $X' \circ Y'$ and $X' \in Y'$.

General Axioms

1. $S(X' \in X)$
2. $S(k \circ l)$
3. $Cl(X') \vee Cl(Y') \supset \sim S(X' \circ Y')$.

Individual Axioms

1. $k \circ l \equiv (\mathbf{S}m)(\mathbf{A}n)(n \circ m \supset n \circ k \mathbin{\&} n \circ l)$
2. $(\mathbf{S}k)(k \in x) \supset (\mathbf{S}l)(lFux)$
3. $(\mathbf{S}x)((\mathbf{A}k)(k \in x \equiv \phi(k, l_1, \ldots, l_m)) \mathbin{\&} (\mathbf{A}y) \sim (y \in x))$, *where* ϕ *is constructed using only* $\circ$, $\sim$, &, $\mathbf{A}$, *and variables quantified over individuals*
4. $k = l \supset . k \in X \equiv l \in X$
5. $(\mathbf{S}X')I(X')$.

Class Axioms

T. $X = Y \supset (\mathbf{A}Z)(X \in Z \equiv Y \in Z)$
P. $(\mathbf{A}x')(\mathbf{A}y')(\mathbf{S}x)(\mathbf{A}u')(u' \in x \equiv T(u' = x' \vee u' = y'))$
N. $(\mathbf{S}x)(\mathbf{A}x')(\sim x' \in x)$
U. $(\mathbf{A}x)(\mathbf{S}y)(\mathbf{A}x')(x' \in y \equiv (\mathbf{S}z)(x' \in z \mathbin{\&} z \in x))$
W. $(\mathbf{A}x)(\mathbf{S}y)(\mathbf{A}x')(x' \in y \equiv T(\mathbf{A}y')(y' \in x' \supset y' \in x))$
S. $(\mathbf{A}x)(\mathbf{A}X)(\mathbf{S}y)(\mathbf{A}x')(x' \in y \equiv x' \in x \mathbin{\&} x' \in X)$.

Before introducing more axioms, I need to prove some theorems and introduce some definitions. To shorten the paper, the theorems are given without proof.

T1 $\sim S(X' \epsilon\, k)$.
T2 $k = l \supset.\ \phi(k) \simeq \phi(1)$, *for any wff* ϕ.
T3 $X = Y \equiv.\ X \subseteq Y \ \&\ Y \subseteq X$.
T4 $X = X$.
T5 $X = Y \supset Y = X$.
T6 $X = Y \supset.\ Y = Z \supset X = Z$.
T7 $X = Y \supset.\ \phi(X) \simeq \phi(Y)$, *for any wff* ϕ.

Define a proper class as a class which is not a set.

$$Pr(X) =_{df} \sim M(X).$$

T8 $Pr(X) \supset F(X \epsilon\, Y)$.
T9 $(\mathbf{A}x')(\mathbf{A}y')(\mathbf{S}!x)(\mathbf{A}u')(u' \epsilon\, x \equiv T(u' = x' \vee u' = y'))$.

We now introduce the definition, $\{x', y'\}$, (the unordered pair of x' and y') for the unique set x such that $(\mathbf{A}u')(u' \epsilon\, x \equiv T(u' = x' \vee u' = y'))$. Also define $\{x'\}$ as $\{x', x'\}$.

T10 $(\mathbf{A}u')(u' \epsilon\, \{x'\} \equiv T(u' = x))$.
T11 $\{x', y'\} = \{y', x'\}$.
T12 $\{x'\} = \{y'\} \supset.\ x' = y'$.
T13 $(\mathbf{S}!x)(\mathbf{A}x')(\sim x' \epsilon\, x)$.

Introduce the definition 0 (the null set) for the unique set x such that $(\mathbf{A}x')(\sim x' \epsilon\, x)$. We now have at least one set as required for the definition of set and class variables. Individual Axiom 5 ensures the existence of at least one individual for the definition of individual variables.

Define an ordered pair, $\langle x', y' \rangle$, of x' and y' as $\{\{x'\}, \{x', y'\}\}$.

T14 $\langle x', y' \rangle = \langle u', v' \rangle \supset x' = u' \ \&\ y' = v'$.

The definition of ordered pairs can be extended as follows:

$$\langle x' \rangle =_{df} x,\ \langle x_1', \ldots, x_{n+1}' \rangle =_{df} \langle \langle x_1', \ldots, x_n' \rangle, x_{n+1}' \rangle.$$

T15 $\langle x_1', \ldots, x_n' \rangle = \langle y_1', \ldots, y_n' \rangle \supset.\ x_1' = y_1' \ \&\ \ldots \ \&\ x_n' = y_n'$.
T16 $(\mathbf{A}x)(\mathbf{S}!y)(\mathbf{A}x')(x' \epsilon\, y \equiv (\mathbf{S}z)(x' \epsilon\, z \ \&\ z \epsilon\, x))$.

Introduce the definition $U(x)$ (the sum set of x) for the unique y such that $(\mathbf{A}x')(x' \epsilon\, y \equiv (\mathbf{S}z)(x' \epsilon\, z \ \&\ z \epsilon\, x))$. Also define $x \cup y$ as $U(\{x, y\})$.

T17 $(\mathbf{A}x')(x' \epsilon\, x \cup y \equiv x' \epsilon\, x \vee x' \epsilon\, y)$.

Define $\{x_1', \ldots, x_n'\}$ inductively as $\{x_1', \ldots, x_{n-1}'\} \cup \{x_n'\}$.

T18 $(\mathbf{A}u')(u' \epsilon\, \{x_1', \ldots, x_n'\} \equiv T(u' = x_1') \vee \ldots \vee T(u' = x_n'))$.
T19 $(\mathbf{A}x)(\mathbf{S}!y)(\mathbf{A}x')(x' \epsilon\, y \equiv T(\mathbf{A}y')(y' \epsilon\, x' \supset y' \epsilon\, x))$.

Define the power set of the set x, $P(x)$, as the unique y such that $(\mathbf{A}x')(x' \epsilon\, y \equiv T(\mathbf{A}y')(y' \epsilon\, x' \supset y' \epsilon\, x))$.

T20 $(\mathbf{A}x)(\mathbf{A}Y)(\mathbf{S}!y)(\mathbf{A}x')(x' \epsilon\, y \equiv x' \epsilon\, x \ \&\ x' \epsilon\, Y)$.

Define the intersection set of the set x and the class Y, $x \cap Y$, as the unique y such that $(\mathbf{A}x')(x' \epsilon y \equiv x' \epsilon x \mathbin{\&} x' \epsilon Y)$.

T21 $\quad X \subseteq x \supset M(X)$.

Define a univocal class (relation) as follows:

$$Un(X) =_{df} (\mathbf{A}x')(\mathbf{A}y')(\mathbf{A}z')(\langle x', y'\rangle \epsilon X \mathbin{\&} \langle x', z'\rangle \epsilon X \supset T(y' = z')).$$

Further Class Axioms

B. $(\mathbf{A}x_1', \ldots, x_l', Y_1', \ldots, Y_m')S\phi(x_1', \ldots, x_l', Y_1', \ldots, Y_m') \supset (\mathbf{S}X)(\mathbf{A}x_1', \ldots, x_l')(\langle x_1', \ldots, x_l'\rangle \epsilon X \equiv \phi(x_1', \ldots, x_l', Y_1', \ldots, Y_m'))$, *where* ϕ *is constructed using* $\circ$, ϵ, $\sim$, $\supset$, T_n, $\mathbf{A}$, $\mathbf{S}$ *such that only variables over sets and individuals are quantified, and* $x_1', \ldots, x_l', Y_1', \ldots, Y_m'$ *are all the free variables of* ϕ *and* X *is not among them.*

R. $(\mathbf{A}x)(Un(X) \supset (\mathbf{S}y')(\mathbf{A}x')(x' \epsilon y \equiv (\mathbf{S}y'(\langle y', x'\rangle \epsilon X \mathbin{\&} y' \epsilon x))))$.

I. $(\mathbf{S}x)(0 \epsilon x \mathbin{\&} (\mathbf{A}y)(y \epsilon x \supset y \cup \{y\} \epsilon x))$.

T22 $\quad (\mathbf{S}!Z)(\mathbf{A}u')(u' \epsilon Z \equiv (\mathbf{S}v')(\mathbf{S}w')(T(u' = \langle v', w'\rangle) \mathbin{\&} v' \epsilon X \mathbin{\&} w' \epsilon Y))$.

Introduce the definition $X \times Y$ (the Cartesian product of classes X and Y), as the unique Z such that $(\mathbf{A}u')(u' \epsilon Z \equiv (\mathbf{S}v')(\mathbf{S}w')(T(u' = \langle v', w'\rangle) \mathbin{\&} v' \epsilon X \mathbin{\&} w' \epsilon Y))$. Let X^2 be defined as $X \times X$ and X^n be defined as $X^{n-1} \times X$.

T23 $\quad (\mathbf{S}!Z)(\mathbf{A}u')(u' \epsilon Z \equiv u' \epsilon X \mathbin{\&} u' \epsilon Y)$.

Define $X \cap Y$ (the intersection of classes X and Y) as the unique Z such that T23 holds.

T24 $\quad (\mathbf{S}!Z)(\mathbf{A}u')(u' \epsilon Z \equiv u' \epsilon X \vee u' \epsilon Y)$.

Define $X \cup Y$ (the union of classes X and Y) as the unique Z such that T24 holds.

T25 $\quad (\mathbf{S}!Y)(\mathbf{A}u')(u' \epsilon Y \equiv \sim u' \epsilon X)$.

Define $\overline{X}$ (the complement of the class X) as the unique Y such that T25 holds. Also define $X - Y$ as $X \cap \overline{Y}$.

T26 $\quad (\mathbf{S}!X)(\mathbf{A}u')(u' \epsilon X \equiv u' = u')$.

Define V (the universal class) as the unique X such that T26 holds.

T27 $\quad (\mathbf{S}!Y)(\mathbf{A}u')(u' \epsilon Y \equiv (\mathbf{S}v')(\langle u', v'\rangle \epsilon X))$.

Define $D(X)$ (the domain of X) as the unique Y such that T27 holds.

T28 $\quad (\mathbf{S}!Y)(\mathbf{A}u')(u' \epsilon Y \equiv (\mathbf{S}v')(\langle v', u'\rangle \epsilon X))$.

Define $R(X)$ (the range of X) as the unique Y such that T28 holds.

T29 $\quad \overline{X \cup Y} = \overline{X} \cap \overline{Y}$.
T30 $\quad \overline{\overline{X}} = X$.
T31 $\quad V = \overline{0}$.
T32 $\quad (\mathbf{A}u')(u' \epsilon V)$.

T33 $(\mathbf{A}u')(u' \epsilon X - Y \equiv u' \epsilon X \,\&\, \sim u' \epsilon Y)$.
T34 $X \cap Y = Y \cap X$.
T35 $X \cap (Y \cap Z) = (X \cap Y) \cap Z$.
T36 $X \cap X = X$.
T37 $X \cap 0 = 0$.
T38 $X \cap V = X$.
T39 $X \cup Y = Y \cup X$.
T40 $X \cup (Y \cup Z) = (X \cup Y) \cup Z$.
T41 $X \cup X = X$.
T42 $X \cup 0 = X$.
T43 $X \cup V = V$.
T44 $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.
T45 $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.
T46 $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$.
T47 $X - X = 0$.
T48 $V - X = \overline{X}$.
T49 $\overline{V} = 0$.

Define a relation as follows: $\mathrm{Rel}(X) =_{df} X \subseteq V^2$.

T50 $(\mathbf{S}!Y)(\mathbf{A}u')(u' \epsilon Y \equiv T(\mathbf{A}v')(v' \epsilon u' \supset v' \epsilon X))$.

Define $P(X)$ (the power class of X) as the unique Y such that T50 holds.

T51 $(\mathbf{S}!Y)(\mathbf{A}u')(u' \epsilon Y \equiv (\mathbf{S}x)(u' \epsilon x \,\&\, x \epsilon X))$.

Define $U(X)$ (the sum class of X) as the unique Y such that T51 holds.

T52 $(\mathbf{S}!X)(\mathbf{A}u')(u' \epsilon X \equiv T(\mathbf{S}v')(u' = \langle v', v'\rangle))$.

Define I_R (the identity relation) as the unique X such that T52 holds.

T53 $(\mathbf{A}x_1', \ldots, x_l', Y_1', \ldots, Y_m')S\phi(x_1', \ldots, x_l', Y_1', \ldots, Y_m') \supset (\mathbf{S}!X)(X \subseteq V^l \,\&\, (\mathrm{A}x_1', \ldots, x_l')(\langle x_1', \ldots, x_l'\rangle \epsilon X \equiv \phi(x_1', \ldots, x_l', Y_1', \ldots, Y_m')))$, *where* ϕ *is constructed as in Axiom B.*

Define $\{\langle x_1', \ldots, x_l'\rangle \mid \phi(x_1', \ldots, x_l', Y_1', \ldots, Y_m')\}$ (the class of ordered l-tuples such that ϕ holds) as the unique X such that T53 holds.
Define the inverse relation of X, $\breve{X}$ as $\{\langle x_1', x_2'\rangle \mid \langle x_2', x_1'\rangle \epsilon X\}$.

T54 $R(X) = D(\breve{X})$.

Define the following:

$Fnc(X) =_{df} \mathrm{Rel}(X) \,\&\, Un(X)$. ($X$ is a function)
$Y 1 X =_{df} X \cap (Y \times V)$. (Restriction of X to the domain Y)
$Un_1(X) =_{df} Un(X) \,\&\, Un(\breve{X})$. ($X$ is one-to-one)

If there is a unique z' such that $\langle y', z'\rangle \epsilon X$ then $X`y' =_{df} z'$. $X``Y =_{df} R(Y 1 X)$.

T55 $(\mathbf{A}x)(Un(X) \supset (\mathbf{S}!y)(\mathbf{A}x')(x' \epsilon y \equiv (\mathbf{S}y')(\langle y', x'\rangle \epsilon X \,\&\, y' \epsilon x)))$.

Define the set, $R(x 1 X)$, as the unique y such that T55 holds.

T56 $\vdash$Axiom R $\Rightarrow$ $\vdash$Axiom S.
T57 $M(D(x))$.
T58 $M(R(x))$.
T59 $M(x \times y)$.
T60 $M(D(X))$ & $M(R(X))$ & $Rel(X) \supset M(X)$.
T61 $Fnc(X) \supset M(x\ 1\ X)$.
T62 $Pr(V)$.
T63 $(\mathbf{S}!x)((\mathbf{A}k)(k \in x \equiv k \circ k)$ & $(\mathbf{A}y)(\sim y \in x))$.

Define I (the set of all individuals) as the unique x such that T63 holds.

T64 $(\mathbf{A}u', Y_1', \ldots, Y_m')S\phi(u', Y_1', \ldots, Y_m') \supset (\mathbf{S}!x)((\mathbf{A}k)(k \in x \equiv \phi(k, Y_1', \ldots, Y_m'))$ & $(\mathbf{A}y)(\sim y \in x))$, *where ϕ is any wff, not containing x free and containing quantification over set and individual variables only, and where $k, Y_1', \ldots, Y_m'$ are all the free variables in ϕ.*

Define $\{k \mid \phi(k, Y_1', \ldots, Y_m')\}$ (the set of all individuals k such that ϕ holds) as the unique x such that T64 holds. This is a more general form than can be derived from Individual Axiom 3 alone.

Individual Axioms 1 and 2 yield the Goodman theory of individuals as in [6]. Leonard and Goodman [7] develop a theory of individuals using sets of individuals as well, but the set theory is taken from *Principia Mathematica* and is not independently axiomatised. However, using Individual Axioms 1, 2, 3, and 4, one can develop a theory of individuals and sets of individuals,[4] which is stronger than Leonard and Goodman's theory. The set variables would range over sets with individual members only and Individual Axiom 4 would have to be changed to:

4′. $k = l \supset . k \in x \equiv l \in x$.

The theory would develop the relationships between $\circ$, $\leqslant$, $<$, $=$, and $\wr$ such as:

$k \circ l \equiv (\mathbf{S}m)(m \leqslant k \ \&\ m \leqslant l)$.
$k \leqslant l \equiv . k < l \vee k = l$.
$k \leqslant l \equiv . (\mathbf{A}m)(m \wr l \supset m \wr k)$.

The theory would define the fusion of the set x, $Fu`x$, as the unique $l$ such that $l\ Fu`x$, on the condition that x is nonempty. It would establish results about fusions such as:

$(\mathbf{S}k)(k \in x) \supset (\mathbf{A}m)(m \wr Fu`x \equiv (\mathbf{A}l)(l \in x \supset m \wr l))$.
$(\mathbf{S}k)(k \in x) \supset . x \subseteq y \supset Fu`x \leqslant Fu`y$.

It would introduce the set of all individuals k such that ϕ, $\{k: \phi(k)\}$, with results like:

$$(\mathbf{S}k)(\phi(k) \ \&\ l \leqslant k) \supset l \leqslant Fu`\{k: \phi(k)\}.$$

It would define the sum of two individuals k and l, $k + l$, as the fusion of $\{m: m = k \vee m = l\}$, with results like:

$$m \wr (k + l) \equiv m \wr k \ \&\ m \wr l.$$
$$(k + l) \leqslant m \equiv k \leqslant m \ \&\ l \leqslant m.$$

It would introduce the negate of the individual k, $\overline{k}$, as $Fu`\{l: l \mathrel{Z} k\}$ on the condition that $(\mathbf{S}l)(l \mathrel{Z} k)$ holds, with results like:

$$(\mathbf{S}l)(l \mathrel{Z} k) \supset. l \mathrel{Z} \overline{k} \equiv l \leqslant k.$$
$$(\mathbf{S}l)(l \mathrel{Z} k) \supset. \overline{\overline{k}} = k.$$

It would introduce the universal individual, $\cup$, as $Fu`\{k: k = k\}$, with results like:

$k \circ \cup$
$(\mathbf{A}k)(k \leqslant l) \equiv l = \cup$
$(\mathbf{S}l)(l \mathrel{Z} k) \supset. k + \overline{k} = \cup.$
$(\mathbf{S}k)(\sim k = \cup \;\&\; \phi(k)) \equiv (\mathbf{S}k)(\sim k = \cup \;\&\; \phi(\overline{k})).$

The theory would define the nucleus of a set x, $Nu`x$ as $\overline{Fu}`\{k: (\mathbf{S}l)(l \in x \;\&\; \sim l = \cup \;\&\; k = \overline{l})\}$, provided that

(i). $(\mathbf{S}k)(\mathbf{S}l)(l \in x \;\&\; \sim l = \cup \;\&\; k = \overline{l})$
(ii). $\sim Fu\{k: (\mathbf{S}l)(l \in x \;\&\; \sim l = \cup \;\&\; k = \overline{l})\} = \cup$
(iii). $\sim(\mathbf{A}k)(k \in x \supset k = \cup)$

all hold. In the case of (iii) failing to hold, $Nu`x =_{df} \cup$, provided $(\mathbf{A}k)(k \in x \supset k = \cup)$ holds.

It would establish results about nuclei such as:

$(\mathbf{S}l)(\mathbf{A}k)(k \in x \supset l \leqslant k) \supset (\mathbf{A}m)(m \leqslant Nu`x \equiv (\mathbf{A}l)(l \in x \supset m \leqslant l)).$
$(\mathbf{S}l)(\mathbf{A}k)(k \in y \supset l \leqslant k) \supset. x \subseteq y \supset Nu`y \leqslant Nu`x.$

It would introduce the product of two individuals k and l, kl, as $Nu`\{m: m = k \vee m = l\}$, provided $k \circ l$ holds, with results like:

$k \circ l \supset. m \leqslant k \;\&\; m \leqslant l \equiv m \leqslant kl.$
$(\mathbf{S}n)(n \leqslant k \;\&\; n \leqslant l \;\&\; n \leqslant m) \supset. k(lm) = (kl)m.$
$k \circ l \;\&\; \sim k = \cup \;\&\; \sim l = \cup \supset. \overline{kl} = \overline{k} + \overline{l}.$
$k \circ l \;\&\; k \circ m \supset kl + km = k(l + m).$
$l \circ m \supset k + lm = (k + l)(k + m).$
$\sim k = \cup \;\&\; \sim l = \cup \;\&\; k \circ l \;\&\; \sim(k \leqslant l) \;\&\; \sim(l \leqslant k) \supset k + l = kl + \overline{k}l + k\overline{l}.$

It would introduce the union and intersection, $x \cup y$ and $x \cap y$, with results like:

$(\mathbf{S}k)(k \in x) \;\&\; (\mathbf{S}k)(k \in y) \supset. Fu`(x \cup y) = Fu`x + Fu`y.$
$(\mathbf{S}l)(\mathbf{A}k)(k \in x \vee k \in y \supset l \leqslant k) \supset Nu`(x \cup y) = (Nu`x)(Nu`y).$

It would introduce the complement, $\overline{x}$, and the universal set, V, with results like:

$Fu`\mathrm{V} = \cup$
$(\mathbf{S}k)(k \in x) \;\&\; \sim Fu`x = \cup \supset. \overline{Fu`x} \leqslant Fu`\overline{x}.$

It would introduce the definition of an atomic individual as follows:

$\mathbf{A}t(k) =_{df} \sim(\mathbf{S}l)(l < k).$

Then these can be proved:

$\mathbf{A}t(k) \supset. l \leqslant k \supset l = k.$
$\mathbf{A}t(k) \;\&\; \mathbf{A}t(l) \supset. k \not\geq l \vee k = l.$

The 3-valued class theory can be continued as in Mendelson [8], pp. 170-197. Some minor differences[5] need to be noted:

$X We Y =_{df} XIrrY \;\&\; (\mathbf{A}Z)(Z \subseteq Y \;\&\; Z \neq 0 \supset (\mathbf{S}y')(y' \in Z \;\&\; (\mathbf{A}v')$ $(v' \in Z \;\&\; \sim T(v' = y') \supset. \langle y', v'\rangle \in X \;\&\; \sim\langle v', y'\rangle \in X)))$. ($X$ well-orders Y)

$X Con Y =_{df} Rel(X) \;\&\; (\mathbf{A}u')(\mathbf{A}v')(u' \in Y \;\&\; v' \in Y \;\&\; \sim T(u = v') \supset$ $\langle u', v'\rangle \in X \vee \langle v', u'\rangle \in X)$. ($X$ is a connected relation on Y)

$E =_{df} \{\langle x', y'\rangle \mid T(x' \in y')$. (The membership relation)

$Trans_1(X) =_{df} (\mathbf{A}y)(y \in X \supset y \subseteq X)$. ($X$ is transitive over sets)

$Trans_2(X) =_{df} (\mathbf{A}k)(\mathbf{A}l)(k \leqslant l \;\&\; l \in X \supset k \in X)$. ($X$ is transitive over individuals)

$Ord(X) =_{df} E\ We\ X \;\&\; Trans_1(X) \;\&\; \sim(\mathbf{S}k)(k \in X)$. ($X$ is an ordinal)

Note the restriction here on the definition of an ordinal. This is necessary to prevent each individual from generating a sequence, $\{k\}$, $\{k, \{k\}\}$, $\{k, \{k\}, \{k, \{k\}\}\}$, etc., which would satisfy the definition of ordinals but would not satisfy the uniqueness requirement.

$Ord(k) =_{df} \sim(k \circ k).$
$Suc(k) =_{df} \sim(k \circ k).$
$\omega =_{df} \{x' \mid x' \in K_I \;\&\; T(\mathbf{A}y)(y \in x' \supset y \in K_I)\}.$
$X^Y =_{df} \{u' \mid (\mathbf{S}x)(T(x = u') \;\&\; Fnc(x) \;\&\; D(x) = Y \;\&\; R(x) \subseteq X)\}.$

Further Class Axioms

AC. $(\mathbf{A}u)(u \in x \supset u \neq 0 \;\&\; (\mathbf{A}v)(v \in x \;\&\; v \neq u \supset v \cap u = 0)) \supset (\mathbf{S}y)$ $(\mathbf{A}u)(u \in x \supset (\mathbf{S}!x')(x' \in u \cap y)).$

D. $(\mathbf{A}X)((\mathbf{S}x)(x \in X) \supset (\mathbf{S}x)(x \in X \;\&\; \sim(\mathbf{S}y)(y \in x \;\&\; y \in X))).$

GCH. $(\mathbf{A}x) \sim (\mathbf{S}y)(x \prec y \prec P(x)).$

C. $(\mathbf{A}x)$ (x is constructible). (to be defined).

As in Mendelson [8], pp. 198-199, AC is equivalent to the four other forms of the Axiom of Choice. Notice the difference between my Axiom D and Mendelson's Axiom of Restriction. Individuals may belong to the intersection of x and X.

Now I will define the notion of constructible set, which is similar to that on p. 87 of Cohen [2]. Define the set M_0 as follows: $u' \in M_0 \equiv (\mathbf{S}k)\, T(u' = \{k\})$. I is a set by Individual Axiom 3, and hence M_0 is a set, using the one-to-one correspondence between I and M_0 and using Axiom R. If α is a limit ordinal, then the set M_α is defined as the union of all the sets M_β, for $\beta <_0 \alpha$, i.e., $u' \in M_\alpha \equiv (\mathbf{S}\beta)(\beta <_0 \alpha \;\&\; u' \in M_\beta)$. The set $M_{\alpha+1}$ is defined as the union of the set M_α and the set of all sets x for which there is a formula

$A(z', w'_1, \ldots, w'_l)$, which is significant for all substitutions into its free variables, such that if $A_{M_\alpha \cup I}$ denotes A with all bound variables restricted to $M_\alpha \cup I$, where I is the set of all individuals, then for some (constant) $\overline{w}'_i$ in $M_\alpha \cup I$, for each i, $x = \{z' \epsilon M_\alpha \cup I / A_{M_\alpha \cup I}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$. $[\{z' \epsilon M_\alpha \cup I / A_{M_\alpha \cup I}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\} = \{z'/z' \epsilon M_\alpha \cup I \,\&\, A_{M_\alpha \cup I}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}]$.

Now we show that $M_{\alpha+1}$ can be defined in the formal system, given that M_α can be so defined. This proof follows that of Cohen [2], p. 92. For each $r \geqslant 0$ let x_r denote the set of all ordered triples $\langle z_1, z_2, z_3\rangle$ where z_1, z_2, and z_3 are sets of ordered n-tuples $\langle x'_1, \ldots, x'_n\rangle$ for which there is a formula $A(x'_1, \ldots, x'_n, t'_1, \ldots, t'_m)$, with exactly r quantifiers, but where A can be nonsignificant for some substitutions into its free variables, such that $z_1 = \{\langle x'_1, \ldots, x'_n\rangle \epsilon (M_\alpha \cup I)^n / TA_{M_\alpha \cup I}(x'_1, \ldots, x'_n, \overline{t}'_1, \ldots, \overline{t}'_m)\}$, $z_2 = \{\langle x'_1, \ldots, x'_n\rangle \epsilon (M_\alpha \cup I)^n / FA_{M_\alpha \cup I}(x'_1, \ldots, x'_n, \overline{t}'_1, \ldots, \overline{t}'_m)\}$, and $z_3 = \{\langle x'_1, \ldots, x'_n\rangle \epsilon (M_\alpha \cup I)^n / {\sim} SA_{M_\alpha \cup I}(x'_1, \ldots, x'_n, \overline{t}'_1, \ldots, \overline{t}'_m)$, where $\overline{t}'_i \epsilon M_\alpha \cup I$, for all i. We show that x_r is expressible in the formal system by an induction on r. Firstly, in order to define x_0, we define y_0 as follows: $u' \epsilon y_{0,n} \equiv (\mathbf{S}y')(y' \epsilon M_\alpha \cup I \,\&\, (\mathbf{S}z_1)(\mathbf{S}z_2)(\mathbf{S}z_3)(u' = \langle z_1, z_2, z_3\rangle \,\&\, (\mathbf{A}w')(w' \epsilon z_1 \equiv (\mathbf{S}x'_1)(\mathbf{S}x'_2)(T(w' = \langle x'_1, x'_2\rangle) \,\&\, x'_1 \epsilon M_\alpha \cup I \,\&\, x'_2 \epsilon M_\alpha \cup I \,\&\, T(x' \epsilon y'))) \,\&\, (\mathbf{A}w')(w' \epsilon z_2 \equiv (\mathbf{S}x'_1)(\mathbf{S}x'_2)(T(w' = \langle x'_1, x'_2\rangle) \,\&\, x'_1 \epsilon M_\alpha \cup I \,\&\, x'_2 \epsilon M_\alpha \cup I \,\&\, F(x'_1 \epsilon y'))) \,\&\, (\mathbf{A}w')(w' \epsilon z_3 \equiv (\mathbf{S}x'_1)(\mathbf{S}x'_2)(T(w' = \langle x'_1, x'_2\rangle) \,\&\, x'_1 \epsilon M_\alpha \cup I \,\&\, x'_2 \epsilon M_\alpha \cup I \,\&\, {\sim}S(x'_1 \epsilon y'))))) \vee \ldots$ (for other types of formulas using ordered tuples from 1 to n).

This example is for the formula $x'_1 \epsilon \overline{y}'$ with ordered pairs $\langle x'_1, x'_2\rangle$, this particular disjunct yielding a set because of the Axiom R and the assumption that M_α is a set. $y_{0,n}$ will be a set because it is a finite union of sets. Define y_0 as the union of all the $y_{0,n}$'s where $n \epsilon \omega - \{0\}$. So y_0 is the set of all ordered triples $\langle z_1, z_2, z_3\rangle$ where z_1, z_2, and z_3 are sets of ordered n-tuples $\langle x'_1, \ldots, x'_n\rangle$ for which there is a formula $A(x'_1, \ldots, x'_n, t'_1, \ldots, t'_m)$ with no connectives or quantifiers and such that z_1, z_2, and z_3 are defined as above.

Using an induction on the length of formulas without quantifiers assume, for all $k < 1$, the set y_k has been constructed to deal with all formulas without quantifiers and with k connectives. To construct y_l we need ordered triples corresponding to formulas with l connectives and obtained from previous formulas by the use of one of $\sim$, $\supset$, and T_n. $u' \epsilon y_l \equiv (\mathbf{S}z_1)(\mathbf{S}z_2)(\mathbf{S}z_3)(\langle z_1, z_2, z_3\rangle \epsilon y_{l-1} \,\&\, T(u' = \langle z_1, z_2, z_3\rangle)) \vee (\mathbf{S}k_1)(\mathbf{S}k_2)(k_1 + k_2 = l - 1 \,\&\, (\mathbf{S}z_1)(\mathbf{S}z_2)(\mathbf{S}z_3)(\mathbf{S}z_4)(\mathbf{S}z_5)(\mathbf{S}z_6)(\langle z_1, z_2, z_3\rangle \epsilon y_{k_1} \,\&\, \langle z_4, z_5, z_6\rangle \epsilon y_{k_2} \,\&\, T(u' = \langle (z_1 \cap z_4) \cup \overline{z}_1, z_1 \cap z_5, z_1 \cap z_6\rangle))) \vee (\mathbf{S}z_1)(\mathbf{S}z_2)(\mathbf{S}z_3)(\langle z_1, z_2, z_3\rangle \epsilon y_{l-1} \,\&\, T(u' = \langle z_1, 0, z_2 \cup z_3\rangle))$, where complements are taken with respect to $(M_\alpha \cup I)^n$ for n-tuples. y_l is a set because the y_k's for $k < l$ are sets and the z_i's are sets. Define x_0 as the union of all y_l's such that $l \epsilon \omega$. Now by induction on r we will define x_r. A set $\langle z_1, z_2, z_3\rangle$ will be a member of x_r *either* if there is a set $\langle z_4, z_5, z_6\rangle \epsilon x_{r-1}$ such that z_4, z_5, and z_6 are sets of $(n+1)$-tuples and such that $\langle x'_1, \ldots, x'_n\rangle \epsilon z_1 \equiv (\mathbf{S}x'_0)(x'_0 \epsilon M_\alpha \cup I \,\&\, \langle x'_0, x'_1, \ldots, x'_n\rangle \epsilon z_4)$, $\langle x'_1, \ldots, x'_n\rangle \epsilon z_2 \equiv (\mathbf{S}x'_0)(x'_0 \epsilon M_\alpha \cup I \,\&\, \langle x'_0, x'_1, \ldots, x'_n\rangle \epsilon z_5) \,\&\, {\sim}(\mathbf{S}x'_0)(x'_0 \epsilon M_\alpha \cup I \,\&\, \langle x'_0, x'_1, \ldots, x'_n\rangle \epsilon z_4)$ and $\langle x'_1, \ldots, x'_n\rangle \epsilon z_3 \equiv (\mathbf{A}x'_0)(x'_0 \epsilon M_\alpha \cup I \supset \langle x'_0, x'_1, \ldots, x'_n\rangle \epsilon z_6)$ *or* if there is a set $\langle z_4, z_5, z_6\rangle \epsilon x_{r-1}$ such that z_4, z_5, and z_6 are sets of $(n+1)$-tuples and such that $\langle x'_1, \ldots, x'_n\rangle \epsilon z_1 \equiv (\mathbf{A}x'_0)(x'_0 \epsilon M_\alpha \cup I \supset \langle x'_0, x'_1, \ldots, x'_n\rangle \epsilon z_4)$, $\langle x'_1, \ldots, x'_n\rangle \epsilon z_2 \equiv (\mathbf{S}x'_0)(x'_0 \epsilon M_\alpha \cup I \,\&$

$\langle x_0', \ldots, x_n'\rangle \epsilon z_5$ & $\sim(\mathbf{S}x_0')(x_0' \epsilon M_\alpha \cup I$ & $\langle x_0', \ldots, x_n'\rangle \epsilon z_6)$ and $\langle x_1', \ldots, x_n'\rangle \epsilon z_3 \equiv (\mathbf{S}x_0')(x_0' \epsilon M_\alpha \cup I$ & $\langle x_0', \ldots, x_n'\rangle \epsilon z_6)$. Then the set $M_{\alpha+1}$ is defined as the union of the set M_α with the set of all sets z_1, where, in the ordered triple $\langle z_1, z_2, z_3\rangle$ which belongs to some x_r, z_3 is the null set and z_1 and z_2 are sets of 1-tuples. Thus the Axiom of Constructibility (Axiom C), in the form $(\mathbf{A}x)(\mathbf{S}\alpha)(x \epsilon M_\alpha)$, can be formally defined in the system.

We now prove a theorem showing that only the connectives $\sim$, &, and T and quantifier $\mathbf{A}$ need be used in the predicate ϕ of the Axiom B to generate all the classes that Axiom B generates.

Theorem 1 *If ϕ is significant for all substitutions into its free variables, then there is a ϕ' such that $\phi \equiv \phi'$ and ϕ' contains only the connectives $\sim$, &, and T and the quantifier* $\mathbf{A}$.

Proof: This proof is similar to the proof that $M_{\alpha+1}$ can be defined formally, given M_α.

There are finitely many atomic formulas occurring in ϕ. Corresponding to each one there are three classes defined as follows: If the atomic formula is $x' \epsilon \bar{y}$, say, then $(\mathbf{A}x')(x' \epsilon Z_1 \equiv T(x' \epsilon \bar{y}))$, $(\mathbf{A}x')(x' \epsilon Z_2 \equiv F(x' \epsilon \bar{y}))$ and $(\mathbf{A}x')(x' \epsilon Z_3 \equiv \sim S(x' \epsilon \bar{y}))$ give definitions of the three classes, Z_1, Z_2, and Z_3. If the atomic formula is $x_1' \epsilon x_2'$, say, then $(\mathbf{A}x_1')(\mathbf{A}x_2')(\langle x_1', x_2'\rangle \epsilon Z_1 \equiv T(x_1' \epsilon x_2'))$, $(\mathbf{A}x_1')(\mathbf{A}x_2')(\langle x_1', x_2'\rangle \epsilon Z_2 \equiv F(x_1' \epsilon x_2'))$ and $(\mathbf{A}x_1')(\mathbf{A}x_2')(\langle x_1', x_2'\rangle \epsilon Z_3 \equiv \sim S(x_1' \epsilon x_2'))$ give definitions of Z_1, Z_2, and Z_3. And so on for any atomic formula appearing in ϕ. If the atomic formula has no or one free variable then the Z's have 1-tuples for members and if the atomic formula has two free variables then the Z's have 2-tuples for members.

We now assume that Z_1, Z_2, and Z_3 have been found for any predicate ϕ with fewer than n connectives and quantifiers and take the quantifiers and connectives in turn.

Let Z_1, Z_2, and Z_3 be the classes for ϕ and form $\sim\phi$. $(\mathbf{A}x_1', \ldots, x_m')(\langle x_1', \ldots, x_m'\rangle \epsilon Z_4 \equiv \langle x_1', \ldots, x_m'\rangle \epsilon Z_2)$, $(\mathbf{A}x_1', \ldots, x_m')(\langle x_1', \ldots, x_m'\rangle \epsilon Z_5 \equiv \langle x_1', \ldots, x_m'\rangle \epsilon Z_1)$ and $(\mathbf{A}x_1', \ldots, x_m')(\langle x_1', \ldots, x_m'\rangle \epsilon Z_6 \equiv \langle x_1', \ldots, x_m'\rangle \epsilon Z_3)$ define the classes Z_4, Z_5, and Z_6 for $\sim\phi$.

Let Z_1, Z_2, and Z_3 be the classes for ϕ_1 (where Z_1, Z_2, and Z_3 have members of the form $\langle x_{i_1}', \ldots, x_{i_k}'\rangle$) and let Z_4, Z_5, and Z_6 be the classes for ϕ_2 (where Z_4, Z_5, and Z_6 have members of the form $\langle x_{j_1}', \ldots, x_{j_l}'\rangle$). $(\mathbf{A}x_{i_1}', \ldots, x_{j_l}')(\langle x_{i_1}', \ldots, x_{j_l}'\rangle \epsilon Z_7 \equiv (\langle x_{i_1}', \ldots, x_{i_k}'\rangle \epsilon Z_1$ & $\langle x_{j_1}', \ldots, x_{j_l}'\rangle \epsilon Z_4) \vee \sim\langle x_{i_1}', \ldots, x_{i_k}'\rangle \epsilon Z_1)$, $(\mathbf{A}x_{i_1}', \ldots, x_{j_l}')(\langle x_{i_1}', \ldots, x_{j_l}'\rangle \epsilon Z_8 \equiv \langle x_{i_1}', \ldots, x_{i_k}'\rangle \epsilon Z_1$ & $\langle x_{j_1}', \ldots, x_{j_l}'\rangle \epsilon Z_5)$ and $(\mathbf{A}x_{i_1}', \ldots, x_{j_l}')(\langle x_{i_1}', \ldots, x_{j_l}'\rangle \epsilon Z_9 \equiv \langle x_{i_1}', \ldots, x_{i_k}'\rangle \epsilon Z_1$ & $\langle x_{j_1}', \ldots, x_{j_l}'\rangle \epsilon Z_6)$ define the classes Z_7, Z_8, and Z_9 for $\phi_1 \supset \phi_2$, where $x_{i_1}', \ldots, x_{j_l}'$ contains no repetition of variables.

Let Z_1, Z_2, and Z_3 be the classes for ϕ and form $T_n\phi$. $(\mathbf{A}x_1', \ldots, x_p')(\langle x_1', \ldots, x_p'\rangle \epsilon Z_4 \equiv \langle x_1', \ldots, x_p'\rangle \epsilon Z_1)$, $(\mathbf{A}x_1', \ldots, x_p')(\langle x_1', \ldots, x_p'\rangle \epsilon Z_5 \equiv 0 \epsilon 0)$ and $(\mathbf{A}x_1', \ldots, x_p')(\langle x_1', \ldots, x_p'\rangle \epsilon Z_6 \equiv \sim\langle x_1', \ldots, x_p'\rangle \epsilon Z_1)$ define the classes Z_4, Z_5, and Z_6 for $T_n\phi$. Let Z_1, Z_2, and Z_3 be the classes for $\phi(x')$ and form $(\mathbf{A}x')\phi(x')$. $(\mathbf{A}x_1', \ldots, x_k')(\langle x_1', \ldots, x_k'\rangle \epsilon Z_4 \equiv (\mathbf{A}x')(\langle x_1', \ldots, x', \ldots, x_k'\rangle \epsilon Z_1))$, $(\mathbf{A}x_1', \ldots, x_k')(\langle x_1', \ldots, x_k'\rangle \epsilon Z_5 \equiv (\mathbf{S}x')(\langle x_1', \ldots, x', \ldots, x_k'\rangle \epsilon Z_2)$ & $\sim(\mathbf{S}x')(\langle x_1', \ldots, x', \ldots, x_k'\rangle \epsilon Z_3))$ and $(\mathbf{A}x_1', \ldots, x_k')(\langle x_1', \ldots, x_k'\rangle \epsilon Z_6 \equiv (\mathbf{S}x')(\langle x_1',$

$\ldots, x', \ldots, x_k'\rangle \in Z_3))$ define the classes Z_4, Z_5, and Z_6 for $(\mathbf{A}x')\phi(x')$. $(\mathbf{A}x_1', \ldots, x_k')(\langle x_1', \ldots, x_k'\rangle \in Z_4 \equiv (\mathbf{S}x')(\langle x_1', \ldots, x', \ldots, x_k\rangle \in Z_1))$, $(\mathbf{A}x_1', \ldots, x_k')(\langle x_1', \ldots, x_k'\rangle \in Z_5 \equiv (\mathbf{S}x')(\langle x_1', \ldots, x', \ldots, x_k'\rangle \equiv Z_2) \,\&\, \sim(\mathbf{S}x')(\langle x_1', \ldots, x', \ldots, x_k'\rangle \in Z_1))$ and $(\mathbf{A}x_1', \ldots, x_k')(\langle x_1', \ldots, x_k'\rangle \in Z_6 \equiv (\mathbf{A}x')(\langle x_1', \ldots, x', \ldots, x_k'\rangle \in Z_3))$ define the classes Z_4, Z_5, and Z_6 for $(\mathbf{S}x')\phi(x')$.

Hence, for any formula ϕ there are corresponding classes Z_1, Z_2, and Z_3 such that $\langle x_1', \ldots, x_n'\rangle \in Z_1 \equiv T\phi$, $\langle x_1', \ldots, x_n'\rangle \in Z_2 \equiv F\phi$ and $\langle x_1', \ldots, x_n'\rangle \in Z_3 \equiv \sim S\phi$, because of the method of constructing the Z's for the formula ϕ. Since ϕ is significant for all substitutions into its free variables, Z_3 is the null class. Hence $\langle x_1', \ldots, x_n'\rangle \in Z_1 \equiv \phi$, where Z_1 was constructed using only $\sim$, &, T, and $\mathbf{A}$, the uses of the quantifier $\mathbf{S}$ being replacable by $\sim\mathbf{A}\sim$ because $\mathbf{S}$ only quantifies two-valued formulas. Hence the ϕ' required can be taken as $\langle x_1', \ldots, x_n'\rangle \in Z_1$.

3 *The meta-theory* The next task is to prove that the formal theory is relatively consistent to an applied **NBG.** This is more difficult than would first appear since, in usual set or class theories containing individuals, there is no axiom asserting the existence of at least one individual and hence one can ignore individuals when constructing a model or showing consistency in any way. But in this theory containing Individual Axiom 5 (necessary of course for the development of a theory of individuals) we cannot ignore individuals when constructing a model for the theory. Since the theory of individuals can be shown to be consistent using a model consisting of only one individual, we will construct a model for the class theory also containing only one individual. The model cannot be an inner model of any standard class theory because there is no such class theory explicitly containing individuals.

We first construct a model N for the individuals and sets of the theory and then extend it to a model N' for the individuals and classes of the theory. The domain of N and the valuations of the membership statements are constructed by a transfinite induction on the ordinals. This is similar to the construction of the constructible sets of the inner model of **ZF**, that appears in [2], p. 87. The final aim is to establish a domain with the following members: k, $\{k\}$, M_β, for all ordinals β, and all expressions of the form: $\{z' \in M_\alpha \cup \{k\}/A_{M_\alpha \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\}$, where $\overline{w}_i'$ is k or $\overline{w}_i' \in M_\alpha$ has the value 1 in the value assignment to follow, where all the bound variables in $A_{M_\alpha \cup \{k\}}$ are restricted to $M_\alpha \cup \{k\}$, and $A_{M_\alpha \cup \{k\}}$ has the value 1 or 0 for all substitutions into its free variable z'.

The restrictions of variable to $M_\alpha \cup \{k\}$ are done as follows:

$(\mathbf{A}x')(x' \in M_\alpha \supset f(x')) \,\&\, f(k)$.
$(\mathbf{S}x')(T_n(x' \in M_\alpha) \,\&\, f(x')) \vee f(k)$.

Assume that these restrictions to $M_\alpha \cup \{k\}$ apply for the whole construction of the domain N.

The transfinite induction is as follows: We shall use the notation, v (expression) = 1, 0, or n. We will construct a transfinite sequence of domains, $D_0 \subseteq D_1 \subseteq D_2 \subseteq \ldots \subseteq D_\alpha \subseteq D_{\alpha+1} \subseteq \ldots \subseteq D^U \subseteq D^0 \subseteq D^1 \subseteq \ldots \subseteq$

$D^n \subseteq \ldots \subseteq D^S$, where D^U will be the domain for sets and individual and D^S will be the domain for classes and individual. The valuations made for each domain will hold good for all domains containing it.

Let the domain D_0 consist of k (the individual) and $\{k\}$. Then $v(k \circ k) = 1$, $v(\{k\} \circ k) = v(k \circ \{k\}) = v(\{k\} \circ \{k\}) = n$, $v(k \in k) = v(\{k\} \in k) = n$, $v(k \in \{k\}) = 1$, $v(\{k\} \in \{k\}) = 0$. Let the domain D_1 consist of k, $\{k\}$, M_0, and all expressions of the form: $\{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$, where $\overline{w}'_i \in D_0$, for all i, and, for all $z' \in D_0$, $A_{M_0 \cup \{k\}}$ has the value 1 or 0. If y and $z \in D_1 - D_0$, then $v(k \circ y) = v(y \circ k) = v(y \circ z) = v(\{k\} \circ y) = v(y \circ \{k\}) = n$. Also $v(k \in M_0) = 0$, $v(\{k\} \in M_0) = 1$, $v(M_0 \in k) = n$, $v(M_0 \in \{k\}) = 0$, $v(M_0 \in M_0) = 0$. If $y \in D_1 - (D_0 \cup \{M_0\})$, then $v(M_0 \in y) = 0$, $v(y \in k) = n$, $v(y \in \{k\}) = 0$. $v(x' \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}) = v(A_{M_0 \cup \{k\}}(x', \overline{w}'_1, \ldots, \overline{w}'_l))$, for all $x' \in D_0$, where the range of bound variables is taken as D_0 for the valuation. If $y \in D_1 - (D_0 \cup \{M_0\})$, then if $v(x' \in y) = v(x' \in \{k\})$ for all $x' \in D_0$, then $v(y \in M_0) = 1$. Call $\{k\}$ *the corresponding member of* D_0 *for* y. If it is not the case that $v(x' \in y) = v(x' \in \{k\})$ for all $x' \in D_0$, then $v(y \in M_0) = 0$.

Let $x \in D_1 - (D_0 \cup \{M_0\})$. Let $v(x \in M_0) = 1$. Then $v(x \in \{z' \in M_0 \cup \{k\}/B_{M_0 \cup \{k\}}(z', \overline{x}'_1, \ldots, \overline{x}'_m)\}) = v(B_{M_0 \cup \{k\}}(\{k\}, \overline{x}'_1, \ldots, \overline{x}'_m))$, where $\overline{x}'_i \in D_0$ for all i, and the range of bound variables in $B_{M_0 \cup \{k\}}$ is taken as D_0. Now let $v(x \in M_0) = 0$. Then $v(x \in \{z' \in M_0 \cup \{k\}/B_{M_0 \cup \{k\}}(z', \overline{x}'_1, \ldots, \overline{x}'_m)\}) = 0$. This completes the valuation for D_1.

We shall now show that the Axiom of Extensionality holds in D_1. Let $v(x' \in x) = v(x' \in y)$ for all $x' \in D_1$, where $x, y \in D_1 - \{k\}$.

$v(x \in \{k\}) = 0 = v(y \in \{k\})$.

(i) Let $v(x \in M_0) = 1$ and $x \in D_0$. Then x is $\{k\}$.

a. Let $y \in D_0$, then y is $\{k\}$, which is x. Hence $v(y \in M_0) = 1$ and $v(x \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}) = v(y \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\})$.

b. Let $y \in D_1 - (D_0 \cup \{M_0\})$. Then x is a corresponding member of D_0 for y, $v(y \in M_0) = 1$ and $v(y \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}) = v(A_{M_0 \cup \{k\}}(x, \overline{w}'_1, \ldots, \overline{w}'_l)) = v(x \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\})$.

c. Let y be M_0. This cannot be the case because $v(k \in x) = 1$ and $v(k \in M_0) = 0$.

(ii) Let $v(x \in M_0) = 1$ and $x \in D_1 - D_0$. x cannot be M_0 because $v(M_0 \in M_0) = 0$.

a. Let $y \in D_0$, then this case has already been treated in (i).

b. Let $y \in D_1 - (D_0 \cup \{M_0\})$. x has a corresponding member, $\{k\}$, of D_0. Hence $v(x' \in x) = v(x' \in \{k\})$ for all $x' \in D_0$, and $v(x \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}) = v(A_{M_0 \cup \{k\}}(\{k\}, \overline{w}'_1, \ldots, \overline{w}'_l))$. $v(x' \in y) = v(x' \in \{k\})$ for all $x' \in D_0$ and $v(y \in M_0) = 1$. $v(y \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}) = v(A_{M_0 \cup \{k\}}(\{k\}, \overline{w}'_1, \ldots, w'_l)) = v(x \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\})$.

c. y cannot be M_0 because $v(k \in M_0) = 0$ and $v(k \in \{k\}) = 1$.

(iii) Let $v(x \in M_0) = 0$. Then it is not the case that $v(x' \in x) = v(x' \in \{k\})$ for all $x' \in D_0$.

a. Let $y \in D_1 - (D_0 \cup \{M_0\})$. The above holds for y and hence $v(y \in M_0) = 0$. Hence $v(x \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}) = 0 = v(y \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\})$.

b. Let y be M_0. Then $v(y \in M_0) = 0$ and $v(y \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}) = 0 = v(x \in \{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\})$.

c. Let $y \in D_0$. Hence y is $\{k\}$ and $v(x' \in y) = v(x' \in \{k\})$ for all $x' \in D_0$, which yields a contradiction.

This completes the proof.

Let $x \in D_1 - (D_0 \cup \{M_0\})$ and let $v(x' \in x) = v(x' \in \{k\})$ for all $x' \in D_0$. So $v(x \in M_0) = 1$. Let $y \in D_1 - D_0$.

(I) Let $v(y \in M_0) = 1$. Then $v(x' \in y) = v(x' \in \{k\})$, for all $x' \in D_0$. Let the x above be $\{z' \in M_0 \cup \{k\}/A_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$. Then $v(y \in x) = v(A_{M_0 \cup \{k\}}(\{k\}, \overline{w}'_1, \ldots, \overline{w}'_l)) = v(\{k\} \in x) = v(\{k\} \in \{k\}) = 0$. $v(y \in \{k\}) = 0 = v(y \in x)$.

(II) Let $v(y \in M_0) = 0$. Then $v(y \in x) = 0 = v(y \in \{k\})$. Hence, by the Axiom of Extensionality, in all contexts, x can be replaced by $\{k\}$, its corresponding member of D_0. Hence $v(B_{M_0 \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l))(z', \overline{w}'_i$ all $\in D_0)$ is the same whether the range of the bound variables in $B_{M_0 \cup \{k\}}$ is taken as D_0 or D_1.

This completes the initial stage of the transfinite induction. The next step is to assume for some ordinal α that domains D_β, for all $\beta \leqslant \alpha$, have been constructed and that all valuations of the expressions constructed from the members of these domains have been obtained in a way similar to the valuation of expressions from D_1. D_β consists of k, $\{k\}$, M_γ, for all γ such that $0 \leqslant \gamma < \beta$, and all expressions of the form: $\{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$, where $\overline{w}'_i \in D_{\beta-1}$ (if β is a successor ordinal) or $\overline{w}'_i \in D_\beta$ (if β is a limit ordinal) and $v(\overline{w}'_i \in M_\gamma) = 1$ or $\overline{w}'_i$ is k, for all i, for γ such that $0 \leqslant \gamma < \beta$, and where $A_{M_\gamma \cup \{k\}}$ has the value 1 or 0 for all $z' \in D_{\beta-1}$ (if β is a successor ordinal) or $z' \in D_\beta$ (if β is a limit ordinal). The Axiom of Extensionality holds in D_β and if $x \in D_\beta - D_\gamma$ and $v(x \in M_\gamma) = 1$ then x can be replaced by any of its corresponding members, in all contexts with the domain D_β. Also $v(B_{M_\gamma \cup \{k\}}(z', \overline{x}'_1, \ldots, \overline{x}'_m))$ is the same whether the range of the bound variables in $B_{M_\gamma \cup \{k\}}$ is taken as D_γ or D_β.

We now define $D_{\alpha+1}$ as all members of the D_β's, for all $\beta \leqslant \alpha$, M_α, and all expressions of the form: $\{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$, where $\overline{w}'_i \in D_\alpha$ such that $v(\overline{w}'_i \in M_\gamma) = 1$ or $\overline{w}'_i$ is k, for all i, where $0 \leqslant \gamma \leqslant \alpha$, and where $A_{M_\gamma \cup \{k\}}$ has the value 1 or 0 for all $z' \in D_\alpha$. [If α is a limit ordinal, then $\gamma = \alpha$ is the only case we need to consider.]

If y and $z \in D_{\alpha+1} - D_\alpha$, then $v(k \circ y) = v(y \circ k) = v(y \circ z) = n$. If $x \in D_\alpha - \{k\}$, then $v(y \circ x) = v(x \circ y) = n$. Also $v(k \in M_\alpha) = 0$, $v(M_\alpha \in k) = n$ and $v(M_\alpha \in M_\alpha) = 0$. If $x \in D_\alpha - \{k\}$, then $v(x \in M_\alpha) = 1$ and $v(M_\alpha \in x) = 0$. If $y \in D_{\alpha+1} - (D_\alpha \cup \{M_\alpha\})$ then $v(M_\alpha \in y) = 0$, $v(y \in k) = n$ and $v(y \in \{k\}) = 0$.

$v(x' \in \{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\}) = v(A_{M_\gamma \cup \{k\}}(x', \overline{w}_1', \ldots, \overline{w}_l'))$, for all $x' \in D_\gamma$, where $\overline{w}_i' \in D_\alpha$ such that $v(\overline{w}_i' \in M_\gamma) = 1$ or $\overline{w}_i'$ is k, for all i, and where the range of the bound variables in $A_{M_\gamma \cup \{k\}}$ is taken as D_α for the valuation. [If α is a limit ordinal, then $\gamma = \alpha$ is the only case we need to consider.]

Let $x' \in D_\alpha - D_\gamma$. If $v(x' \in M_\gamma) = 1$, then $v(z' \in x') = v(z' \in y')$ for all $z' \in D_\alpha$, for some $y' \in D_\gamma$, where y' is a corresponding member of D_γ for x'. Then $v(x' \in \{z' \in M_\gamma \in \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\}) = v(A_{M_\gamma \cup \{k\}}(y', \overline{w}_1', \ldots, \overline{w}_l'))$. If $v(x' \in M_\gamma) = 0$ then $v(x' \in \{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\}) = 0$. If $y \in D_{\alpha+1} - (D_\alpha \cup \{M_\alpha\})$ and $0 \leqslant \gamma \leqslant \alpha$, then if $v(x' \in y) = v(x' \in z)$ for all $x' \in D_\alpha$, for some $z \in D_\gamma$, then $v(y \in M_\gamma) = 1$ and z is called *a corresponding member of D_γ for y*.

If it is not the case that $v(x' \in y) = v(x' \in z)$ for all $x' \in D_\alpha$, for some $z \in D_\gamma$, then $v(y \in M_\gamma) = 0$.

Let $x \in D_{\alpha+1} - (D_\alpha \cup \{M_\alpha\})$. Let $v(x \in M_\gamma) = 1$. Then $v(x \in \{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\}) = v(A_{M_\gamma \cup \{k\}}(z, \overline{w}_1', \ldots, \overline{w}_l'))$, where z is a corresponding member of D_γ for x and where the range of the bound variables is taken as D_α.

Now let $v(x \in M_\gamma) = 0$. Then $v(x \in \{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\}) = 0$. Since the Axiom of Extensionality holds in D_α, any corresponding member, z, of D_γ for x can be substituted in the above expression.

We shall now show that the Axiom of Extensionality holds in $D_{\alpha+1}$. Let $v(x' \in x) = v(x' \in y)$ for all $x' \in D_{\alpha+1}$, where $x, y \in D_{\alpha+1} - \{k\}$. $v(x \in \{k\}) = 0 = v(y \in \{k\})$. Let $0 \leqslant \gamma \leqslant \alpha$. Let $\overline{w}_i' \in D_\alpha$ and $v(\overline{w}_i' \in M_\gamma) = 1$ or $\overline{w}_i'$ be k, for all i.

(i) Let $v(x \in M_\gamma) = 1$ and $x \in D_\gamma$.

a. Let $v \in D_\gamma$. Then $v(y \in M_\gamma) = 1$ and $v(y \in \{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\}) = v(A_{M_\gamma \cup \{k\}}(y, \overline{w}_1', \ldots, \overline{w}_l')) = v(A_{M_\gamma \cup \{k\}}(x, \overline{w}_1', \ldots, \overline{w}_l')) = v(x \in \{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\})$, using the Axiom of Extensionality in D_α.

b. Let $y \in D_{\alpha+1} - D_\gamma$. Then x is a corresponding member of D_γ for y, $v(y \in M_\gamma) = 1$, and $v(y \in \{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\}) = v(A_{M_\gamma \cup \{k\}}(x, \overline{w}_1', \ldots, \overline{w}_l)) = v(x \in \{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\})$.

(ii) Let $v(x \in M_\gamma) = 1$ and $x \in D_{\alpha+1} - D_\gamma$.

a. Let $y \in D_\gamma$. This case has already been treated in (i).

b. Let $y \in D_{\alpha+1} - D_\gamma$. x has a corresponding member, w, of D_γ. Hence $v(x' \in x) = v(x' \in w)$ for all $x' \in D_\alpha$, and $v(x \in \{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\}) = v(A_{M_\gamma \cup \{k\}}(w, \overline{w}_1', \ldots, \overline{w}_l'))$. $v(x' \in y) = v(x' \in w)$ for all $x' \in D_\alpha$ and $v(y \in M_\gamma) = 1$. $v(y \in \{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\}) = v(A_{M_\gamma \cup \{k\}}(w, \overline{w}_1', \ldots, \overline{w}_l)) = v(x \in \{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\})$.

(iii) Let $v(x \in M_\gamma) = 0$. Then it is not the case that $v(x' \in x) = v(x' \in w)$ for all $x' \in D_\alpha$, for some $w \in D_\gamma$.

a. Let $y \in D_\gamma$. Then $v(x' \in x) = v(x' \in y)$ for all $x' \in D_\alpha$. This is a contradiction.

b. Let $y \epsilon D_{\alpha+1} - D_\gamma$. Then it is not the case that $v(x' \epsilon y) = v(x' \epsilon w)$ for all $x' \epsilon D_\alpha$, for some $w \epsilon D_\gamma$. Hence $v(y \epsilon M_\gamma) = 0$ and $v(y \epsilon \{z' \epsilon M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}) = 0 = v(x \epsilon \{z' \epsilon M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\})$.

This completes the proof.

Let $x \epsilon D_{\alpha+1} - D_\gamma$ and let $v(x' \epsilon x) = v(x' \epsilon z)$ for all $x' \epsilon D_\alpha$, for some $z \epsilon D_\gamma (0 \leqslant \gamma \leqslant \alpha)$. So $v(x \epsilon M_\gamma) = 1$. This covers all cases of $v(x \epsilon M_\gamma) = 1$, where $x \epsilon D_\alpha - D_\gamma$, because x and z can be interchanged in all contexts in D_α. Also x cannot take the form M_δ, $\gamma \leqslant \delta \leqslant \alpha$, because $v(M_\delta \epsilon M_\gamma) = 0$. Let x be $\{z' \epsilon M_\kappa \cup \{k\}/A_{M_\kappa \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$, where $0 \leqslant \kappa \leqslant \alpha$ and $\overline{w}'_i \epsilon D_\alpha$ and $v(\overline{w}'_i \epsilon M_\kappa) = 1$ or $\overline{w}_i$ is k, for all i.

Let $y \epsilon D_{\alpha+1} - D_\alpha$.

(I) Let $v(y \epsilon M_\kappa) = 1$. Then $v(x' \epsilon y) = v(x' \epsilon u)$ for all $x' \epsilon D_\alpha$, for some $u \epsilon D_\kappa$. Then $v(y \epsilon x) = v(A_{M_\kappa \cup \{k\}}(u, \overline{w}'_1, \ldots, \overline{w}'_l)) = v(u \epsilon x) = v(u \epsilon z)$. Let z be $\{z' \epsilon M_\delta \cup \{k\}/B_{M_\delta \cup \{k\}}(z', \overline{x}'_1, \ldots, \overline{x}'_m)\}$, where $0 \leqslant \delta < \gamma \leqslant \alpha$ and $\overline{x}'_i \epsilon D_{\gamma-1}$ (or D_γ, if γ is a limit ordinal) such that $v(\overline{x}'_i \epsilon M_\delta) = 1$ or $\overline{x}'_i$ is k, for all i.

(i) Let $v(y \epsilon M_\delta) = 1$. Then $v(x' \epsilon y) = v(x' \epsilon w)$ for all $x' \epsilon D_\alpha$, for some $w \epsilon D_\delta$. Hence $v(y \epsilon z) = v(B_{M_\delta \cup \{k\}}(w, \overline{x}'_1, \ldots, \overline{x}'_m)) = v(w \epsilon z)$.

a. Let $\delta \leqslant \kappa$. Then $D_\delta \subseteq D_\kappa$ and w could have been used as the corresponding member of D_κ for y and hence $v(y \epsilon x) = v(w \epsilon z) = v(y \epsilon z)$.

b. Let $\delta > \kappa$. Then $D_\kappa \subseteq D_\delta$ and u could have been used as the corresponding member of D_δ for y and hence $v(y \epsilon z) = v(u \epsilon z) = v(y \epsilon x)$.

(ii) Let $v(y \epsilon M_\delta) = 0$. Then $v(y \epsilon z) = 0$. It is not the case that $v(x' \epsilon y) = v(x' \epsilon w)$ for all $x' \epsilon D_\alpha$, for some $w \epsilon D_\delta$. This also follows for u. Hence $v(u \epsilon M_\delta) = 0$ and $v(u \epsilon z) = 0$. Hence $v(y \epsilon x) = 0 = v(y \epsilon z)$.
Let z be $\{k\}$. Then $v(y \epsilon z) = 0$ and $v(y \epsilon x) = v(u \epsilon z) = 0$.
Let z be M_δ, $0 \leqslant \delta < \gamma \leqslant \alpha$. As above, if $v(y \epsilon M_\delta) = 1$ (or 0) then $v(u \epsilon M_\delta) = 1$ (or 0) and hence $v(y \epsilon x) = v(y \epsilon z)$.

(II) Let $v(y \epsilon M_\kappa) = 0$. Then it is not the case that $v(x' \epsilon y) = v(x' \epsilon u)$ for all $x' \epsilon D_\alpha$, for some $u \epsilon D_\kappa$. $v(y \epsilon x) = 0$.
Let z be $\{z' \epsilon M_\delta \cup \{k\}/B_{M_\delta \cup \{k\}}(z', \overline{x}'_1, \ldots, \overline{x}'_m)\}$, where $0 \leqslant \delta < \gamma \leqslant \alpha$ and $\overline{x}'_i \epsilon D_{\gamma-1}$ (or D_γ, if γ is a limit ordinal) such that $v(\overline{x}'_i \epsilon M_\delta) = 1$ or $\overline{x}'_i$ is k, for all i.

(i) Let $v(y \epsilon M_\delta) = 1$. Then $v(x' \epsilon y) = v(x' \epsilon w)$ for all $x' \epsilon D_\alpha$, for some $w \epsilon D_\delta$. If $\delta \leqslant \kappa$, then $D_\delta \subseteq D_\kappa$ and $v(x' \epsilon y) = v(x' \epsilon w)$ for all $x' \epsilon D_\alpha$, for some $w \epsilon D_\kappa$, which yields a contradiction. Hence, let $\delta > \kappa$. $v(y \epsilon z) = v(w \epsilon z) = v(w \epsilon x)$. It is not the case that $v(x' \epsilon w) = v(x' \epsilon u)$ for all $x' \epsilon D_\alpha$, for some $u \epsilon D_\kappa$. Hence $v(w \epsilon M_\kappa) = 0$ and $v(w \epsilon x) = 0$. Hence $v(y \epsilon z) = 0 = v(y \epsilon x)$.

(ii) Let $v(y \epsilon M_\delta) = 0$. Then $v(y \epsilon z) = 0 = v(y \epsilon x)$.
Let z be $\{k\}$. Then $v(y \epsilon z) = 0 = v(y \epsilon x)$.
Let z be M_δ, $0 \leqslant \delta < \gamma \leqslant \alpha$. Let $v(y \epsilon M_\delta) = 1$. Then $v(x' \epsilon y) = v(x' \epsilon w)$ for

all $x' \in D_\alpha$, for some $w \in D_\delta$. If $\delta \leq \kappa$, then $D_\delta \subseteq D_\kappa$ and $v(x' \in y) = v(x' \in w)$ for all $x' \in D_\alpha$, for some $w \in D_\kappa$, which yields a contradiction. Hence, let $\delta > \kappa$. $v(w \in M_\delta) = 1 = v(w \in x)$. It is not the case that $v(x' \in w) = v(x' \in u)$ for all $x' \in D_\alpha$, for some $u \in D_\kappa$. Hence $v(w \in M_\kappa) = 0$. Hence $v(w \in x) = 0$, which is a contradiction. Hence $v(y \in M_\delta) = 0$ and $v(y \in z) = 0 = v(y \in x)$. Hence, by the Axiom of Extensionality, in all contexts, x can be replaced by a corresponding member of D_γ, $0 \leq \gamma \leq \alpha$. Hence $v(B_{M_\gamma \cup \{k\}}(z', \overline{x}'_1, \ldots, \overline{x}'_m))(z', \overline{w}'_i \text{ all} \in D_\alpha)$ is the same whether the range of the bound variables in $B_{M_\gamma \cup \{k\}}$ is D_γ or $D_{\alpha+1}$, where $0 \leq \gamma \leq \alpha$.

The next stage of the transfinite induction is to consider the formation of D_α, α a limit ordinal. D_α consists of k, $\{k\}$, M_γ, for all γ such that $0 \leq \gamma < \alpha$, and all expressions of the form: $\{z' \in M_\gamma \cup \{k\}/A_{M_\gamma \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}_l)\}$, where $\overline{w}'_i \in D_\alpha$ and $v(\overline{w}'_i \in M_\gamma) = 1$ or $\overline{w}'_i$ is k, for all i, where $0 \leq \gamma < \alpha$, and where $A_{M_\gamma \cup \{k\}}$ has the value 1 or 0 for all $z' \in D_\alpha$. That is, $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$. By the induction hypothesis, all the valuations for D_α have been made, the Axiom of Extensionality holds in D_α and if $x \in D_\alpha - D_\gamma$ and $v(x \in M_\gamma) = 1$ then x can be replaced by any of its corresponding members, in all contexts with the domain D_α. Also $v(B_{M_\gamma \cup \{k\}}(z', \overline{x}'_1, \ldots, \overline{x}'_m))$ is the same whether the range of the bound variables in $B_{M_\gamma \cup \{k\}}$ is taken as D_γ or D_α.

Now define $D^U = \bigcup_\alpha D_\alpha$. D^U consists of k, $\{k\}$, M_α, for all α, and all expressions of the form: $\{z' \in M_\alpha \cup \{k\}/A_{M_\alpha \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$, where $\overline{w}'_i \in D^U$ and $v(\overline{w}'_i \in M_\alpha) = 1$ or $\overline{w}'_i$ is k, for all i, where α is any ordinal, and where $A_{M_\alpha \cup \{k\}}$ has the value 1 or 0 for all $z' \in D^U$. By the transfinite induction, all the valuations for D^U have been made, the Axiom of Extensionality holds in D^U, if $x \in D^U - D_\alpha$ and $v(x \in D_\alpha) = 1$ then x can be replaced by any of its corresponding members, in all contexts with the domain D^U, and $v(A_{M_\alpha \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l))$ is the same whether the range of the bound variables in $A_{M_\alpha \cup \{k\}}$ is taken as D_α or D^U.

The following valuations hold in D^U:

If $x \in D^U - \{k\}$ and $y \in D^U - \{k\}$ then $v(k \circ k) = 1$, $v(x \circ k) = v(k \circ x) = v(x \circ y) = n$, $v(k \in M_\alpha) = 0$, $v(M_\beta \in M_\alpha) = 1$ if $\beta < \alpha$, $v(M_\gamma \in M_\alpha) = 0$ if $\gamma \geq \alpha$. If $x' \in D^U$ then $v(x' \in k) = n$. If $y \in D^U - \{k\}$, then $v(y \in \{k\}) = 0$ and $v(k \in \{k\}) = 1$.

If x is $\{z' \in M_\alpha \cup \{k\}/A_{M_\alpha \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$, where $\overline{w}'_i \in D^U$ and $v(\overline{w}_i \in M_\alpha) = 1$ or $\overline{w}'_i$ is k, for all i, then $v(x \in M_\delta) = 1$ for all $\delta > \alpha$, and $v(M_\tau \in x) = 0$ for all $\tau \geq \alpha$. If $v(x' \in M_\alpha) = 1$ or x' is k, then $v(x' \in x) = v(A_{M_\alpha \cup \{k\}}(x', \overline{w}'_1, \ldots, \overline{w}'_l))$. If $v(x' \in M_\alpha) = 0$ and x' is not k, then $v(x' \in x) = 0$. $v(\{k\} \in M_\alpha) = 1$, for all α. If $v(x' \in x) = v(x' \in z)$ for all $x' \in D^U$, for some z such that $v(z \in M_\tau) = 1$ for some $0 \leq \tau \leq \alpha$, then $v(x \in M_\tau) = 1$. If it is not the case that $v(x' \in x) = v(x' \in z)$ for all $x' \in D^U$, for some z such that $v(z \in M_\tau) = 1$, for some $0 \leq \tau \leq \alpha$, then $v(x \in M_\tau) = 0$. If $v(x' \in M_\delta) = 1$ then $v(x' \in M_\alpha) = 1$, for all $x' \in D^U$, for all δ and κ such that $\delta \leq \kappa$. Hence for any $x' \in D^U$, except for k, there is a least ordinal α such that $v(x' \in M_\alpha) = 1$. Hence $v(x' \in M_\gamma) = 0$, for all $\gamma < \alpha$ and $v(x' \in M_\gamma) = 1$ for all $\gamma \geq \alpha$. Call this least ordinal, $\alpha_{x'}$. Note that $\alpha_{M_\beta} = \beta + 1$, $\alpha_{\{k\}} = 0$, and $\alpha_{x'}$ is always a successor ordinal. If $\alpha_x \leq \alpha_y$, then $v(y \in x) = 0$. If x is $\{z' \in M_\alpha \cup \{k\}/A_{M_\alpha \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$, then $\alpha_x \leq \alpha + 1$. If $v(y \in x) = 1$ then $\alpha_y \leq \alpha_x - 1$.

The domain D^U and its valuations will form the model for the axioms involving sets and individuals, which will be shown later. We now construct a domain D^S which, with its valuations, will form the model for the axioms involving classes and individuals.

Let D^0 consist of all the members of D^U and all the expressions of the form: $\{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m')\}$, where $\overline{x}_i' \in D^U$, for all i, where A contains quantification only over sets and individuals, and where A has the value 1 or 0 for all $z' \in D^U$. In the following, all quantification over sets and individuals that occurs in predicates A will be evaluated over D^U.

If $\{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m')\} \in D^0 - D^U$, then $v(y' \in \{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m')\}) = v(A(y', \overline{x}_1', \ldots, \overline{x}_m'))$, for all $y' \in D^U$. If $v(y' \in \{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m')\}) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(\{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m')\} \in U') = v(w' \in U')$, for all $U' \in D^0$. If it is not the case that $v(y' \in \{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m')\}) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(\{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m')\} \in U') = 0$, for all $U' \in D^0$.

Given that D^n and its valuations have been determined, D^{n+1} consists of all of the members of D^n and all the expressions of the form: $\{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\}$, where $\overline{x}_i' \in D^U$, for all i, and $\overline{Y}_j \in D^n$, for all j, where A contains quantification only over sets and individuals, and where A has the value 1 or 0 for all $z' \in D^U$.

If $\{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\} \in D^{n+1} - D^n$ then $v(y' \in \{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\}) = v(A(y', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p'))$, for all $y' \in D^U$. If $v(y' \in \{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\}) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(\{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\} \in U') = v(w' \in U')$, for all $U' \in D^{n+1}$. If it is not the case that $v(y' \in \{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\}) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(\{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\} \in U') = 0$ for all $U' \in D^{n+1}$. If $v(y' \in V') = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, where $V' \in D^n - D^U$, then $v(V' \in U') = v(w' \in U')$ for all $U' \in D^{n+1} - D^n$. If it is not the case that $v(y' \in V') = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(V' \in U') = 0$, for all $U' \in D^{n+1} - D^n$.

We need to show that if $v(y' \in w') = v(y' \in w_1')$ for all $y' \in D^U$, then $v(w' \in U') = v(w_1' \in U')$, for all $U' \in D^{n+1} - D^n$, where w' and $w_1' \in D^U$. By the Axiom of Extensionality for D^U, the above holds for all $U' \in D^0$. Let us assume that the above holds for $U' \in D^n$. Now let $U' \in D^{n+1} - D^n$. For some predicate A, U' is $\{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\}$. $v(w' \in U') = v(A(w', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p'))$. If $\overline{y}_i \in D^n - D^U$, then either $v(\overline{y}_i \in w') = v(\overline{y}_i \in w_1') = 0$ or $v(\overline{y}_i \in w') = v(\overline{y}_i' \in w') = v(\overline{y}_i' \in w_1')$, for some $\overline{y}_i' \in D^U$. Hence $v(A(w', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')) = v(A(w_1', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p'))$ and $v(w' \in U') = v(w_1' \in U')$. This completes the proof.

Let $D^S = \underset{n}{U}\, D^n$ and let D^S have the valuations obtained by induction on the D^n's. Hence D^S consists of all the members of D^U and all expressions of the form: $\{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\}$, where $\overline{x}_i' \in D^U$, for all i, and $\overline{y}_j \in D^S$, for all j, where A contains quantification over sets and individuals only, and where A has the value 1 or 0 for all $z' \in D^U$.

If $\{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\} \in D^S - D^U$, then $v(y' \in \{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\}) = v(A(y', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p'))$, for all $y' \in D^U$. If $v(y' \in \{z'/A(z', \overline{x}_1', \ldots, \overline{x}_m', \overline{Y}_1', \ldots, \overline{Y}_p')\}) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \epsilon D^U$, then $v(\{z'/A(z', \overline{x}'_1, \ldots, \overline{x}'_m, \overline{Y}'_1, \ldots, \overline{Y}'_p)\} \epsilon U') = v(w' \epsilon U')$, for all $U' \epsilon D^S$. If it is not the case that $v(y' \epsilon \{z'/A(z', \overline{x}'_1, \ldots, \overline{x}'_m, \overline{Y}'_1, \ldots, \overline{Y}'_p)\}) = v(y' \epsilon w')$ for all $y' \epsilon D^U$, for some $w' \epsilon D^U$, then $v(\{z'/A(z', \overline{x}'_1, \ldots, \overline{x}'_m, \overline{Y}'_1, \ldots, \overline{Y}'_p)\} \epsilon U') = 0$, for all $U' \epsilon D^S$. If $v(y' \epsilon w') = v(y' \epsilon w'_1)$ for all $y' \epsilon D^U$, then $v(w' \epsilon U') = v(w'_1 \epsilon U')$ for all $U' \epsilon D^S$, where w' and $w'_1 \epsilon D^U$. This follows by induction using an above argument.

Now we will show that D^S and its valuations form a model, N', for all the axioms.

The domain for classes and individuals is D^S, the domain for individuals is $\{k\}$, the domain for sets and individuals is D^U, the domain for classes is $D^S - \{k\}$, and the domain for sets is $D^U - \{k\}$.

The General Axioms 1, 2, and 3 are obviously valid in the model N'. Individual Axiom 1 is valid because there is only one individual, k, in the model. Individual Axiom 2 is valid because the fusion of x is k. Individual Axiom 3 is valid because x is either $\{k\}$ or 0, where 0 can be taken as $\{z' \epsilon M_0 \cup \{k\}/{\sim}(k \epsilon \{k\})\}$. Individual Axioms 4 and 5 are valid because there is one individual, k.

For showing the validity of Axiom T, let $v(x \epsilon X) = v(x \epsilon Y)$ for all $x \epsilon D^S$, where X and $Y \epsilon D^S$. If X and $Y \epsilon D^U$, then we have already shown that $v(X \epsilon U') = v(Y \epsilon U')$, for all $U' \epsilon D^S$. If $X \epsilon D^S - D^U$ and $Y \epsilon D^U$, then by the construction of D^S, $v(X \epsilon U') = v(Y \epsilon U')$ for all $U' \epsilon D^S$. Similarly, if $X \epsilon D^U$ and $Y \epsilon D^S - D^U$. Let $X \epsilon D^S - D^U$ and $Y \epsilon D^S - D^U$.

a. If $v(y' \epsilon X) = v(y' \epsilon w')$ for all $y' \epsilon D^U$, for some $w' \epsilon D^U$, then $v(y' \epsilon Y) = v(y' \epsilon w')$ for all $y' \epsilon D^U$, for some $w' \epsilon D^U$. Hence $v(X \epsilon U') = v(w' \epsilon U') = v(Y \epsilon U')$, for all $U' \epsilon D^S$.

b. If it is not the case that $v(y' \epsilon X) = v(y' \epsilon w')$ for all $y' \epsilon D^U$, for some $w' \epsilon D^U$, then it is not the case that $v(y' \epsilon Y) = v(y' \epsilon w')$ for all $y' \epsilon D^U$, for some $w' \epsilon D^U$. Hence $v(X \epsilon U') = 0 = v(Y \epsilon U')$, for all $U' \epsilon D^S$. Hence Axiom T is valid in the model.

To show that the Axiom P is valid in the model, let x' and y' be unequal to k. Let $\alpha_{x'} \leqslant \alpha_{y'}$. Then $v(x' \epsilon M_{\alpha_{y'}}) = v(y' \epsilon M_{\alpha_{y'}}) = 1$. The required x is then $\{z' \epsilon M_{\alpha_{y'}} \cup \{k\}/(\mathbf{A}w')(w' \epsilon M_{\alpha_{y'}} \supset T(w' \epsilon z' \doteqdot w' \epsilon x')) \;\&\; T(k \epsilon z' \doteqdot k \epsilon x') \,.\!\vee.\, (\mathbf{A}w')(w' \epsilon M_{\alpha_{y'}} \supset T(w' \epsilon z' \doteqdot w' \epsilon y')) \;\&\; T(k \epsilon z' \doteqdot k \epsilon y')\}$. Now let x' be unequal to k and let y' be k. Then the required x is $\{z' \epsilon M_{\alpha_{x'}} \cup \{k\}/ z' \epsilon \{k\} \vee. (\mathbf{A}w')(w' \epsilon M_{\alpha_{x'}} \supset T(w' \epsilon z' \doteqdot w' \epsilon x')) \;\&\; T(k \epsilon z' \doteqdot k \epsilon x')\}$. If x' and y' are both k, then the required x is $\{k\}$.

Axiom N is valid as the required x can be taken as $\{z' \epsilon M_0 \cup \{k\}/{\sim}(k \epsilon \{k\})\}$. As before, call this 0.

For showing the validity of Axiom B, consider the predicate $\phi(x'_1, \ldots, x'_l, Y'_1, \ldots, Y'_m)$, where only variables over sets and individuals are quantified, and where ϕ is significant for all substitutions into its free variables. The required x is $\{z'/(\mathbf{S}x'_1) \ldots (\mathbf{S}x'_l)((\mathbf{A}w')\,T(w' \epsilon z' \doteqdot w' \epsilon \langle x'_1, \ldots, x'_l\rangle) \;\&\; \phi(x'_1, \ldots, x'_l, \overline{Y}'_1, \ldots, \overline{Y}_m))\}$. The $\langle x'_1, \ldots, x'_l\rangle$ is defined the same way as earlier in this section, using the set x, used to show the validity of Axiom P.

For the Axiom U, the required x is $\{z' \in M_{\alpha_x} \cup \{k\}/(\mathbf{S}v')(T_n(v' \in M_{\alpha_x}) \;\&\; z' \in v' \;\&\; v' \in x) \vee . z' \in k \;\&\; k \in x\}$.

For the Axiom W, consider all the members x' of x in the model, i.e., such that $v(x' \in x) = 1$. All of these members will be members of M_{α_x-1}, i.e., such that $v(x' \in M_{\alpha_x-1}) = 1$, or be k. Consider a set S of only those members of x which are members of the set D_{α_x-1}. Any member of x not in D_{α_x-1} will be replaceable by a member of x in D_{α_x-1} in all contexts. Let the set T be the set of all subsets of S. Form a subset R of T such that $X \in R \equiv X \in T \;\&\; (\mathbf{S}z)(z \in D^U \;\&\; (\mathbf{A}w')(w' \in D^U \supset . \; v(w' \in z) = 1 \equiv w' \in X))$. [By transfinite induction, the class V of ordered pairs $\langle x', y'\rangle$, $x', y' \in D^U$, such that $v(x' \in y') = 1$ can be constructed so that '$v(w' \in z) = 1$' can be replaced by '$\langle w', z'\rangle \in V$'.] For each member X of R, since there is a member z of D^U there is a least ordinal α_z such that $v(z \in M_\alpha) = 1$. Hence choose a z from D_{α_z} satisfying the above property. So to each member X of R we can choose a corresponding z from D^U. Since there is a set of such z's, there is an ordinal β which is the sup of all ordinals α_z. Hence the required y can be taken as $\{z' \in M_\beta \cup \{k\}/(\mathbf{A}w')(w' \in M_\beta \supset . \; T(w' \in z' \supset w' \in x)) \;\&\; T(k \in z' \supset k \in x)\}$. This is the required power set of x because, by the above argument, all possible subsets of x will be members of M_β.

Next we will show that Axiom R is valid but firstly in a form applicable to sets and individuals only. That is, if $A(x', y', u_1', \ldots, u_m')$ is univocal then $(\mathbf{S}y)(\mathbf{A}y')(y' \in y \equiv (\mathbf{S}x')(A(x', y', u_1', \ldots, u_m') \;\&\; x' \in x))$, where quantification in A is over sets and individuals only, and where A is significant for all substitutions into its free variables. By Theorem 1, we need only consider wffs A such that A contains only the connectives $\sim$, &, and T and the quantifier $\mathbf{A}$. By the result in the appendix of this paper we need only consider wffs A such that A has all of its quantifiers, $\mathbf{A}$ and $\mathbf{E}$, at the beginning of the formula. [In the proof in the appendix Sp can be defined as $\sim(\sim Tp \;\&\; \sim T \sim p)$ and $T(\mathbf{E}x)A(x) \simeq (\mathbf{E}x)(\mathbf{A}y)(TA(x) \;\&\; SA(y))$.] The proof will follow that in [2], pp. 90-92.

Lemma 1 *Let $y' = \phi(x')$ be a univocal function defined by a formula $A(x', y', u_1', \ldots, u_m')$ for some $u_i' \in D^U$ and such that $x' \in D^U$ implies $\phi(x') \in D^U$. If $u' \in D^U$ then there is a $w' \in D^U$ such that if v' is the range of ϕ on u', then $y' \in v' \supset . \; v(y' \in w') = 1$, for all $y' \in D^U$. [v' is a set of members of D^U, and cannot belong to D^U.]*

Proof: Note that if $v(w_1' \in z') = v(w_1' \in x')$ for all $w_1' \in D^U$ and where z' and $x' \in D^U$, then $v(w_2' \in \phi(z')) = v(w_2' \in \phi(x'))$, for all $w_2' \in D^U$. For each x' such that $v(x' \in u') = 1$ and $x' \in D_{\alpha_{u'}-1}$, let $g(x')$ be the least ordinal α such that $v(\phi(x') \in M_\alpha) = 1$, if $\phi(x') \in D^U - \{k\}$, and let $g(x')$ be 0 if $\phi(x')$ is k. Let β be the sup of all these $g(x')$'s. Clearly $y' \in v' \supset v(y' \in M_\beta \cup \{k\}) = 1$, for all $y' \in D^U$. [$M_\beta \cup \{k\}$ can be taken as $\{z' \in M_\beta \cup \{k\}/k \in \{k\}\}$.]

Lemma 2 *Let $A(x_1', \ldots, x_n')$ be a wff with the above restrictions on connectives and quantifiers and with its quantifiers, $\mathbf{A}$ and $\mathbf{E}$, at the beginning of the formula. Let $\overline{y}' \in D^U - \{k\}$. There is an $M_\mu \cup \{k\} \in D^U$ such that if $v(z' \in \overline{y}') = 1$ then $v(z' \in M_\mu \cup \{k\}) = 1$, for all $z' \in D^U$, and for all*

$\bar{x}_i' \in D^U$ *such that* $v(\bar{x}_i' \in M_\mu \cup \{k\}) = 1$, $v(A(\bar{x}_1', \ldots, \bar{x}_n')) = v(A_{M_\mu \cup \{k\}}(\bar{x}_1', \ldots, \bar{x}_n'))$, *where* $A_{M_\mu \cup \{k\}}$ *is* A *with all its bound variables restricted to* $M_\mu \cup \{k\}$. [$M_\mu \cup \{k\}$ *can be taken as* $\{z' \in M_\mu \cup \{k\}/k \in \{k\}\}$.]

Proof: By the above conditions, A is of the form $\mathbf{Q}_1 y_1' \ldots \mathbf{Q}_m y_m' B(x_1', \ldots, x_n', y_1', \ldots, y_m')$, where $\mathbf{Q}_r (1 \leq r \leq m)$ is either $\mathbf{A}$ or $\mathbf{E}$. Let $\bar{u}' \in D^U - \{k\}$. For $1 \leq r \leq m$ there are functions $f_r(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r-1}')$ defined for $\bar{x}_i', \bar{y}_j'$ in $\bar{u}'$ (i.e., $v(\bar{x}_i' \in \bar{u}') = 1$ and $v(\bar{y}_j' \in \bar{u}') = 1$) with the following property: If $\mathbf{Q}_r$ is $\mathbf{E}$ and there is a set or individual $\bar{y}_r' \in D^U$ such that:

(1) $\ldots \mathbf{Q}_{r+1} y_{r+1}' \ldots \mathbf{Q}_m y_m' B(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r', y_{r+1}', \ldots, y_m')$

is nonsignificant in the model, or given that there are no sets or individuals $\bar{y}_r'$ in D^U such that (1) is nonsignificant in the model, there is a set or individual $\bar{y}_r'$ in D^U such that (1) is true in the model, then $f_r = \alpha$ where α is the least ordinal such that there is a $\bar{y}_r' \in M_\alpha \cup \{k\}$ (i.e., $v(\bar{y}_r' \in M_\alpha \cup \{k\}) = 1$) satisfying either of the above conditions. If no such $\bar{y}_r'$ exists, put $f_r = 0$. If $\mathbf{Q}_r$ is $\mathbf{A}$ (i.e., $\sim\mathbf{E}\sim$), then f_r is defined the same way as for $\mathbf{E}$ except that (1) is replaced by its negation.

Let β be the sup of $f_r(\bar{x}_i', \bar{y}_j')$ for all $\bar{x}_i'$ and $\bar{y}_j' \in D_{\alpha_{\bar{u}'}-1}$, such that $v(\bar{x}_i' \in \bar{u}') = 1$ and $v(\bar{y}_j' \in \bar{u}') = 1$, and all r for $1 \leq r \leq m$. Put $(\bar{u}')^* = \bar{u}' \cup M_\beta \cup \{k\}$. This union can be formed using the Pairing Axiom and Sum Set Axiom which have already been shown to be valid in the model. So $(\bar{u}')^* \in D^U$. Now we define a sequence $\bar{z}_n'$ with $\bar{z}_0'$ as $M_\alpha \cup \{k\}$, if α is the least ordinal such that $v(\bar{y}' \in M_\alpha) = 1$, and $\bar{z}_{n+1}' = (\bar{z}_n')^*$. So $\bar{z}_n' \in D^U$, for all n. Let $\bar{z}' = \underset{n}{U}\, \bar{z}_n'$. This requires the validity of the Axiom of Infinity, which will be shown later. Assuming this, $\bar{z}' \in D^U$. So $\bar{z}' = U\{M_\beta/\beta < \alpha'\} \cup \{k\}$, for some α'. $\bar{z}' = M_{\alpha'} \cup \{k\}$ if α' is a limit ordinal or $\bar{z}' = M_{\alpha'-1} \cup \{k\}$ if α' is a successor ordinal. If $v(x' \in M_\alpha) = 1$ then $v(x' \in \bar{z}') = 1$ for all $x' \in D^U$, and hence if $v(x' \in \bar{y}') = 1$ then $v(x' \in \bar{z}') = 1$ for all $x' \in D^U$.

Now we need to show that $v(A(\bar{x}_1', \ldots, \bar{x}_n')) = v(A_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n'))$ for all $\bar{x}_i' \in D^U$ such that $v(\bar{x}_i' \in \bar{z}') = 1$. Let $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ denote the statement (1). Assume that we have shown that, for $r > r_0$, $C \simeq C_{\bar{z}'}$ is true in the model, for all $\bar{x}_i'$ and $\bar{y}_j \in D^U$ such that $v(\bar{x}_i' \in \bar{z}') = 1$ and $v(\bar{y}_j' \in \bar{z}') = 1$. Certainly this is the case for $r_0 = m$. Then with $r = r_0$, given that $v(\bar{x}_i' \in \bar{z}') = 1$ and $v(\bar{y}_j' \in \bar{z}') = 1$, they must all lie in $\bar{z}_k'$ for some k.

Let $\mathbf{Q}_{r+1}$ be $\mathbf{E}$. Then if $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is true in the model then there is a $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}_{k+1}') = 1$ and $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is true in the model and for all $\bar{y}_{r+1}' \in D^U$, $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is significant in the model. By assumption, $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is true for the chosen $\bar{y}_{r+1}'$ and significant for all $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}') = 1$, and hence $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is true. If $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is false (in the model) then for all $\bar{y}_{r+1}' \in D^U$, $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is false. By assumption, $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is false for all $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}') = 1$ and hence $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is false. If $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is nonsignificant (in the model) then there is a $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}_{k+1}') = 1$ and $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is nonsignificant. By assumption, $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is nonsignificant for the chosen $\overline{y}'_{r+1}$, and hence $C_{\overline{z}'}(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_r)$ is nonsignificant.

Let $\mathbf{Q}_{r+1}$ be $\mathbf{A}$ (i.e., $\sim\mathbf{E}\sim$). If $C(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_r)$ is true (in the model) then for all $\overline{y}'_{r+1} \in D^U$, $C(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_{r+1})$ is true. By assumption, $C_{\overline{z}'}(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_{r+1})$ is true for all $\overline{y}'_{r+1} \in D^U$ such that $v(\overline{y}'_{r+1} \in \overline{z}') = 1$ and hence $C_{\overline{z}'}(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_r)$ is true. If $C(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_r)$ is false (in the model) then there is a $\overline{y}'_{r+1} \in D^U$ such that $v(\overline{y}'_{r+1} \in \overline{z}'_{k+1}) = 1$ and $C(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_{r+1})$ is false and, for all $\overline{y}'_{r+1} \in D^U$, $C(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_{r+1})$ is significant. By assumption, $C_{\overline{z}'}(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_{r+1})$ is false for the chosen $\overline{y}'_{r+1}$ and significant for all $\overline{y}'_{r+1} \in D^U$ such that $v(\overline{y}'_{r+1} \in \overline{z}') = 1$, and hence $C_{\overline{z}'}(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_r)$ is false. If $C(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_r)$ is nonsignificant (in the model) then there is a $\overline{y}'_{r+1} \in D^U$ such that $v(\overline{y}'_{r+1} \in \overline{z}'_{k+1}) = 1$ and $C(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_{r+1})$ is nonsignificant. By assumption, $C_{\overline{z}'}(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_{r+1})$ is nonsignificant for the chosen $\overline{y}'_{r+1}$, and hence $C_{\overline{z}'}(\overline{x}'_1, \ldots, \overline{x}'_n, \overline{y}'_1, \ldots, \overline{y}'_r)$ is nonsignificant. This completes the proof.

Theorem 2 *The Axiom of Replacement in the form:* $(\mathbf{A}x')(\mathbf{S}!y')A(x', y', u'_1, \ldots, u'_m)$ & $(\mathbf{A}x', y', u'_1, \ldots, u'_m)SA(x', y', u'_1, \ldots, u'_m) \supset (\mathbf{S}y)(\mathbf{A}y')(y' \in y \equiv (\mathbf{S}x')(A(x', y', u'_1, \ldots, u'_m) \mathbin{\&} x' \in x))$, *where A contains quantification over sets and individuals only, is valid in the model.*

Proof: Let $A(x', y', u'_1, \ldots, u'_m)$ define a univocal function in D^U: $y' = \phi(x')$, for particular $\overline{u}'_1, \ldots, \overline{u}'_m$ in D^U. Let $\overline{x} \in D^U$ and let $\overline{v}'$ be the range of ϕ on $\overline{x}$. [$\overline{v}'$ is a set of members of D^U but does not itself belong to D^U.] By Lemma 1, there is an α such that $z' \in \overline{v}' \supset. v(z' \in M_\alpha \cup \{k\}) = 1$, for all $z' \in D^U$. We can assume that $\overline{x}, \overline{u}'_1, \ldots, \overline{u}'_m$ all belong to $M_\alpha \cup \{k\}$, i.e., $v(\overline{x} \in M_\alpha \cup \{k\}) = 1$ and $v(\overline{u}'_i \in M_\alpha \cup \{k\}) = 1$ for all i. Taking $M_\alpha \cup \{k\}$ as the $\overline{y}'$ of Lemma 2, it follows that for some μ, $v(A(x', y', \overline{u}'_1, \ldots, \overline{u}'_m)) = v(A_{M_\mu \cup \{k\}}(x', y', \overline{u}'_1, \ldots, \overline{u}'_m))$, for all $x', y' \in D^U$ such that $v(x' \in M_\mu \cup \{k\}) = 1$ and $v(y' \in M_\mu \cup \{k\}) = 1$. Also $v(z' \in M_\alpha \cup \{k\}) = 1$ implies that $v(z' \in M_\mu \cup \{k\}) = 1$, for all $z' \in D^U$. Hence, for all $z' \in D^U$, $z' \in \overline{v}' \supset v(z' \in M_\mu \cup \{k\}) = 1$. Also, $v(\overline{x}' \in M_\mu \cup \{k\}) = 1$ and $v(\overline{u}'_i \in M_\mu \cup \{k\}) = 1$, for all i. Hence the required $\overline{y}$ can be taken as $\{y' \in M_\mu \cup \{k\}/(\mathbf{S}x')(T_n(x' \in M_\mu) \mathbin{\&} x' \in \overline{x} \mathbin{\&} A_{M_\mu \cup \{k\}}(x', y', \overline{u}'_1, \ldots, \overline{u}'_m)) \vee (k \in \overline{x} \mathbin{\&} A_{M_\mu \cup \{k\}}(k, y', \overline{u}'_1, \ldots, \overline{u}'_m))\}$. For arbitrary $\overline{u}'_1, \ldots, \overline{u}'_m, \overline{x}$, an ordinal μ can be found so that the above $\overline{y}$ represents the set $\overline{v}'$ in D^U, in that $z' \in \overline{v}'$ iff $v(z' \in \overline{y}) = 1$, for all $z' \in D^U$. Hence the above form of the Axiom of Replacement is valid in the model.

Lemma 3 *If $X \in D^S - D^U$, then there is a $Y \in D^0$ such that, for all $Z' \in D^S$, $v(Z' \in X) = v(Z' \in Y)$.*

Proof: Let $X \in D^{n+1} - D^n$ and assume that the lemma holds for all members of D^n. Let X be $\{z'/A(z', \overline{u}'_1, \ldots, \overline{u}'_m, \overline{V}'_1, \ldots, \overline{V}'_l)\}$, where $\overline{u}'_i \in D^U$, for all i, and $\overline{V}'_j \in D^n$, for all j. By the assumption, for each $\overline{V}'_j \in D^n$ there is a $\overline{W}'_j \in D^0$ such that $v(Z' \in \overline{V}'_j) = v(Z' \in \overline{W}'_j)$ for all $Z' \in D^S$. Let X_1 be $\{z'/A(z', \overline{u}'_1, \ldots, \overline{u}'_m, \overline{W}'_1, \ldots, \overline{W}'_l)\}$. Then, by the Axiom of Extensionality, which is valid in the model, $v(Z' \in X) = v(Z' \in X_1)$ for all $Z' \in D^S$. If $v(Z' \in \overline{W}'_j) = v(Z' \in \overline{y}')$ for all $Z' \in D^S$, for some $\overline{y}' \in D^U$, then replace the $\overline{W}'_j$ in A by the $\overline{y}'$. If there is no such $\overline{y}' \epsilon D^U$ then replace any statement of the form $\overline{W}_j' \epsilon X'$ in A by any false statement and replace any statement of the form $x' \epsilon \overline{W}_j'$ by its equivalent predicate expression, i.e., if $\overline{W}_j'$ is $\{z'/B(z')\}$ then $x' \epsilon \overline{W}_j'$ is replaced by $B(x')$. For statements in A of the form $\overline{X}' \epsilon \overline{W}_j'$, where $\overline{X}' \epsilon D^S - D^U$, if $v(Z' \epsilon \overline{X}') = v(Z' \epsilon \overline{y}')$ for all $Z' \epsilon D^S$, for some $\overline{y}' \epsilon D^U$ then replace $\overline{X}'$ by $\overline{y}'$ and replace $\overline{y}' \epsilon \overline{W}_j'$ by its equivalent predicate expression, and if there is no such $\overline{y}'$ then replace $\overline{X}' \epsilon \overline{W}_j'$ by any false statement. Let A' be the resulting form of A after these replacements have been made. Let Y be $\{z'/A'(z', \overline{u}_1', \ldots, \overline{u}_m', \overline{x}_1', \ldots, \overline{x}_p')\}$. $Y \epsilon D^0$, $v(Z' \epsilon Y) = v(Z' \epsilon X_1)$ for all $Z' \epsilon D^S$ and hence $v(Z' \epsilon Y) = v(Z' \epsilon X)$ for all $Z' \epsilon D^S$.

Theorem 3 *The Axiom of Replacement* (R) *in the form:* $Un(X) \supset (\mathbf{S}y)(\mathbf{A}x')(x' \epsilon y \equiv (\mathbf{S}y')(\langle y', x'\rangle \epsilon X \;\&\; y' \epsilon x))$, *is valid in the model.*

Proof: Let $X \epsilon D^S - D^U$ and let X be univocal. By Lemma 3, there is a $Y \epsilon D^0$ such that $v(Z' \epsilon X) = v(Z' \epsilon Y)$, for all $Z' \epsilon D^S$. Let Y be $\{z'/A(z', \overline{u}_1', \ldots, \overline{u}_m')\}$. So $\langle y', x'\rangle \epsilon X \simeq (\mathbf{S}z')(T(\mathbf{A}w')(w' \epsilon z' \doteq w' \epsilon \langle y', x'\rangle) \;\&\; A(z', \overline{u}_1', \ldots, \overline{u}_m'))$ is valid in the model. Let the expression on the right-hand side of the '$\simeq$' be called $B(y', x', \overline{u}_1', \ldots, \overline{u}_m')$. Since X is univocal, so is $B(y', x', \overline{u}_1', \ldots, \overline{u}_m')$. Hence, by Theorem 2, the Axiom R is valid in the model.

Since Axiom R implies Axiom S formally, Axiom S is valid in the model.

We will now test the validity of Axiom I (Axiom of Infinity). If $v(y \epsilon M_\alpha) = 1$ then $v(y \cup \{y\} \epsilon M_{\alpha+1}) = 1$ since $y \cup \{y\}$ can be taken as $\{z' \epsilon M_\alpha \cup \{k\}/z' \epsilon y \vee . (\mathbf{A}w')(w' \epsilon M_\alpha \supset T(w' \epsilon z' \doteq w' \epsilon y)) \;\&\; T(k \epsilon z' \doteq k \epsilon y)\}$. Also $v(\{z' \epsilon M_0 \cup \{k\}/\sim(k \epsilon \{k\})\} \epsilon M_1) = 1$. Hence the required x can be taken as M_ω.

We will now test the validity of Axiom D, the Axiom of Regularity. Since X has at least one member, which is a member of D^U, let α be the least ordinal such that some member of X is a member of M_α. Let $v(y \epsilon X) = 1$ and $v(y \epsilon M_\alpha) = 1$. Then y is either $M_{\alpha-1}$, $\{k\}$ (if $\alpha = 0$), or of the form $\{z' \epsilon M_{\alpha-1} \cup \{k\}/A_{M_{\alpha-1} \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_l')\}$, where $v(\overline{w}_i' \epsilon M_{\alpha-1}) = 1$ or w_i' is k, for all i. Hence any member of y will be a member of $M_{\alpha-1}$ or be k. Hence there are no set members of y that are members of X and the Axiom D is valid in the model.

We will now test the validity of Axiom C, the Axiom of Constructibility. Formally this is $(\mathbf{A}x)(\mathbf{S}\alpha)(x \epsilon M_\alpha)$. Firstly we need to show that the ordinals defined according to the formal theory and interpreted in the model are in one-one correspondence with the M_α's of the model, that is, with the ordinals used to set up the model. Before doing this, we need the following lemma:

Lemma 4 *'y' is an ordinal' is absolute, i.e., if $Trans_1(z) \;\&\; y' \epsilon z \;\&\; (y'$ is an ordinal$)_z$ then y' is an ordinal, where $(y'$ is an ordinal$)_z$ means that all the bound variables in 'y' is an ordinal' are restricted to z.*

Proof: Absoluteness can be shown for $x = y$, $z = \{x, y\}$, $z = \langle x, y\rangle$, etc., as in Cohen [2], p. 94. However, in the definition of 'y' is an ordinal', we need to replace 'EWey'' by 'ECony''. This can be done as in [10], pp. 35-36. Then the absoluteness of 'y' is an ordinal' will follow.

We will now define the ordinals in the model. Let 0 be $\{z' \epsilon M_0 \cup \{k\}/ \sim(k \epsilon \{k\})\}$. Then $v(0 \epsilon M_1) = 1$ and the smallest ordinal α such that $v(0 \epsilon M_\alpha) = 1$ is 1. Let α be defined in the model and let the smallest ordinal β such that $v(\alpha \epsilon M_\beta) = 1$ be $\alpha + 1$. Let $\alpha + 1$ be $\{z' \epsilon M_{\alpha+1} \cup \{k\}/z' \epsilon \alpha \vee . (\mathbf{A}w')(w' \epsilon M_{\alpha+1} \supset T(w' \epsilon z' \doteq w' \epsilon \alpha))\ \&\ T(k \epsilon z' \doteq k \epsilon \alpha)\}$. Clearly $v(\alpha + 1 \epsilon M_{\alpha+1}) = 1$. If $v(\alpha + 1 \epsilon M_{\alpha+1}) = 1$ then $\alpha + 1$ is either M_α or a subset of $M_\alpha \cup \{k\}$. Since $v(\alpha \epsilon \alpha + 1) = 1$ then $v(\alpha \epsilon M_\alpha) = 1$, which is a contradiction. Hence $\alpha + 2$ is the smallest ordinal such that $v(\alpha + 1 \epsilon M_{\alpha+2}) = 1$. Now let α be a limit ordinal and assume that for all $\beta < \alpha$, β is defined in the model such that the smallest ordinal γ such that $v(\beta \epsilon M_\gamma) = 1$ is $\beta + 1$. Let α be $\{z' \epsilon M_\alpha \cup \{k\}/(z'$ is an ordinal$)_{M_\alpha \cup \{k\}}\}$. Since '$x$ is an ordinal' is absolute and $M_\alpha \cup \{k\}$ is transitive then α is the set of all ordinals in $M_\alpha \cup \{k\}$ and is hence the required limit ordinal. Clearly $v(\alpha \epsilon M_{\alpha+1}) = 1$. If $v(\alpha \epsilon M_\alpha) = 1$, then $v(\alpha \epsilon M_\beta) = 1$ for some $\beta < \alpha$. Since $v(\beta \epsilon \alpha) = 1$ then $v(\beta \epsilon M_\beta) = 1$, which is a contradiction. Hence $\alpha + 1$ is the smallest ordinal such that $v(\alpha \epsilon M_{\alpha+1}) = 1$.

Hence all the ordinals can be defined in the model satisfying the properties of the ordinals and such that the smallest ordinal β such that $v(\alpha \epsilon M_\beta) = 1$ is $\alpha + 1$, for all the ordinals α. Hence the ordinals α defined in the model are in one-one correspondence with the M_α's of the model.

To show the validity of Axiom C, in the model, we must show that the M_α's of the formal theory, when interpreted in the model, have the same members as the M_α's of the model. This is shown by transfinite induction on the ordinals, the one-one correspondence above dispelling any ambiguity between the ordinals defined in the model and the ordinals used to construct the model.

Clearly M_0 of the model can be taken as a member of D^U with the same members as that of the formally defined M_0, interpreted in the model. Assume that the same holds for M_α. $M_{\alpha+1}$ is formally defined as the union of M_α and the set of all sets x such that there is a predicate A, which is significant for all substitutions into its free variables, and $x = \{z' \epsilon M_\alpha \cup I/ A_{M_\alpha \cup I}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$, where $\overline{w}'_i \epsilon M_\alpha \cup I$, for all i. Since $M_\alpha \cup \{k\}$ can be taken as $M_\alpha \cup I$, interpreted in the model, $\{z' \epsilon M_\alpha \cup \{k\}/A_{M_\alpha \cup \{k\}}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$ can be taken as $\{z' \epsilon M_\alpha \cup I/A_{M_0 \cup I}(z', \overline{w}'_1, \ldots, \overline{w}'_l)\}$, interpreted in the model. Hence $M_{\alpha+1}$ of the model can be taken as the formal $M_{\alpha+1}$, interpreted in the model. If α is a limit ordinal and the above property holds for all $\beta < \alpha$, then the M_α of the model, satisfying the property of being the union of all the M_α's such that $\beta < \alpha$, can be taken as the formal M_α, interpreted in the model.

Since there is an α such that $v(x \epsilon M_\alpha) = 1$, for all $x \epsilon D^U$, $(\mathbf{A}x)(\mathbf{S}\alpha)(x \epsilon M_\alpha)$ is valid in the model. Hence Axiom C is valid in the model.

The next step is to show that the Axiom of Choice (A.C.) is valid in the model, using Axiom C. There are various equivalents of the Axiom of Choice, which can be shown by the methods in Mendelson, [8], pp. 197-199, with little or no modification to allow for individuals. One of these equivalents is the Well-Ordering Principle: $(\mathbf{A}x)(\mathbf{S}y)(yWex)$, and so it is sufficient to prove it. This proof follows that in Cohen, [2], p. 95.

Lemma 5 *There is a wff $A(z, w, M_\alpha \cup \{k\}, y)$ such that if y is a well-ordering of the set $M_\alpha \cup \{k\}$, the relation $z < w = A(z, w, M_\alpha \cup \{k\}, y)$ induces a well-ordering of the set $M_{\alpha+1} \cup \{k\}$, where A is significant for all substitutions into its free variables.*

Proof: Enumerate the countably many formulas $B_n(x', t_1', \ldots, t_k')$. We have already essentially shown how to express the relation $C(z, n, t_1', \ldots, t_k')$: $z = \{z' \epsilon M_\alpha \cup \{k\}/(B_n)_{M_\alpha \cup \{k\}}(z', t_1', \ldots, t_k')$. Now the well-ordering y induces a natural well-ordering on the set of all possible $(k+1)$-tuples $\langle n, t_1', \ldots, t_k'\rangle$ where $t_i' \epsilon M_\alpha \cup \{k\}$, for all i. For each $z \epsilon M_{\alpha+1}$ we can define $\phi(z)$ as the first $(k+1)$-tuple, for some k, under this well-ordering, such that $C(z, \phi(z), t_1', \ldots, t_k')$ holds. Now we can define A by having $z < w$ mean $\phi(z) < \phi(w)$. One can easily add k at the beginning of the well-ordering so that k is the first member of $M_{\alpha+1} \cup \{k\}$. Thus $M_{\alpha+1} \cup \{k\}$ can be well-ordered.

By transfinite induction, we can define a well-ordering on $M_\alpha \cup \{k\}$ as follows: $M_0 \cup \{k\}$ is $\{k, \{k\}\}$ and so can be well-ordered. If α is a limit ordinal and the well-ordering has been defined for all $M_\beta \cup \{k\}$ with $\beta < \alpha$, we well-order $M_\alpha \cup \{k\} = \underset{\beta<\alpha}{U} M_\beta \cup \{k\}$ in an obvious manner. By Lemma 5, if $M_\alpha \cup \{k\}$ can be well-ordered then $M_{\alpha+1} \cup \{k\}$ can be well-ordered, and so $M_\alpha \cup \{k\}$ can be well-ordered for all α. Since Axiom C is valid in the model, let $\phi(x)$ be the least ordinal α such that $v(x \epsilon M_\alpha) = 1$. Define $x < y$ if $\phi(x) < \phi(y)$ or if $\phi(x) = \phi(y) = \alpha$ and x precedes y in the well-ordering of $M_\alpha \cup \{k\}$. Thus we have given a single formula $A(x, y)$ which well-orders all sets. Hence Axiom A.C. is valid in the model.

The next step is to show that Axiom GCH is valid in the model, using Axioms C and A.C. The proof follows that in Cohen, [2], pp. 95-98 and 82-83. Instead of using ranks in the Skolem-Lowenheim Thereom on p. 82, use the least ordinal α such that $x \epsilon M_\alpha$. This does the required job of restricting the Axiom of Choice to sets and so the theorem follows similarly to the proof of the validity of the Axiom of Replacement in the model. One does, of course, only need to consider formulas $A(x_1', \ldots, x_n')$ containing only the connectives $\sim$, &, and T and with its quantifiers, $\mathbf{A}$ and $\mathbf{E}$, at the beginning of the formula.

Lemma 6 *For all infinite α, $\overline{\overline{M_\alpha}} = \overline{\overline{\alpha}}$, in the model.*

Proof: $\overline{\overline{M_n}}$ is finite, for all integers n. $\overline{\overline{M_n}} \geq n$ since $\alpha \epsilon M_{\alpha+1}$, for all ordinals α. Hence $\overline{\overline{M_\omega}} = \aleph_0 = \overline{\overline{\omega}}$. If α is a successor ordinal and $\overline{\overline{M_\beta}} = \overline{\overline{\beta}}$ for all $\beta \leq \alpha - 1$, then the number of predicates $A_{M_{\alpha-1}} \cup \{k\}$ is $\overline{\overline{M_{\alpha-1}}}$ and hence $\overline{\overline{M_\alpha}} = \overline{\overline{M_{\alpha-1}}} = \overline{\overline{\alpha - 1}} = \overline{\overline{\alpha}}$. If α is a limit ordinal and $\overline{\overline{M_\beta}} = \overline{\overline{\beta}}$ for all $\beta < \alpha$, then $\overline{\overline{M_\alpha}} = \overline{\overline{\underset{\beta<\alpha}{U} M_\beta}} \leq \overline{\overline{\alpha}} \times \overline{\overline{\alpha}} = \overline{\overline{\alpha}}$. Since $\beta \epsilon M_{\beta+1}$, $\overline{\overline{M_\alpha}} \geq \overline{\overline{\alpha}}$ and hence $\overline{\overline{M_\alpha}} = \overline{\overline{\alpha}}$. Hence, for all infinite α, $\overline{\overline{M_\alpha}} = \overline{\overline{\alpha}}$ in the model.

Thus, in the model, Lemma 1 of [2], p. 96, follows, where a set x is extensional if y and $z \epsilon x$ and $\sim(y = z)$ implies $(\mathbf{S}x')(x' \epsilon x \;\&.\; (x' \epsilon y \;\& \sim x' \epsilon z) \vee (x' \epsilon z \;\& \sim x' \epsilon y))$. In the theorem on p. 73, [2], concerning the unique one-one map ϕ from an extensional set to a transitive set, let the rank of x be the least α such that $x \epsilon M_\alpha$, and if k is an individual $\epsilon\, A$ then let $\phi(k) = k$ and if rank $(x) = 0$ then let $\phi(x) = x = \{k\}$. In the proof of ϕ being one-one, where $\alpha = \max(\text{rank } X', \text{rank } Y') = 0, X' = Y' = \{k\}$, since the proof is being carried out in the model. If X' and Y' are individuals then it is not the case that $\sim(X' = Y')$. If X' is an individual and Y' is a set then $\sim T(\phi(X') = \phi(Y'))$. So the one-one condition is: $T(X' = Y')$ iff $T(\phi(X') = \phi(Y'))$. The rest of the proof follows as in Cohen, [2], and we can use the unique ϵ-isomorphism in the proof of Theorem 1 in [2], pp. 95-97. The next result we need is the absoluteness of '$x' \epsilon M_\alpha$'. Since the proof is being done in the model, if $k \epsilon T$ and $l \circ k$ then $l \epsilon T$ for any transitive set T. Hence we can show that '$x' = y'$' is absolute and use this to show the absoluteness of '$x' \epsilon M_\alpha$', following through the steps in Cohen [2], p. 94, and using my formal definition of the M_α's. Now Theorem 1 ([2], p. 95-7) will follow. The Axiom GCH can now be shown to be valid in the model by the proof at the bottom of [2], p. 98.

Hence all the axioms are valid in the model and the formal system is consistent relative to the theory needed to set up the model. **NBG**, with individuals added in the style of **ZF**, is sufficient to do this, the D_α's being sets of expressions and D^U, D^0, D^1, . . ., D^n, . . ., D^S, all being proper classes of expressions. These expressions are treated as individuals and sets and classes are formed from them, and so the formal system is consistent relative an applied **NBG** set theory.

This leaves a number of questions unanswered. We have not proved formally that Axiom C implies Axiom AC. This however looks very doubtful because Axiom C does not say anything about the well-ordering of the set of all individuals, I. However, if an extra axiom, call it WOI, was added which ensured the possibility of well-ordering the set of all individuals, then it seems likely that Axiom AC would follow.

However, Axiom C formally implies Axiom D.

It also seems likely that Axiom C, together with Axiom WOI, formally implies Axiom GCH. To prove this, it seems, involves dispensing with the individuals altogether in the normal proof of Axiom GCH from Axiom C because they affect the cardinalities in the form of Axiom GCH. It seems the result can be proved by building up a transfinite sequence of N_α's, similar to the M_α's, but with $N_0 = 0$ instead of $\{\{k\}, \{1\}, \text{etc.}\}$, and so the individuals are excluded completely from the construction. Then show that the Axiom GCH holds for the sets belonging to the class $\bigcup\limits_\alpha N_\alpha$ and, using the Axiom AC, that to each set belonging to $\bigcup\limits_\alpha M_\alpha$ there is a set belonging to $\bigcup\limits_\alpha N_\alpha$ with the same cardinality.

However, Axiom GCH formally implies Axiom AC.

The question that now arises is that of the independence of the Axioms

C, GCH and AC. It seems likely that these can be shown by forming *inner* models along similar lines to those of Cohen in [2]. As well as the "generic" sets that Cohen uses, one also needs the generic sets, $\{k\}$, $\{l\}$, etc., for all the individuals, and at each stage in the transfinite construction one should form subsets of $M_\alpha \cup I$, as in the model N of this section.

One could also add to the formal system ordinary language predicates so that these can be used to generate classes. The addition of these does not affect the consistency proof nor the development of the formal theory if they are introduced by adding general predicate variables and general subject variables to the formal theory. For the purpose of proving consistency one can specialise the predicate variables to those concerning membership and overlapping of classes and individuals. In the development of the formal theory, whenever a wff-schema appears, as in forms of the Abstraction Axiom and Axiom of Replacement, a general predicate variable can be substituted. This would then allow one to apply the formal theory in order to generate classes from ordinary language predicates.

Appendix

Theorem *If A is a wff of* **S5** (see [1]) *containing only the connectives* $\sim$, &, $\vee$, T, *and the quantifiers* $\mathbf{A}$, $\mathbf{S}$, *then there is a wff A' of* **S5** *such that* $A \simeq A'$ *and A' has all of its quantifiers,* $\mathbf{A}, \mathbf{S}$, $\forall$, $\mathbf{E}$[6] *at the beginning of the formula.*

Proof: The following are valid and hence provable in **S5**:

$\sim(\mathbf{A}x)A \simeq (\mathbf{E}x)\sim A$.
$\sim(\mathbf{S}x)A \simeq (\forall x)\sim A$.
$\sim(\forall x)A \simeq (\mathbf{S}x)\sim A$.
$\sim(\mathbf{E}x)A \simeq (\mathbf{A}x)\sim A$.
$(\mathbf{A}x)A \mathbin{\&} B \simeq (\mathbf{A}x)(A \mathbin{\&} B)$, where x is not free in B.
$(\mathbf{S}x)A \mathbin{\&} B \simeq (\mathbf{S}x)(A \mathbin{\&} B)$, where x is not free in B.
$(\forall x)A \mathbin{\&} B \simeq (\forall x)(A \mathbin{\&} B)$, where x is not free in B.
$(\mathbf{E}x)A \mathbin{\&} B \simeq (\mathbf{E}x)(A \mathbin{\&} B)$, where x is not free in B.
$(\mathbf{A}x)A \vee B \simeq (\mathbf{A}x)(A \vee B)$, where x is not free in B.
$(\mathbf{S}x)A \vee B \simeq (\mathbf{S}x)(A \vee B)$, where x is not free in B.
$(\forall x)A(x) \vee B \simeq (\forall x)(\mathbf{S}y)(\mathbf{S}z)(\mathbf{A}w) \sim(\sim(A(x) \vee B) \mathbin{\&} \sim(T \sim B \mathbin{\&} \sim SA(y) \mathbin{\&} TA(z) \mathbin{\&} \sim T \sim A(w)))$, where x is not free in B.
$(\mathbf{E}x)A(x) \vee B \simeq (\mathbf{E}x)(\mathbf{A}y)(\mathbf{A}z)((A(x) \vee B) \mathbin{\&} \sim(T \sim B \mathbin{\&} \sim SA(y) \mathbin{\&} TA(z)))$, where x is not free in B.
$T(\mathbf{A}x)A \simeq (\mathbf{A}x)TA$.
$T(\mathbf{S}x)A \simeq (\mathbf{S}x)TA$.
$T(\forall x)A(x) \simeq (\mathbf{S}x)(\mathbf{A}y)(TA(x) \mathbin{\&} \sim T \sim A(y))$.
$T(\mathbf{E}x)A(x) \simeq (\mathbf{S}x)(\mathbf{A}y)(TA(x) \mathbin{\&} SA(y))$.

Applying these equivalences to each connective in turn, one can construct an A' such that $A' \simeq A$ and the quantifiers of A' are in front of a formula containing connectives only.

NOTES

1. See [3], [11], [5], [4] and [1] for accounts of nonsignificance and of 3-valued significance logic.
2. See Leonard and Goodman [7], pp. 47-8.
3. The connectives and quantifiers are those of the system **S5** from [1]. These are given as follows:

$\sim$	
1	0
0	1
n	n

$\supset$	1	0	n
1	1	0	n
0	1	1	1
n	1	1	1

T_n	
1	1
0	n
n	n

('1' denotes truth, '0' denotes falsity, and 'n' denotes nonsignificance)

$(\mathsf{A}X')\phi(X')$ is true if $\phi(X')$ is true for all X'.
$(\mathsf{A}X')\phi(X')$ is nonsignificant if $\phi(X')$ is nonsignificant for some X'.
$(\mathsf{A}X')\phi(X')$ is false, otherwise.
$(\mathsf{S}X')\phi(X')$ is true if $\phi(X')$ is true for some X'.
$(\mathsf{S}X')\phi(X')$ is nonsignificant if $\phi(X')$ is nonsignificant for all X'.
$(\mathsf{S}X')\phi(X')$ is false, otherwise.

Other connectives that are used in the sequel are as follows:

&	1	0	n
1	1	0	n
0	0	0	n
n	n	n	n

$\vee$	1	0	n
1	1	1	1
0	1	0	0
n	1	0	n

$\equiv$	1	0	n
1	1	0	n
0	0	1	1
n	n	1	1

T	
1	1
0	0
n	0

F	
1	0
0	1
n	0

S	
1	1
0	1
n	0

$\dot{\supset}$	1	0	n
1	1	0	n
0	1	1	n
n	n	n	n

$\doteq$	1	0	n
1	1	0	n
0	0	1	n
n	n	n	n

$\simeq$	1	0	n
1	1	0	0
0	0	1	0
n	0	0	1

The definitions of these in terms of $\sim$, $\supset$, and T_n can be found in [1].

4. Such a theory is developed in my Ph.D. thesis, *A 4-valued Theory of Classes and Individuals.*
5. Fuller details can be found in my thesis, in Chapter 4.
6. In the significance logic **S5** in [1], $(\forall x)A =_{df} \sim(\mathsf{S}x)\sim A$ and $(\mathsf{E}x)A =_{df} \sim(\mathsf{A}x)\sim A$.

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