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ON THE EQUIVALENCE OF SYSTEMS OF RULES AND SYSTEMS OF AXIOMS IN ILLATIVE COMBINATORY LOGIC

M. W. BUNDER

The most useful systems of illative combinatory logic contain the primitive Ξ for restricted generality with the rule:

Rule Ξ $\qquad\qquad$ $\Xi XY, XV \vdash YV,$

or alternatively the primitive **F** or the primitives **P** and Π with their appropriate rules.[1,2]

In addition they have for (combinatory) equality:

Rule Eq $\qquad\qquad$ *If* $X = Y$, *then* $X \vdash Y$,

and a deduction rule for Ξ (or for **F** or for **P** and Π) such as for example that of [1]:

DTΞ $\qquad\qquad$ *If* $\Delta, XV \vdash YV$

where Δ is any sequence of terms and V is not free in Δ, X or Y, then

$$\Delta, \mathbf{FAH}X \vdash \Xi XY.^3$$

Often however, such systems are set up using in addition to Rule Ξ and Rule Eq, a set of axioms, and the deduction rule is derived as a theorem.

The curious thing is that no system (other than some proved inconsistent) has been found in which these axioms are in turn derivable from the deduction theorem and the other rules. This is unfortunate particularly as investigations into consistency are simpler for systems of rules. The system of rules corresponding to Curry's system of axioms $\mathcal{F}_{22}$, for example, was proved $\mathbf{Q}$-consistent (see [8] and [9]), but this system is not equivalent to $\mathcal{F}_{22}$. In this paper we examine this problem by looking at a system strong enough for first order predicate calculus (that of [1]) and a generalization of this (that of [4]) which is strong enough for the development of a predicate logic of arbitrarily high order. We then suggest a weaker version of [4] to bring the goal of equivalence between the two kinds of systems nearer, but find that only a radical addition—the allowance of quantification over theorems as well as over various functions over

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theorems—allows us to reach the goal. Although this may seem to make the system excessively strong, it is in fact provably consistent. (This was shown in [6].)

1 *The first order theory* The system of [1] on which first order predicate calculus can be based (see [2]) consists of Rule Ξ, Rule Eq and the following axioms:

Axiom **Q** $\vdash \mathbf{WQ}X$ *whenever X is a primitive or X is an indeterminate*[4]

(We take the primitives to include at least **K**, **S**, Ξ, **A**, **H**, and **Q**).

Axiom 1 $\vdash \mathbf{WQ}x \supset_x . \mathbf{WQ}y \supset_y \mathbf{WQ}(xy)$[5]
Axiom 2 $\vdash \mathbf{FAH}x \supset_x \Xi xx$
Axiom 3 $\vdash \mathbf{FAH}x \supset_x (\mathbf{WQ}y \supset_y : xu \supset_u . yuv \supset_v xu)$
Axiom 4 $\vdash \mathbf{FAH}x \supset_x \therefore \mathbf{WQ}t \supset_t : (xu \supset_u yu(tu)) \supset_y .$
$(xu \supset_u (yuv \supset_v zuv)) \supset_z (xu \supset_u zu(tu)).$
Axiom 5 $\vdash \mathbf{FAH}x \supset_x \Xi x(\mathbf{WQ})$
Axiom 6 $\vdash \Xi \mathbf{IH}$
Axiom 7 $\vdash \mathbf{FAHH}$
Axiom 8 $\vdash \mathbf{FAH}x \supset_x . \mathbf{F}x\mathbf{H}y \supset_y \mathbf{H}(\Xi xy)$
Axiom 9 $\vdash \mathbf{FAHA}$

(DTΞ can be derived from Axioms **Q**, 2, 3, 4, 5, 6, and 7).

In the corresponding system of rules DTΞ replaces Axioms 2, 3, 4, and 5,

Rule **H** $X \vdash \mathbf{H}X,$

replaces Axiom 6, and

Rule **H**Ξ $\mathbf{FAH}X,\ \mathbf{F}X\mathbf{H}Y \vdash \mathbf{H}(\Xi XY),$

replaces Axiom 8. (A rule could also replace Axiom 1 if required.)

It is clear that unless we also have $\vdash \mathbf{FAH}(\mathbf{FAH})$ (which is not provable) none of the axioms is derivable from the rules. This extra axiom as well as Axiom 7 lead to inconsistency if we have $\mathbf{A} = \mathbf{WQ}$ (see [7]) and even without $\mathbf{A} = \mathbf{WQ}$ it leads to inconsistency if $\vdash \mathbf{H}(\mathbf{Q}XY)$ for all X and Y, is present (see [3]). However, even with these axioms we still cannot derive Axiom 6 from Rule **H** and DTΞ (as we do not have $\vdash \mathbf{FAHI}$) and more seriously we cannot derive Axiom 4 as we do not have:

$$\mathbf{FAH}X,\ \mathbf{WQT} \vdash \mathbf{FAH}(\lambda y . Xu \supset_u yu(\mathbf{T}u))$$

or

$$\mathbf{FAH}X,\ \mathbf{WQT},\ Xu \supset_u Yu(\mathbf{T}u) \vdash \mathbf{FAH}(\lambda z . Xu \supset_u . Yuv \supset_v zuv).$$

Also, we cannot derive Axiom 8. We therefore look to a stronger system, but preferably one where Axiom 7 and $\vdash \mathbf{FAH}(\mathbf{FAH})$ are not required.

2 *The higher order predicate calculus* The higher order predicate calculus of [4] allows us to quantify over **H** and **FAH**, but does not require these two axioms. It has Rules Ξ Eq, Axioms **Q**, 1, and 6 and replaces Axioms 2, 3, 4, 5, 7, and 9, and 8 by:

2H $\vdash \mathbf{F}U\mathbf{H}x \supset_x . \, xu \supset_u xu$,
3H $\vdash \mathbf{F}U\mathbf{H}x \supset_x . \, \mathbf{H}y \supset_y (y \supset . \, xu \supset_u y)$,
4H $\vdash \mathbf{F}U\mathbf{H}x \supset_x \therefore \mathbf{WQ}t \supset_t : (xu \supset_u yu(tu)) \supset_y .$
$(xu \supset_u . \, yuv \supset_v zuv) \supset_z (xu \supset_u zu(tu))$,
5H $\vdash \mathbf{F}U\mathbf{H}x \supset_x \Xi x(\mathbf{WQ})$
7H $\vdash \mathbf{F}U\mathbf{H}U$,
8H $\vdash \mathbf{F}U\mathbf{H}x \supset_x \mathbf{F}x\mathbf{H}y \supset_y \mathbf{H}(\Xi xy)$,

where $U \in \mathcal{U} = \{\mathbf{A}, \mathbf{H}, \mathbf{FAA}, \mathbf{FAH}, \mathbf{FHA}, \mathbf{FHH}, \mathbf{FA(FAA)}, \ldots\}$.

We then have as deduction theorems:

DTΞ' If $\Delta, XV \vdash YV$ where V is not free in X, Y, or Δ, then $\Delta, \mathbf{F}U\mathbf{H}X \vdash \Xi XY$ where $U \in \mathcal{U}$.

As for each $U \in \mathcal{U}$, $\mathbf{F}U\mathbf{H} \in \mathcal{U}$ we can quantify over U and over $\mathbf{F}U\mathbf{H}$ for each U in $\mathcal{U}$ and so we can prove Axioms **2H**, **3H**, **5H**, and **7H** using DTΞ', Axiom **Q** and Rule **H**. Axioms **4H**, 6, and **8H** remain unprovable. We note however, that for a higher order predicate calculus the full strength of [4] is not really required. For example we use DTΞ only when

(i) $X = \mathbf{KZ}$ and $Y = \mathbf{KT}$ to give the deduction theorem for **P** (implication),

and

(ii) when $X \in \mathcal{U}$ to allow generalization over individuals (**A**), propositions (**H**), predicates (**FAH**, **FHH**, **F(FAH)H**, . . .) and functions (**FAA**, **F(FAH)A**, . . .).

Thus appropriate deduction theorems would be:

DT**P** *If* $\Delta, X \vdash Y$, *then* $\Delta, \mathbf{H}X \vdash X \supset Y$

and

DTΞ'' *If* Δ, $UV \vdash YV$ *where* V *is not free in* Δ *or* Y *and* $U \in \mathcal{U}$,

then $\Delta \vdash \Xi UY$.

In the same way Rule Ξ can be restricted to:

Rule Ξ' $\qquad \Xi UX,\ UV \vdash XV$ for $U \in \mathcal{U}$

and

Rule **P** $\qquad X \supset Y,\ X \vdash Y$.

Amending the proof of DTΞ in [1] to one of DTΞ' based on Rules **P** and Ξ' we find that we require only the following axioms in addition to Axiom 6:

2R $\vdash Uu \supset_u Uu$
3R $\vdash \mathbf{H}x \supset_x : x \supset Uu \supset_u x$
4R $\vdash (Uu \supset_u V(tu)) \supset_t . \, (Uu \supset_u (Vv \supset_v zuv)) \supset_z (Uu \supset_u zu(tu))$

where $U, V \in \mathcal{U}$.

Similarly to prove DTP we require, besides Axiom 6:

2P $\vdash \mathbf{H}x \supset_x x \supset x$
3P $\vdash \mathbf{H}x \supset_x : \mathbf{H}y \supset_y . (y \supset . x \supset y)$
4P $\vdash \mathbf{H}x \supset_x : (x \supset y) \supset_y . (x \supset . y \supset z) \supset_z (x \supset z)$

Note that now Axiom **7H** becomes superfluous as does **8H** at this stage. However we want the following restricted forms of **8H** for the development of propositional and predicate calculus:

8R $\vdash \mathbf{F}U\mathbf{H}x \supset_x \mathbf{H}(\Xi Ux)$ *for* $U \in \mathcal{U}$
8P $\vdash \mathbf{H}x \supset_x . (x \supset \mathbf{H}y) \supset_y \mathbf{H}(x \supset y)$.

Note also that now that we no longer require **WQ** (or some other universal class) in the statement of any of the axioms, Axioms **Q**, 1 and a restricted version of Axiom 5**H**

5H′ $\vdash \Xi U(\mathbf{WQ})$ for $U \in \mathcal{U}$

become "optional extras".

In order now to prove **4R** using DTΞ'' we require $\lambda t(Uu \supset_u V(tu))$ which is $\mathbf{F}UV$ to be in $\mathcal{U}$ which is the case, but we also require $\lambda z . (Uu \supset_u (Vv \supset_v zuv)) = \mathbf{F}U(\mathbf{F}V\mathbf{I}) \in \mathcal{U}$ which is not the case, in fact, we require this even if we only want to prove using **8R** that the term given in **4R** is a proposition. We can make the term in **4R** a proposition if we alter **4R** to:

$$\vdash (Uu \supset_u V(tu)) \supset_t : (Uu \supset_u . Vv \supset_v \mathbf{H}(zuv)) \supset_z . (Uu \supset_u (Vv \supset_v zuv)) \supset (Uu \supset_u zu(tu)),$$

but then although this can be proved using DTΞ'' and DT**P**, it is no longer strong enough to prove DTΞ'' with 2**R** and 3**R**. We would require in addition:

$$Uu \supset_u (Vv \supset_v zuv) \vdash Uu \supset_u (Vv \supset_v \mathbf{H}(zuv)) \textit{ for all } U, V \in \mathcal{U}.$$

The axiom corresponding to this however, is not provable by DTΞ'' and Rule H and if this rule is added as a primitive rule further axioms are required for the proof of DTΞ'' which are not provable from DTΞ''. If we note further that $\vdash \Xi\mathbf{IH}$ is still not provable using DTΞ'' and Rule **H**, we have a clue to a solution. We can simply add **I** as well as **FAI**, **FIA**, **FHI**, **FIH**, **F(FAA)I** etc. to $\mathcal{U}$. We can then prove **4R** from DTΞ'' and Axiom 6 from DTΞ'' and Rule **H**.

We still have the problem of 4**P**, however. This is provable using DT**P**, 8**P**, and DTΞ', but not using DTΞ''. The only solution seems to be to weaken Axioms 4**P**, 8**P**, and DT**P** as follows:

DT**P′** *If* $\Delta, X \vdash Y$, *then* $\Delta, \mathbf{H}X, \mathbf{H}Y \vdash X \supset Y$
4**P′** $\vdash \mathbf{H}x \supset_x :: \mathbf{H}y \supset_y .: \mathbf{H}z \supset_z : (x \supset y) \supset . (x \supset (y \supset z)) \supset (x \supset z)$
8**P′** $\vdash \mathbf{H}x \supset_x . \mathbf{H}y \supset_y \mathbf{H}(x \supset y)$.

Axioms 2**P**, 3**P**, 4**P′**, and 8**P′** then clearly give DT**P** and DT**P′**, DTΞ'' and

HP $\mathbf{H}X, \mathbf{H}Y \vdash \mathbf{H}(X \supset Y)$

allow us to prove the four axioms.

e will now summarise the equivalent systems:

System of Axioms

;q and:

;' $\Xi UX,\ UT \vdash XT$

s $\vdash \mathbf{H}x \supset_x x \supset x$

$\vdash \mathbf{H}x \supset_x .\ \mathbf{H}y \supset_y (y \supset .\ x \supset y)$

$\vdash \mathbf{H}x \supset_x :: \mathbf{H}y \supset_y \therefore \mathbf{H}z \supset_z : (x \supset y) \supset .\ (x \supset y \supset z) \supset (x \supset z)$

$\vdash \mathbf{H}x \supset_x .\ \mathbf{H}y \supset_y \mathbf{H}(x \supset y)$

$\vdash\ x \supset_x \mathbf{H}x$

$\vdash Uu \supset_u Uu$

$\vdash \mathbf{H}x \supset_x (x \supset .\ Uu \supset_u x)$

$\vdash \mathbf{F}UVt \supset_t .\ (Uu \supset_u (Vv \supset_v zuv)) \supset ._z (Uu \supset_u zu(tu))$

$\vdash \mathbf{F}U\mathbf{H}x \supset_x \mathbf{H}(\Xi Ux)$

$U,\ V \in \mathcal{U}$

›ssibly Axioms **Q**, 1, and 5**H′** for **Q**).

System of Rules

Eq and Ξ' and

If $\Delta, X \vdash Y$, *then* $\Delta, \mathbf{H}X, \mathbf{H}Y \vdash X \supset Y,\ \mathbf{H}X, \mathbf{H}Y \vdash \mathbf{H}(X \supset Y)$.

If Δ, $Uu \vdash Xu$ *where u is not free in* Δ *or X, then* $\Delta \vdash \Xi UX$,

$\mathbf{F}U\mathbf{H}X \vdash \mathbf{H}(\Xi UX)$,

$X \vdash \mathbf{H}X$,

$\mathcal{U}$

ossibly Axiom **Q** and Rules for **Q** corresponding to Axioms 1 and 5**H′**

throughout $\mathcal{U} = \{\mathbf{A}, \mathbf{H}, \mathbf{I}, \mathbf{FAA}, \mathbf{FAI}, \mathbf{FAH}, \mathbf{FIA}, \ldots\}$

we want to have $\mathcal{U}$ formally within the system we require axioms s

$$\vdash \mathcal{U}\mathbf{A},\ \vdash \mathcal{U}\mathbf{H},\ \vdash \mathcal{U}\mathbf{I}$$

$$\vdash \mathcal{U}x \supset_x .\ \mathcal{U}y \supset_y \mathcal{U}(\mathbf{F}xy).$$

n have a finite set of axioms instead of an infinite set. (For example, s 2**R** becomes $\vdash \mathcal{U}x \supset_x .\ xu \supset_u xu$). Our problems, moreover, arise If we want to be able to quantify over $\mathcal{U}$ to prove our new (finite set ɔms we need as axioms:

$\vdash \mathcal{U}u \supset_u \mathcal{U}u$

$\vdash \mathbf{H}y \supset_y (y \supset .\ \mathcal{U}u \supset_u y)$

$\vdash \mathbf{F}\mathcal{U}Vt \supset_t \mathbf{F}_2\mathcal{U}VIz \supset_z (\mathcal{U}u \supset_u zu(tu))$

$\vdash \mathbf{F}\mathcal{U}\mathbf{H}x \supset_x \mathbf{H}(\Xi \mathcal{U}x)$

for $V \in \mathcal{U}$

which will give us a deduction theorem:

If $\Delta, \mathcal{U}x \vdash Vx$ *where* x *is not free in* Δ *and* V, *then* $\Delta \vdash \Xi\mathcal{U}v$.

However, this deduction theorem is now not strong enough to prove the last two of the above axioms and we have a situation similar to the one we struck earlier even if we assume $\vdash \mathcal{U}\mathcal{U}$.

NOTES

1. The systems whose equivalence is discussed in this paper are also called "natural deduction" and "Hilbert-type" systems respectively.

2. The results in this paper can easily be adapted to systems based on **F** and probably to one based on **P** and Π. **F** (functionality), **P** (implication) and Π (universal generality) are defined in terms of Ξ by:

   $\mathbf{F} \equiv \lambda x\lambda y\lambda z \cdot \Xi x(\mathbf{B}yz)$,
   $\mathbf{P} \equiv \lambda x\lambda y \cdot \Xi(\mathbf{K}x)(\mathbf{K}y)$

   and

   $\Pi = \lambda x \cdot \Xi(\mathbf{WQ})x$, where **Q** stands for equality.

3. As in [1], [2] and [4] **A** stands for the class of individuals and **H** for the class of propositions, **FAH** then represents the class of first order predicates.

4. The Axiom $\vdash \mathbf{WQ}X$ where X is indeterminate was left out in [1] but is in fact necessary there.

5. $Xu \supset_u Yu$ is an alternative notation for ΞXY, similarly $X \supset Y$ is an alternative for $\mathbf{P}XY$.

REFERENCES

[1] Bunder, M. W., "A deduction theorem for restricted generality," *Notre Dame Journal of Formal Logic*, vol. XIV (1973), pp. 341-346.

[2] Bunder, M. W., "Propositional and predicate calculuses based on combinatory logic," *Notre Dame Journal of Formal Logic*, vol. XV (1974), pp. 25-34.

[3] Bunder, M. W., "Equality in $\mathcal{F}_{21}^*$ with restricted subjects," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 24 (1978), pp. 125-127.

[4] Bunder, M. W., "Predicate calculus of arbitrarily high order," Not yet published.

[5] Bunder, M. W., "A one axiom set theory," Not yet published.

[6] Bunder, M. W., "The consistency of higher order predicate calculus and set theory based on combinatory logic," Not yet published.

[7] Bunder, M. W. and R. K. Meyer, "On the inconsistency of systems similar to $\mathcal{F}_{21}^*$," *The Journal of Symbolic Logic*, vol. 43 (1978), pp. 1-2.

[8] Curry, H. B., "The consistency of a system of combinatory restricted generality," *The Journal of Symbolic Logic*, vol. 38 (1973), pp. 489-492.

[9] Seldin, J. P., "The **Q**-consistency of $\mathcal{F}_{22}$," *Notre Dame Journal of Formal Logic*, vol. XVIII (1977), pp. 117-127.

The University of Wollongong
Wollongong, New South Wales, Australia