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ARISTOTLE'S SYLLOGISTIC

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I present a syllogistic system simpler and truer to the Aristotelian text than that of [8].

1 *Primitive symbols* The Greek capitals:

$$AB\Gamma\Delta EZH\Theta KMN\Xi\Pi P\Sigma$$

(possibly with numerical subscripts) are *term-variables*. The mnemonic vowels:

aeio

are *functors* which when superscripted to a pair of term-variables form a *protasis*:

AB^a (A belongs to all B) $\quad$ AB^e (A belongs to no B)
AB^i (A belongs to some B) $\quad$ AB^o (A does not belong to some B).

The first variable in a protasis is its *predicate*, the second its *subject*. The a- and e-protases with the same variables in the same order are *contraries* ([3], B8, 59b8-11; B15, 63b23-30). The a- and o-protases (also e- and i-protases) with the same variables in the same order are *contradictories* ([3], *ibid.*).

2 *Formation rules* The theses of the system take such forms as:

If A belongs to some B, B belongs necessarily to some A ([3], A2, 25a20-21).

If A belongs to no B and B belongs to some Γ, necessarily A does not belong to some Γ ([3], A4, 26a25-27).

The one remaining primitive symbol, then, of Aristotle's syllogistic is a connective:

If . . . then necessarily ___

which joins a number of antecedent protases to a consequent one. (If there

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is more than one antecedent, grammar will require that they be linked by the word "and"; so our connective will contain an indeterminate number of "and"s, possibly zero.) I represent this connective by means of a horizontal line:

$$\overline{}$$

which I call the *syllogistic sign*. This connective is of variable polyadicity, always having just one consequent, but having any number of antecedents greater than zero. A *wff* is then defined as consisting of a syllogistic sign with a (non-null) sequence of protases above it (these being called its *premisses*), and just one protasis below it (its *conclusion*).

Among wffs a special place is held by those that *interweave*. I say that a wff interweaves iff the protases occurring in it can be so ordered that, for any one of them, one of its variables occurs in its successor and the other in its predecessor; the *successor* of the j-th protasis being the $(j + 1)$-th and the first being the successor of the last; and the *predecessor* of the j-th being the $(j - 1)$-th and the last being the predecessor of the first.

A pair of protases sharing a variable is said to be in the first, second or third *figure*, depending on whether the shared variable is (I) once subject and once predicate, or (II) twice predicate, or (III) twice subject ([3], A23, 41a13-18). The figure of a 2-premissed wff is the same as that of its premiss-pair. 2-premissed wffs interweave iff they fall into one of these figures. Thus a third figure wff:

$$\frac{\Pi\Sigma \quad P\Sigma}{\Pi P}$$

interweaves because its protases can be ordered $\langle \Pi\Sigma, P\Sigma, \Pi P\rangle$.

3 *Substitution* In [3], A1-2 and A4-7, Aristotle regularly uses variables $AB\Gamma$ for Figure I, $MN\Xi$ for Figure II, and $\Pi\Sigma P$ for Figure III. But the possibility of 'reducing' syllogisms in one figure to syllogisms in another (as outlined in [3], A45) implies that syllogisms with any of these sets of variables will occur in each of the figures. So the choice of variables is logically immaterial.

Indeed, the text of the *Analytics* abounds in examples of different sets of variables being used as variants of one another. For instance the wff Celarent:

$$\frac{AB^e \quad B\Gamma^a}{A\Gamma^e}$$

is first formulated in $AB\Gamma$ ([3], A4, 25b40-a2), but variants occur in $NM\Xi$ ([3], A5, 27a7-8), ΞMN ([3], A5, 27a11-12), $A\Delta B$ ([4], A15, 79b2-4) and $A\Gamma B$ ([4], A16, 80a13). Thus it is clear that Aristotle accepted (though he did not formulate) the rule of Substitution:

U* Any alphabetic variant (in the sense of [6], p. 86) of a thesis is a thesis.

This rule licenses all the substitutions that Aristotle makes in the exposition of his system, but not such ones as the identification of Γ with B in the above example of Celarent, to yield:

$$\frac{AB^e \ BB^a}{AB^e}$$

which is typical of what Abelard called "*syllogismi ridiculosi*" ([1], *Tractatus* II, *liber* iii, 232:26). Occasionally Aristotle does indeed identify distinct variables:

> When A's being necessitates B's, and B's Γ's, A's being will necessitate Γ's. If then A's being necessitates B's, and B's A's . . ., A may be put in the place of Γ . . .. Consequently A's being necessitates that A is. ([4], A3, 72b37-a5)

But he seems to find this process strange. It is more characteristic of him to retain a wff with three distinct term-variables:

$$\frac{\Gamma A^e \ BA^a}{\Gamma B^o}$$

even to cover cases where there are only two distinct terms, such as "If all medicine is science and no medicine is science then some science is not science" ([3], B15, 64a22-27). For this reason I have adopted U^* rather than a full rule of Substitution permitting the identification of distinct variables.

4 *Permutation* Just as with his standard but not invariant variable-sequences for wffs, Aristotle has a standard but flexible premiss-order, with the premiss containing the predicate of the conclusion coming first. The extent to which this order is flexible will be clear from the fact that four of the six theses in Figure III are formulated in the 'wrong' order in [3], A6 (28a26-29; 28b7-11; 28b11-14; 28b17-20).

Further, as Łukasiewicz ([8], p. 34) and Rose ([9], ch. x) have noted, Aristotle sometimes tacitly permutes theses, stating their premisses at one time in one order, at another in another. It is clear then that he accepted (but did not formulate) the rule of Permutation:

M Let p be a protasis and Q, R be sequences of protases which do not differ otherwise than in the ordering of their elements. Then, if $\frac{Q}{p}$ is a thesis so is $\frac{R}{p}$.

I abbreviate this:

$$\frac{Q}{p} \rightarrow \frac{R}{p} \text{ (under the stated conditions)}$$

(The letter "*m*" was used in the medieval mnemonics to indicate the need for this rule in 'reducing' a wff to Figure I.)

5 *Cut* The task of proving that a wff $\frac{Q}{p}$ is a thesis may be approached in two ways. One may try to find wffs which are already known to be theses, and transformation-rules, such that those theses imply the desired thesis in accordance with the transformation-rules. Alternatively, one may try to deduce p from the members of Q by interpolating one or more 'middles' between the members of Q and p in such a way that each step in the deduction is an already-established thesis. In the second case it would be natural to talk of deriving p from the members of Q considered as assumptions or hypotheses; but one would not in fact be assuming the members of Q in any substantive sense—only showing that *if* they are assumed then p can be inferred. This second approach is common in [3]:

> Let M belong to no N, but to all Ξ. Since the negative is convertible N will belong to no M. But M was supposed to belong to all Ξ. Consequently N will belong to no Ξ. This has already been shown. (A5, 27a5-9)

In order to prove that $\dfrac{MN^e\ M\Xi^a}{N\Xi^e}$ is a thesis, Aristotle 'assumes' its premisses; from the first he derives NM^e $\left(\text{for } \dfrac{MN^e}{NM^e} \text{ is an already-established thesis}\right)$; then from this together with $M\Xi^a$ he derives $N\Xi^e$ by the previously established thesis $\dfrac{NM^e\ M\Xi^a}{N\Xi^e}$ (which is what he describes as having already been shown).

A proof like this—a Direct Reduction—can always be changed into a proof of the first type. The known theses, from which the proof starts, will be the individual steps in the derivation; and the transformation-rule will be something like:

> Let p, q, r be protases and Q, R be (possibly null) sequences of protases.
> Then if $\dfrac{pQ}{q}$ and $\dfrac{qR}{r}$ are theses so is $\dfrac{pQR}{r}$.

More precisely, Aristotle uses this rule only when $\langle p, Q\rangle$ and $\langle R, r\rangle$ share no variables not in q. With the stated proviso, the rule may be abbreviated thus:

$$T_1^*:\ \left.\begin{array}{c}\dfrac{pQ}{q}\\[2ex] \dfrac{qR}{r}\end{array}\right\}\rightarrow \dfrac{pQR}{r} \text{ (under the stated conditions).}$$

This is a restricted version of Gentzen's rule of Cut ([7], p. 31). The reason for the restriction is that without it one would be able to derive theses essentially similar to Abelard's "*syllogismi ridiculosi*":

$$\left.\begin{array}{c}\dfrac{AB^a\ B\Gamma^a}{A\Gamma^a}\\[2ex] \dfrac{A\Gamma^a\ \Gamma B^a}{AB^a}\end{array}\right\}\rightarrow \dfrac{AB^a\ B\Gamma^a\ \Gamma B^a}{AB^a}$$

The essential feature of such un-Aristotelian wffs is that they can be got by identifying distinct variables in a wff that interweaves, in this case the wff:

$$\frac{AB^a \quad B\Gamma^a \quad \Gamma\Delta^a}{A\Delta^a}.$$

Now, Aristotle's proof can be represented as a derivation of $N\Xi^a$ from the 'assumptions' MN^e and $M\Xi^a$:

$$\dfrac{\dfrac{MN^e}{NM^e} \quad M\Xi^a}{N\Xi^e}.$$

But it can also be represented as a proof of the wff $\frac{MN^e \quad M\Xi^a}{N\Xi^e}$ using the rule T_1^* (with null Q):

$$\left.\begin{array}{l}\dfrac{MN^e}{NM^e} \\ \dfrac{NM^e \quad M\Xi^a}{N\Xi^e}\end{array}\right\} \xrightarrow{T_1^*} \frac{MN^e \quad M\Xi^a}{N\Xi^e}.$$

The first way of putting it fits better with Aristotle's talk of 'reducing' the wff to be proved to the first figure. But this is just a matter of how we set the proof out. Either way it relies on rule T_1^*—as do all Direct Reductions.

When neither Q nor R is null T_1^* is what permits the generation of a many-premissed wff from a number of 2-premissed ones, as in [3], A23, 41a18-20 and A25, 42a6-8. When just one of them is null the rule permits the Direct Reduction of one wff to another, including the case of syllogisms with 'weakened' conclusions, as in [3], B1, 53a3-14. When both Q and R are null the rule states the transitivity of implication, which is explicitly maintained by Aristotle in [3], B4, 57b6-9. So there seems to be nothing in it that is not safely attributable to him.

6 *Indirect Reduction* His other method of 'reducing' one wff to another—Indirect Reduction—relies on the rules:

$$C:\ \frac{pQ}{q} \to \frac{\sim qQ}{\sim p} \qquad\qquad K:\ \frac{pQ}{q} \to \frac{\urcorner qQ}{\sim p}$$

(where p, q are protases; $\sim p$, $\sim q$ being their contradictories; $\urcorner q$ being the contrary of q, if there is one; and Q being a possibly null sequence of protases).

Rule C, for non-null Q, is stated by Aristotle when he says that if the conclusion of a syllogism is 'transposed' (*metatithenta*), i.e., denied, and one of the premisses retained, the remaining premiss must be abandoned ([3], B8, 59b1-5). It is also stated for null Q (in which case it expresses the law of Transposition for implication):

> . . . if A's being necessitates B's being, B's not being necessitates A's not being ([3], B2, 53b12-13; B4, 57b1-2).

It is rule C that is being relied on in the proof:

> If P belongs to all Σ and Π does not belong to some Σ it is necessary that Π does not belong to some P. For if Π belongs to all P and P to all Σ, Π will belong to all Σ; but we supposed that it did not belong to it. ([3], A6, 28b17-20)

Rule K is not stated by Aristotle; but it is used in the proof:

> . . . if A and B belong to all Γ it results that A belongs to some B. For if A belongs to no B and B belongs to all Γ, A will belong to no Γ; but we said that it belonged to all Γ. ([3], A7, 29a37-39)

$$\frac{AB^e \ \ B\Gamma^a}{A\Gamma^e} \rightarrow \frac{A\Gamma^a \ \ B\Gamma^a}{AB^i}.$$

Moreover, K follows from C given that $\neg q$ always implies $\sim q$; but this implication just summarizes the laws of Subalternation, which are stated, for instance, at [3], A4, 26b14-15 and A5, 27b21-22. (The letter "c" was used in the medieval mnemonics to indicate the need for C in 'reducing' a wff to Figure I.)

Aristotle calls Indirect Reductions proofs *per impossibile* of a syllogism. Łukasiewicz writes in [8], p. 55, as if he finds this use of the expression "*per impossibile*" unwarranted. But it is connected to the central use of the expression in the following way: anyone who argues *per impossibile* for the conclusion $\sim p$ by 'supposing' p can turn his 'indirect' argument into a 'direct' one by applying rule C or rule K to it.

7 *Axioms* The process of Direct Reduction requires 1-premissed theses in addition to the rule T_1^*. The basic one of these is:

e-conversion $\dfrac{AB^e}{BA^e}$ ([3], A2, 25a15-16).

The two theses $\dfrac{AB^e}{AB^e}$ and $\dfrac{AB^i}{AB^i}$ follow from e-conversion by U^*, T_1^* and C:

$$\left.\begin{array}{l} (e\text{-conversion}) \ \dfrac{AB^e}{BA^e} \\ \dfrac{AB^e}{BA^e} \xrightarrow{U^*} \dfrac{BA^e}{AB^e} \end{array}\right\} \xrightarrow{T_1^*} \frac{AB^e}{AB^e} \xrightarrow{C} \frac{AB^i}{AB^i}.$$

Now, since Aristotelian methods deliver up theses according to which both e- and i-protases imply themselves, it seems reasonable to adopt the corresponding axiom for a-forms:

Repetition $\dfrac{AB^a}{AB^a}$ $\left(\text{whence } \dfrac{AB^o}{AB^o} \text{ follows by } C\right)$.

The laws of Subalternation follow from Repetition by rules K and C.

To these two axioms, following Aristotle, we add two more:

Barbara $\dfrac{AB^a \ \ B\Gamma^a}{A\Gamma^a}$ ([3], A4, 25b37-39)

Celarent $\dfrac{AB^e \ \ B\Gamma^a}{A\Gamma^e}$ ([3], A4, 25b40-a2).

All theses of the Aristotelian syllogistic can be derived from these four axioms by means of rules U^*, M, C, K, T_1^*.

8 *Łukasiewicz's syllogistic* This system is simpler than Łukasiewicz's, in that it contains fewer kinds of primitive symbol, lacking propositional variables and propositional negation. Analogues of these occur only in our metalanguage. His system is also more complicated than the one described here, in having 14 more axioms, drawn from propositional logic ([8], p. 89).

It is true that, against our five transformation-rules, he has only four (a full rule of Substitution, the rule of Detachment, and two rules allowing the intersubstitutability of negative protases with the negations of their contradictories, *ibid.*, p. 88). But this economy produces complications in the construction of proofs. For example, transformation-rules are applied 26 times in his proof of Cesare (pp. 91-92), but three times in ours:

$$\left.\begin{array}{r}(e\text{-conversion})\ \dfrac{AB^e}{BA^e} \xrightarrow{U*} \dfrac{MN^e}{NM^e} \\ (\text{Celarent})\ \dfrac{AB^e\ \ B\Gamma^a}{A\Gamma^e} \xrightarrow{U*} \dfrac{NM^e\ \ M\Xi^a}{N\Xi^e}\end{array}\right\} \xrightarrow{T_1^*} \dfrac{MN^e\ \ M\Xi^a}{N\Xi^e}\ (\text{Cesare})\ .$$

Our proofs, moreover, are based directly on Aristotle's own. Like Aristotle, we derive i-conversion from e-conversion ([3], A2, 25a20-22); and we derive all second and third figure syllogisms from Barbara and Celarent along with the laws of conversion, using the processes of Direct and Indirect Reduction. But Łukasiewicz ([8], p. 88) uses Datisi in his proof of i-conversion, Datisi and not Celarent being axiomatic.

Łukasiewicz's system is untrue to the Aristotelian text in other ways too. He includes truth-functional conjunction in the system, as a defined connective ([8], p. 88), because he assumes that the form "If p and q then r" has to be understood as an implication with a conjunctive antecedent ([8], p. 20). But there is in the *Analytics* no logic of conjunction. And the above form does not have to be understood in Łukasiewicz's way, but can be taken as a triadic implication, with one consequent and two antecedents. Moreover, this analysis is not new, being implicit in the following remarks of Walter Burleigh:

> . . . it is not in every valid consequence that the opposite of the antecedent follows from the opposite of the consequent, but only in non-syllogistic consequences. For in syllogistic consequences the antecedent does not have an opposite, because a syllogistic antecedent is several unconjoined propositions, and such an antecedent just does not have an opposite, because it is not one proposition simply nor one by conjunction. ([5], 207:31-208:3)

Aristotle, too, describes the antecedent of a syllogism as "several things" ([3], A1, 24b19), not as a conjunction of several things.

Another divergence of Łukasiewicz's from Aristotle's syllogistic is its reliance on laws of the propositional calculus. Aristotle's ignorance of many of these laws is indeed to be lamented, but the proofs in his system remain perfectly rigorous in spite of it.

Łukasiewicz's system even has theses containing term-variables that are not Aristotelian, for his rule of Substitution allows the identification of distinct variables, proving "*syllogismi ridiculosi*" as theses. And he includes two laws of Identity among his axioms ([8], p. 88). Now, while one passage ([3], B15, 64b7-9) implies that the contradictories of the forms AA^a and AA^i are unsatisfiable, Aristotle's systematic exposition of syllogistic in no way relies on them. They were formulated, in fact, not by Aristotle but by his commentator Alexander of Aphrodisias ([2], 34:15).

However, the inclusion of "*syllogismi ridiculosi*" and the laws of Identity is a relatively minor matter. We could easily extend our system to include both by adopting a full rule of Substitution and adding the axiom:

$$\frac{}{AA^a}\ .$$

This would involve the possibility of wffs with a null premiss sequence, so that Cut would have to be re-stated:

$$\left.\begin{array}{c} \dfrac{Q}{q} \\ \dfrac{qR}{r} \end{array}\right\} \longrightarrow \frac{QR}{r}\ .$$

And the axiom of Repetition would become redundant, following from Identity and Barbara:

$$\begin{array}{ll} \text{(Identity)}\ \dfrac{}{AA^a} & \\ \text{(Barbara)}\ \dfrac{AB^a\ \ B\Gamma^a}{A\Gamma^a} \xrightarrow{\text{Substitution}} \dfrac{AA^a\ \ AB^a}{AB^a} & \end{array} \Bigg\} \xrightarrow{\text{Cut}} \frac{AB^a}{AB^a}\ .$$

Now, even in this expanded system, every thesis interweaves, and thus every 2-premissed thesis is in one of the figures. But Łukasiewicz has 2-premissed theses that are not in a figure, e.g., $\dfrac{AB^a\ \ \Gamma\Delta^a}{AB^a}$ (or the analogue in his notation). This thesis is totally un-Aristotelian: the conclusion of a syllogism is required by definition to be other than the premisses ([3], A1, 24b19); and this thesis, unlike any of Aristotle's, is a palpable *petitio principii*.

Finally, a system containing non-interweaving theses cannot give an adequate account of the many syntactical metatheorems formulated by Aristotle in [3], e.g., at A24, 41b6; 41b6-23; 41b23-24; 41b27-31; B1; B5-17, for many of these metatheorems assume that all theses interweave.

REFERENCES

[1] Abaelardus, Petrus, *Dialectica*, ed. L. M. de Rijk, Van Gorcum, Assen, 1956.

[2] *Alexandri in Aristotelis Analyticorum Priorum Librum I Commentarium*, ed. M. Wallies, Berlin, 1883.

[3] Aristotle, *Prior Analytics*.

[4] Aristotle, *Posterior Analytics*.

[5] Burleigh, W., *De Puritate Artis Logicae Tractatus Longior, with a Revised Edition of the Tractatus Brevior*, ed. P. Boehner, St. Bonaventure, New York, 1955.

[6] Church, A., *Introduction to Mathematical Logic*, Vol. I, Princeton University Press, Princeton, New Jersey, 1956.

[7] *The Collected Papers of Gerhard Gentzen*, ed. M. E. Szabo, North-Holland, Amsterdam, 1969.

[8] Łukasiewicz, J., *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Second Edition, Oxford University Press, Oxford, 1957.

[9] Rose, L. E., *Aristotle's Syllogistic*, Springfield, Illinois, 1968.

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