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ŁUKASIEWICZ'S TWIN POSSIBILITY FUNCTORS

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Jan Łukasiewicz, in his paper "A system of Modal logic",[1] introduces what he calls "the twin possibilities",[2] viz. the 'Δ' and '∇' functors. The existence of these two functors is, he claims, something of a logical paradox, for they are identical when defined apart, i.e., as parts of separate matrices, but non-identical when defined together, i.e., as parts of one matrix. It is my purpose in this paper to point out that Łukasiewicz's arguments in this matter are faulty, and that the apparent paradox dissolves in the light of a reasonable criterion for determining identity of matrices. Łukasiewicz creates his $\mathfrak{M}9$ matrix, i.e.,

	C	(5,7)	(5,8)	(6,7)	(6,8)	N	Δ
	*(5,7)	(5,7)	(5,8)	(6,7)	(6,8)	(6,8)	(5,7)
$\mathfrak{M}9$	(5,8)	(5,7)	(5,7)	(6,7)	(6,7)	(6,7)	(5,7)
	(6,7)	(5,7)	(5,8)	(5,7)	(5,8)	(5,8)	(6,7)
	(6,8)	(5,7)	(5,7)	(5,7)	(5,7)	(5,7)	(6,7)

by multiplying the matrices

	C	5	6	N
$\mathfrak{M}7$	*5	5	6	6
	6	5	5	5

	C	7	8	N
$\mathfrak{M}8$	*7	7	8	8
	8	7	7	7

together using the equalities

(a) $C(a,x)(b,y) = (Cab,Cxy)$
(b) $N(a,x) = (Na,Nx)$
(c) $\Delta(a,x) = (a,Cxx)$

where 'a' and 'b' represent elements of $\mathfrak{M}7$ and 'x' and 'y' represent elements of $\mathfrak{M}8$. He abbreviates the $\mathfrak{M}9$ matrix by allowing '1' to stand for

1. *The Journal of Computing Systems*, vol. 1 (1953), pp. 111-149.

2. *Ibid*., p. 127.

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'(5,7)', '2' for '(5,8)', '3' for '(6,7)' and '4' for '(6,8)', producing the following matrix:

	C	1	2	3	4	N	Δ
	*1	1	2	3	4	4	1
$\mathfrak{M}6$	2	1	1	3	3	3	1
	3	1	2	1	2	2	3
	4	1	1	1	1	1	3

He then introduces a new equality

(d) $\nabla(a,x) = (Caa,x)$,

which, when used with (a) and (b) in multiplying $\mathfrak{M}7$ and $\mathfrak{M}8$ together, helps produce

	C	(5,7)	(5,8)	(6,7)	(6,8)	N	∇
	*(5,7)	(5,7)	(5,8)	(6,7)	(6,8)	(6,8)	(5,7)
$\mathfrak{M}10$	(5,8)	(5,7)	(5,7)	(6,7)	(6,7)	(6,7)	(5,8)
	(6,7)	(5,7)	(5,8)	(5,7)	(5,8)	(5,8)	(5,7)
	(6,8)	(5,7)	(5,7)	(5,7)	(5,7)	(5,7)	(5,8)

The question naturally arises as to the relationship between $\mathfrak{M}9$ and $\mathfrak{M}10$. Łukasiewicz attempts to provide an answer to this question by abbreviating $\mathfrak{M}10$ using '1' to stand for '(5,7), '2' for '(6,7)', '3' for '(5,8)' and '4' for '(6,8)'. The result of this abbreviation schema is

	C	1	3	2	4	N	∇
	*1	1	3	2	4	4	1
$\mathfrak{M}6a$	3	1	1	2	2	2	3
	2	1	3	1	3	3	1
	4	1	1	1	1	1	3

which, when rearranged so that the numerals on the outside of the matrix are in numerical order, takes the form

	C	1	2	3	4	N	∇
	*1	1	2	3	4	4	1
$\mathfrak{M}6b$	2	1	1	3	3	3	1
	3	1	2	1	2	2	3
	4	1	1	1	1	1	3

This latter matrix is obviously identical with $\mathfrak{M}6$ except for the ideographic replacement of 'Δ' with '∇'. Łukasiewicz notes this and concludes that since $\mathfrak{M}6$ and $\mathfrak{M}6b$ are identical and are abbreviation of $\mathfrak{M}9$ and $\mathfrak{M}10$ respectively, $\mathfrak{M}9$ and $\mathfrak{M}10$ are themselves identical. He further concludes that since $\mathfrak{M}9$ and $\mathfrak{M}10$ are identical, the functor 'Δ', defined as in $\mathfrak{M}9$, is identical with the functor '∇', defined as in $\mathfrak{M}10$.[3]

3. *Ibid.*, p. 128.

I wish to contest these conclusions. In the first place, Łukasiewicz, as far as I can see, claims that $\mathfrak{M}9$ and $\mathfrak{M}10$ are identical on the principle that if two items have the same abbreviation, then they are identical. This line of reasoning is clearly fallacious: "ass." is an abbreviation of both "assistant" and "association", but we would not conclude on that basis that the latter two words are identical. Furthermore, while it is possible to produce matrix $\mathfrak{M}6\mathfrak{b}$ as an abbreviation of $\mathfrak{M}10$, it would have been just as easy to let '1' stand for '(5,7)', '2' for '(5,8)', '3' for '(6,7)' and '4' for '(6,8)' and produce the following abbreviation of $\mathfrak{M}10$:

$\mathfrak{M}11$

C	1	2	3	4	N	∇
*1	1	2	3	4	4	1
2	1	1	3	3	3	2
3	1	2	1	2	2	1
4	1	1	1	1	1	2

In this case, the definition of '∇' has not changed, since it is $\mathfrak{M}10$ which is still being abbreviated, but this functor is certainly not identical to the 'Δ' functor as defined in $\mathfrak{M}9$. Thus, it is not enough to point to the identity of $\mathfrak{M}6$ and $\mathfrak{M}6\mathfrak{b}$ to establish that the two functors in question are identical when defined apart. One could just as easily point to the differences in $\mathfrak{M}6$ and $\mathfrak{M}11$ to establish that they (the functors) are quite different when so defined.

The crux of the problem is to decide whether $\mathfrak{M}9$ and $\mathfrak{M}10$ are identical. One obvious way of deciding this is to count these two matrices as identical if, *sans abbreviation*, they verify or reject the same wffs. The restriction against abbreviation is included since, as we have just seen, such abbreviation tends to obscure rather than clarify the issue. Proceeding in this way, we can quickly see that these two matrices are not identical. Let 'D' represent the constant (6,7) functor. Let us assume that $\mathfrak{M}9$ and $\mathfrak{M}10$ are identical: $\ulcorner CDp\Delta p$ and $\ulcorner CDp\nabla p\urcorner$ are then really the same wff with a mere ideographic difference ('Δ' for '∇'), and $\mathfrak{M}9$ and $\mathfrak{M}10$ will either both verify or both reject this wff. But this is not the case, since while $\mathfrak{M}9$ does verify $\ulcorner CDp\Delta p\urcorner$, $\mathfrak{M}10$ rejects $\ulcorner CDp\nabla p\urcorner$—when $p = (5,8)$, $\ulcorner CDp\nabla p\urcorner = (5,8)$. Thus, the two matrices are not identical and, as is easy to see, they are not identical precisely becuase their possibility functors are not equal.

With this Łukasiewicz's paradox dissolves, since it is no longer the case that there are two functors which are identical when defined apart but different when defined together. The 'Δ' and '∇' functors are different when defined together *and* when defined apart, a not too surprising result.

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