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# THE APPROACHES TO SET THEORY

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*Introduction*

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Our aim is to consider and criticise the different approaches to set theory; we often seem to disagree with the accepted views.

We describe Cantor's work in chapter **1**. Then we emphasize its second order nature, indicate how it seems to have been misunderstood and suggest that a lot of later work was motivated by such misunderstandings. Part of chapter **2** gives a justification of **ZF** in Cantorian terms and in the remainder of that chapter we consider related problems and quasi-constructive approaches. In chapter **3** we consider set theories with a universal set; the most well known of these being **NF**. Other sections of that chapter concern approaches via theories of properties.

## CHAPTER 1—CANTOR'S WORK

**1.1** *Introduction* Although it seems possible to trace the notion of a set back for an indefinite period, it is indisputable that Cantor's work made the greatest step, by far, in the development of the idea. This is one of the reasons why we think it important to consider his work here. The other is that its nature is often misrepresented in textbooks and mythology today. Basically we shall give an account of Cantor's work on the notion of a set and, from his publications, we can discern three stages in the development of his ideas. It is quite possible that Cantor's views remained constant and that we are really only considering different stages of presentation, but we shall always write as if his papers correspond to his ideas. The main references which we shall use are [5], [6], [7] and [17], and we shall usually refer to Cantor's papers just by the year in which they were first published.

As well as describing Cantor's ideas we shall often comment on points at which various problems arise and sometimes we shall investigate them further. Also, we shall try to show how, in the development of set theory,

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some people have gone astray (knowingly, or otherwise) from the original ideas. Frequently, we shall impose certain ways of thinking on the published work so that we cannot be sure that we are faithfully presenting Cantor's work, but we leave others to argue over such problems.

Actually, Cantor has written relatively little on the notion of a set (or aggregate, as it was called at the end of the nineteenth century: we shall always update such terminology without further mention) and most of his work concerns infinite ordinals and cardinals. He did not view these in the current way, but firstly as newly postulated entities and later as abstractions from ordered sets. During this chapter the terms ordinal and cardinal have a variable status (among the three meanings) and we hope that the intended usage will be clear from the context.

A reasonable introduction to Cantor's earlier work and some indications of his motivation are given in [17]. This also describes his first work on powers of sets (two sets were said to have the same power if there is a bijection between them, so this corresponds to cardinality) and we shall not discuss this. For a discussion of the prior opinions and uses of the notion of infinity in mathematics and philosophy [1883] is very good.

**1.2** *Early work on ordinals* In the last part of [1883] Cantor explains certain principles by which, he argues, we can form new infinite ordinals. His language is very suggestive of one's creating new objects in time and we shall discuss this interpretation in section **9.4**. Cantor's considerations start with the sequence of natural numbers

(I) $1, 2, 3, \ldots, \nu, \ldots$

In this sequence each element is obtained from the previous element by adding a unit to it, and this process is called the first principle of generation. Cantor then argues that we can posit a new number, $\omega$, which is the least number greater than all of the elements of (I). Then, applying the first principle of generation repeatedly, we obtain the new sequence

$$\omega + 1, \omega + 2, \omega + 3, \ldots, \omega + \nu, \ldots$$

On the basis of this, and other, examples Cantor defined the second principle of generation as follows.

> If any definite succession of ordinals is given, for which there is no greatest, a new number can be created on the basis of the second principle, which is defined to be the least number greater than all of the elements of the sequence.

Using this principle Cantor then introduced $\omega . n$ and $\omega^n$ in the obvious way and he proceeded to illustrate the dazzling array of small countable ordinals. Cantor then defined the totality of all numbers of the same power as (I) as the second number class, (II) ((I) was called the first number class). From the existence of (II) and the second principle Cantor then obtained a least member of the third number class, and so on. In making these definitions Cantor has used the third principle which takes the form of a restricting, or limiting, principle on the second one. This states that the numbers to be next formed using the second principle are all to be of the power of a smaller number. To be precise, the 1883 paper does not actually state the third principle, but it is said that (II) has the required property and hence it is said to satisfy the third principle. From the introductory part of [1883] (see page 547) it seems that Cantor might have wanted the third principle to give the number classes rather than to restrict all uses of the second principle in this way.

Some theorems on ordinal arithmetic and a proof that the power of (II) is the next greater cardinal to that of (I) form the remaining technical results of [1883]. These proofs are always of a higher order nature (i.e., they consider sets of ordinals, etc.) but we shall consider this point again with respect to the later work.

We learn from [17] that in 1883 the above approach to ordinals had already been replaced by Cantor (probably for reasons which we shall outline in the next section) and the notion of an order type was introduced as an abstraction from an ordered set. Further details of Cantor's work between 1883 and 1890 are given in [17] and we only note that some of the work which was published in [1895] (which we call later work) had been completed ten years earlier.

**1.3** *Some comments on the early work* The main criticisms of Cantor's earlier work on ordinals seem to concern certain uses of the second principle and we find it convenient to split the uses of this principle into the following cases.

(2a) When we apply it to a countable, increasing sequence of ordinals which have already been introduced and for which we have a notation. Such sequences are called fundamental ones.

(2b) When we are producing a least ordinal of the next higher cardinality.

(2a) leaves no doubts that we have a definite succession of ordinals, but this does not seem to be true of (2b). The third principle might have been intended just as an assertion that (2b) is dealing with a definite succession of ordinals, but this still gives no reason for believing it. It seems intuitely reasonable that however we describe any procedure which only uses fundamental sequences of ordinals we shall never be able to generate the first uncountable ordinal. Thus, if the second number class is to be thought of as a completed totality we seem to require a more detailed description of the process by which it is to be generated. In particular, what is a definite, uncountable process? It is hard to imagine an answer to this question which does not use an uncountable set to index the process, and the only way to get such a set, at the moment, seems to be by using the power set axiom. We cannot assume that Cantor had such a scheme in mind as no indication of it is given and it would hardly have been obvious to his readers. An alternative solution to this question would be to allow (2b) without the power set axiom by adding the proviso that the class of all ordinals less than the new one is essentially incomplete. However, we do not think that such an approach is intuitively very plausible.

On the basis of the above arguments we suggest that Cantor's justification of the existence of the second number class is not completely convincing. It is equally possible to advance analogous criticisms of the notion of a set which was given in [1882]. In that paper the concept of power was considered as an attribute of "well defined collections", where

> A collection of elements belonging to any well defined sphere of thought is said to be well defined when, in consequence of its definition and the logical principle of the excluded middle, it must be considered as intrinsically determined whether any object belonging to this sphere of thought belongs to the collection, or not, and, secondly, whether two objects belonging to the collection are equal or not, in spite of formal differences in the manner in which they are given.

Cantor went on to emphasize that "intrinsically determined" does not mean that we can actually find the answer. With this notion of a set it is hard not to jump to the conclusion that all sets are definable, in some sense, so that there cannot be a first uncountable ordinal, all of whose members are sets. It might be worthwhile to consider how far one could go in formalizing a system of sets and objects where all sets are definable, and we mention this again in section **1.8**. This notion is also slightly evident in the following definition of a set which Cantor gives in a note to the 1883 paper. It is also possible to see the later ideas developing here.

> "By a set I understand, generally, any multiplicity which can be thought of as one, that is to say, any totality of definite elements which can be bound up into a whole by means of a law."

**1.4** *Cantor's later work* By the later work we mean [1895] and [1897]. Here, the main aims are to establish a rigorous basis for the ordinals and cardinals, and to start the development of their theories. Throughout these papers set theory is not treated in general although Cantor says that he intended to formulate this theory later. The [1895] paper starts with the oft quoted "definition" of a set,

> "By a set we are to understand any collection into a whole of definite and separate objects of our intuition or thought."

It seems highly unlikely that Cantor intended this to be anything more than a heuristic guideline as he frequently explains why certain sets can be said to exist. Consequently, we shall not treat this statement as a definition. We take it to mean that any collection which can be consistently "visualized", in some sense, can be thought of as a set.

Next in [1895] Cantor explained his basic ideas about cardinality and the relationships between cardinals. He also defined arithmetic operations on the cardinals, proved some results about $\aleph_0$ and indicated some results concerning increasing sequences of cardinals. The most important point, from our point of view, is that Cantor no longer based these ideas on direct intuition, but says that for a rigorous foundation of these matters we must turn to the theory of order types, which he considered next.

Cantor starts from the notion of a linearly ordered set. He considered this as a set with a separate ordering relation rather than the current view which includes the ordering as a set. Order types are considered as abstractions from these ordered sets where the abstraction is thought of as a set, all of whose elements are "unity", which has the same order precedence as the given set. Cantor then discussed similarity or order types and finite order types. Finally, in [1895], addition and multiplication of order types are considered and the order types of the rationals and the reals are discussed. The results include the well known characterizations of the latter two order types.

This work continues in [1897] where Cantor defines well ordered sets as linearly ordered ones for which

(i) there is a least element,
(ii) if a part, $f$, of the set has one or more elements of the set above it, then there is an element of the set which follows immediately after $f$.

It is clear that this is equivalent to the usual definition of a well ordering. Cantor then proved the results on well orderings which now form a well known part of courses on set theory. Ordinals are defined as the order types of well ordered sets and the law of trichotomy for ordinals is proved rigorously. Then, at the beginning of section 15 of [1897], there comes what, from our point of view, is the most important definition in the paper.

> "The second number class, $Z(\aleph_0)$, is the set of all order types of well ordered sets of cardinality $\aleph_0$."

In effect, this is allowing us to gather into a whole all the different well orderings of $\omega$ and, as such, it is a new principle which has not been previously used in these two papers. It is quite clear when an ordering of $\omega$ is a well ordering and, although we cannot give a process which enumerates the well orderings of $\omega$, we are allowed to gather them all together at one sweep. Thus $Z(\aleph_0)$ is defined in a single second order way (we take *all* well orderings of $\omega$—these can obviously be obtained from *all* subsets of $\omega$), rather than by a vague belief that the building up processes for obtaining ordinals can be continued through all countable ordinals.

Cantor then proceeded to analyze $Z(\aleph_0)$ and he proved that its cardinality is the next greater one to $\aleph_0$. He also proved his normal form theorem and this illustrates Cantor's approach to set theory: he studied the structure of $Z(\aleph_0)$ in some detail, rather than getting involved in vaguer macro problems.

**1.5** *The second order nature of the later work* We think that, at the moment, the second order nature of Cantor's work cannot be overemphasized. If we were to begin to formalize his work on ordinals, then the principles akin to (2a) could easily be handled within a first order system, but this does not seem to be true when it comes to the existence of $Z(\aleph_0)$ and the power set axiom. We do not think that Cantor would have assented to founding set theory on full second order logic, where the variables $X, Y, \ldots$ range over subcollections of the "universe of all sets", for reasons which we shall discuss in the next section. We suggest that a suitable form of second order logic (we call it a mild second order theory) would be one where $X$, $Y$, . . . range over *all* those collections of sets which are equipotent to a set, and $x$, $y$, . . . range over sets, as usual. Then the power set axiom (the existence of $Z(\aleph_0)$ can be derived from this) would take the form

$$\forall x \exists y (\forall t \in y \; t \subseteq x \wedge \forall X \subseteq x \exists t \in y \; t = X)$$

This essential viewpoint gets lost in first order axiomatizations of set theory, such as **ZF**.

It is also important to notice how, on the basis of the above ideas, we can justify the comprehension axiom of **ZF** without any reference to truth considerations, as follows. We consider a set $y$ and, for convenience, a formula $\phi$ with exactly one free variable. If $x \in y$, Cantor would argue that by the logical principle of the excluded middle, we would have $\phi(x)$ or $\neg\phi(x)$. Then, as the power set of $y$ contains *all* subcollections of $y$, there must be one, $z$ say, for which $\forall x(x \in z \leftrightarrow x \in y \wedge \phi(x))$. Hence the comprehension axiom holds. This reduces the truth of comprehension to a question of logic, and although people can, and do, work in non classical logics, classical logic is presupposed in all of Cantor's work.

This justification of the comprehension axiom runs counter to what some people have recently suggested and we think that model theory is partly to blame for this shift of emphasis. Here, one frequently considers first order **ZF** (a quaint theory, as it only ensures that certain definable subsets exist although it is not at all clear what the variables range over so that we do not know in what sense these subsets are definable anyway) and then from Skolem's work we know that there are countable models of **ZF** so that people get very worried about which subsets of $\omega$, for instance, "really exist". They also begin to think that comprehension is true because, for a given formula $\phi$, they can check the truth definition of $\phi$ in the model, whereas questions of truth in set theory cannot use Tarski's truth definition for it assumes that the universe is a single consistent totality.

A good example of bad motivation which follows from such misunderstandings is Barwise's paper [1]. In the concluding remarks of that work he says that to allow all first order formulae to occur in the comprehension axiom (a suggestion due to Skolem which is obviously inadequate for giving all subsets) assumes that we can form a true universe of all sets. Why this should be true, unless Barwise is worried about truth definitions, remains a mystery. Barwise considers restricting the comprehension axiom to $\Delta_0(\mathcal{P})$ formulae (i.e. those formulae which are $\Delta_0$ when we allow $\mathcal{P}$, the power set operator, as a new basic symbol) and he seems quite willing to believe these instances. But now if one is willing to believe the power set axiom in its mild second order form then all instances of comprehension follow, and if one believes it in some other form it seems to be a harder problem to say which subsets exist than to accept the comprehension axiom.

It seems that [40] is the origin of such heresies and the presupposition of this paper is that set theory is a first order theory rather than a mild second order one. This fallacious belief seems to be held largely by people who publish in logic journals: mathematicians, in general, seem quite happy to believe in a genuine power set operation which cannot be first order. In [40] Zermelo talks of comprehension holding for "definite properties" and this notion is an open ended extension of Skolem's restriction to first order formulae. Although all instances of Zermelo's comprehension axiom will be true from a Cantorian viewpoint, there does not seem to be any reason for supposing that these ideas suffice for describing the true power set operation.

Finally, in this section, we note that, in [4], Borel criticized Cantor's work on ordinal numbers and he was probably referring to the earlier work so that his reasons might have been similar to those of section **3**. Borel acknowledged Cantor's proof that $\mathcal{P}\omega$ was larger than $\aleph_0$, but he did not believe in the existence of $\omega_1$. This was the motivation for his later (famous) work. In a footnote Borel asks why there should be a least cardinal greater than $\aleph_0$, although from Cantor's later work and the Schroder-Bernstein theorem (both of which had been published before [4]) there seems to be a convincing proof of this fact. Of course, we do not know that Borel was aquainted with these results and, as he offered no criticisms of them, we assume that he was not. Thus his work was motivated by doubts about the principle (2b) and we shall later suggest that other work also arose in this way.

**1.6** *Inconsistent multiplicities* A letter which Cantor wrote in 1899 [7] contains what we consider to be his final conclusions about the notions of set, ordinal and cardinal. The discussion in the letter assumes that there are multiplicities (we hope that this word does not have any connotations of oneness) which are not sets. The main point of the letter is to show that all cardinal numbers are alephs, or, in effect, that every set can be well ordered. However, Cantor firstly outlines his general ideas.

Cantor says that it is necessary to distinguish between two sorts of multiplicities (he always assumes that we are considering only definite multiplicities) and he says that for some multiplicities the assumption that "all of its elements are together" leads to a contradiction, so that it cannot be conceived of as "one finished thing". On the other hand, if the elements of a multiplicity can be thought of as "being together", then it is called a consistent multiplicity, or a set. Thus all notions of processes and building up are eliminated and the whole of set theory is given in one psychological (though not obvious) swoop.

Then Cantor gives informal versions of the axioms of **ZF** as ways of getting from one set to another. Hence it would seem more reasonable for this theory to be called **CZF** than **ZF**. Two of the statements which are of interest to us are

(a) Two equivalent multiplicities are either both sets or both inconsistent.
(b) Every submultiplicity of a set is a set.

(a) obviously implies the replacement axiom and (b) suggests that our mild second order theory is a reasonable formalization of part of Cantor's ideas. Cantor probably believed these axioms because of considerations of the Absolute, although he does not explicitly say this.

As examples of inconsistent multiplicities of Cantor gives "the totality of all things thinkable" and $\Omega$, which is the system of all ordinals under their natural ordering. The proofs that these multiplicities are inconsistent are, of course, the usual paradoxes. Cantor then reiterates his work on ordinals and gives the following proof that if $v$ is a definite multiplicity and no aleph corresponds to it as its cardinal number, then $v$ must be inconsistent.

Suppose that $v$ is a definite multiplicity and that no aleph corresponds to it as its cardinal number. Then "we readily see that, on the assumption made, the whole system $\Omega$ is projectible into the multiplicity $v$, that is, there must exist a submultiplicity $v'$ of $v$ that is equipotent to the system $\Omega$. $v'$ is inconsistent because $\Omega$ is and the same must therefore be asserted of $v$."

From this Cantor proved the law of trichotomy for cardinals. The quoted proof was objected to by Zermelo as it used inconsistent multiplicities: we consider this further in section **8**.

Cantor's considerations of inconsistent multiplicities can be argued to follow logically from his earlier work as, in [1883], he says that considering the infinite in the sense of finite increasing without bound implies the existence of the truly infinite as the domain for the variables. In this way, the use of variables over sets necessitates the existence of inconsistent multiplicities as their domains.

In the introduction to Cantor's letter in [15], van Heijenoort says that Cantor's inconsistent multiplicities prefigure the distinction between sets and classes which was introduced by von Neumann. This seems to be untrue as the nature of proper classes assumes that they are definite, fixed totalities which are not inconsistent by their very existence. The idea of a proper class seems far more likely to have originated with Zermelo's definite properties.

**1.7** *Cantor's notions and set theoretic developments* Before we consider some of the interelations between Cantor's notions and set theoretic developments, we shall return to the so called definition in [1895], which says

> By a set we are to understand any collection into a whole of definite and separate objects of our intuition or thought.

It is often claimed that this leads to an inconsistent theory and, as an example of this, we quote from pages 285-6 of [18]. We do not think that the sense is altered by the omissions.

> Cantor's definition has not been retained in quite its original form by later authors, but was replaced at an early stage by a more abstractly conceived principle, or axiom, that has become known as the principle of comprehension [We refer to it as the abstraction principle so as not to confuse it with the axiom of comprehension] . . . [This] can be expressed in the following form

$$\exists z \forall x (x \epsilon z \leftrightarrow H(x))$$

> . . . The formal system which we have obtained in this way [the abstraction principle and extensionality formulated in the first order predicate calculus with $\epsilon$] . . . may indeed be regarded as a reasonable formalisation of Cantor's naive theory of sets.

This argument simply does not seem to be valid. Presumably the variables of the formal system are ranging over sets, but then the abstraction principle shows certain objects to be sets whilst Cantor showed that they were not sets. The formal system has more in sympathy with Frege than with Cantor as it ignores Cantor's insistence on our being able to visualize all the members of a set being together.

Also, on page 262 of [13], Godel suggests that "a satisfactory foundation of Cantor's theory in its whole original extent and meaning" can be given on the basis of iterations of the notion of "set of", and this contrasts sharply with the suggestion that a reasonable formalization of Cantor's theory is inconsistent.

Next we point out three areas where people have extended set theory using new principles which run contrary to Cantor's ideas. Their justifications do not seem to be as well motivated as Cantor's work.

The first example is Ackermann's set theory which we discussed in [24]. The second is the notion of building up sets "in time"; [29] and [30] being examples of this. On page 573 of [1883] Cantor says that, in his opinion, it is wrong to use the concept of time to explain the much more basic concept of a continuum and hence it is reasonable to suggest that this is also true for the notion of a set. Thirdly, there is the topic of reflection principles and their connections with the Absolute. In [3] and [28], for instance, axioms are asserted which suggest that there exist sets (or at least consistent multiplicities for the notion of set in such theories is often weaker than Cantor's notion) which resemble (e.g. are elementary substructures of) the Absolute. It is quite clear that Cantor believed we could not have any good approximation to the Absolute and on page 587 of [1883] he says

> There is no doubt in my mind that in this way [producing new number classes] we may mount even higher, never arriving at any approximate comprehension of the Absolute. The Absolute can only be recognised, never known, not even approximately.

Thus if we are to have any strong reflection principles and to maintain a Cantorian viewpoint then we must believe that the expressive power of the language under consideration is hopelessly inadequate for truth in the Absolute. However, such ideas do not seem to be considered at all in the works on reflection principles. One way of making reflection principles and Ackermann's set theory more reasonable is to consider them as ways of picking out certain ordinals which occur in their natural models, but this was not the original motivation for these ideas.

Comparing the kind of results which Cantor proved with those which are proved today we get another contrast, this time in methodology. He concentrated on structural problems for small sets rather than larger cardinals, for instance. Although Cantor was investigating problems which occur in nature (specifically the continuum hypothesis, of course) perhaps we could still gain much guidance from small, structural considerations.

Sierpinski is one of the very few mathematicians who have continued to work in Cantor's original spirit. Some further topics for structural considerations are countable order types (although there is quite a bit in the literature on this topic) and other countable partial orderings. Another topic which seems to have been neglected is n dimensional order types (for $n \epsilon \omega$ see page 80 of [6]) and higher dimensional ones. It might be possible to show that all interesting questions concerning these objects can be reduced, in some uniform way, to questions about ordinary order types, but we know of no such results.

**1.8** *Formalizing parts of Cantor's work* Here, we shall briefly outline three problems connected with formalizing parts of Cantor's work. Firstly, there is the "constructive" notion of building up sets by a definite process, which we shall again refer to in the next chapter. These ideas have been considered by Lorenzen, [25], Wang, [39], Borel, [4], and many others. We consider all this work to be motivated by Cantor's ideas which lead to the first principle and the principle (2a). Is it possible to isolate a definite part of set theory which results from just these principles (when (2a) is modified to deal with sets as well as ordinals)?

Our next considerations concern the interpretation of Cantor's earlier work, mentioned in section **3**, which suggests that all sets are definable. Although we cannot easily formalize such statements in a first order system we indicate how a first order system, analogous to **ZF**, could be set up, the axioms of which would be true under this interpretation. It would not be assumed that all members of sets are sets so that an additional predicate, $M(x)$, would be introduced for "$x$ is a (definable) set". We then let $\Phi_i(x)$ stand for $\exists ! y \, \phi_i(y) \wedge \phi_i(x)$, where $\phi(x)$ is an $\epsilon$-formula with one free variable, and we would have the schema

$$\Phi(x) \rightarrow M(x).$$

The other axioms would be obvious variants of those of **ZF** and, for instance, the comprehension axiom would take the form

$$\Phi_1(x) \wedge \Phi_2(y) \rightarrow \exists z \, (M(z) \wedge \forall t \, (t \epsilon z \longleftrightarrow t \epsilon x \wedge \phi(t,y))).$$

This system would be quite similar to one which Friedman introduced in [11] and if we add $\forall x \, M(x)$ (which is false under our intended interpretation) to our system it becomes Friedman's. Obvious questions which one could ask for this system are its relative consistency and the structure of its models, but we shall not pursue these questions.

Our final considerations in this chapter concern Cantor's notions of inconsistent multiplicities and the Absolute. We hope to consider, elsewhere, the general problems of formalizing these notions and here we only consider the conversion of Cantor's proof that every set has a cardinality which is an aleph (see section **6**) into a proof which would be acceptable in a **ZF** like system.

We assume that all variables range over sets and then the hypothesis of the proof is

$$\neg\exists\alpha v \approx \aleph_\alpha \tag{*}$$

Cantor then considered it obvious that we could project the whole of $\Omega$ into $v$. If we interpret this as meaning that there is an injection from $\Omega$ into $v$, then this leads to a contradiction in **ZF**. Hence the question reduces to showing that $\Omega$ can be projected into $v$.

Cantor seems to have used the axiom of choice as a logical principle so that we feel it is reasonable to assume the existence of a choice function $F: \mathcal{P}(v) - \{\emptyset\} \to v$ with $F(x) \in x$. Now the argument that $\Omega$ can be projected into $v$ can be represented by defining the following function by recursion

$$g(0) = F(v)$$
$$g(\gamma) = F(v - g[\gamma]),$$

and then we know that $g$ must be defined on all ordinals as, otherwise, consideration of the least ordinal for which $g$ is not defined contradicts (*).

Thus it is possible to get a proof of the well ordering theorem from Cantor's proof (by eliminating one of the reductio ad absudrums) so that there are grounds for believing his proof. However, it remains true that Zermelo was the first person to rigorously prove the well ordering theorem without using inconsistent multiplicities.

## CHAPTER 2—ZF AND QUASI-CONSTRUCTIVE APPROACHES TO SET THEORY

**2.1** *Historical developments of* **ZF** *and* **NBG** Briefly, Zermelo [40] first axiomatized part of Cantor's work and then Fraenkel [9] noted the omission of the replacement axiom. However, Zermelo's axiomatization included the notion of a "definite property", or definite assertion, so that his comprehension axiom took the form

For every definite propostional function $F(x)$,

$$\forall y \exists z \forall t (t \in z \leftrightarrow F(t) \wedge t \in y).$$

It is not completely clear what Zermelo meant by a definite property, but Skolem [38] suggested that it could be taken as any first order expression, giving us the theory which is now known as **ZF**. We believe that Skolem's suggestion is, essentially, a correct interpretation of Zermelo's ideas, except that Zermelo wanted to allow all (definite) predicates to appear in the comprehension axiom rather than just $\in$, so that his notion is open ended.

Another line of development from Zermelo's axioms is that which considers definite properties as objects in themselves. This started with von Neumann [26] and his justification of this step seems to be somewhat formalistic as he talks of how far the abstraction principle can be extended without generating the paradoxes. We shall ignore the fact that von Neumann's work is couched in terms of functions, but just note that the theory was put nearer modern **NBG** by Bernays [2]: his theory explicitly considers two types of individuals, sets and classes, adopting an extensional view of both. For the rest of this chapter we shall use the term class for proper classes (i.e., those classes for which there is no set which has the same members). The obvious question which we must now consider is what these classes are supposed to be.

From the Cantorian viewpoint it would seem natural to think of classes as inconsistent multiplicities, but this is alien to their appearing as definite collections in a formal system. The next alternatives is to think of classes as genuine properties (rather than collections of sets) or as the extensions of properties, possibly over some given collection. One criticism of both these approaches is that the notion of a property seems to be at least as complex as that of a set so that it is just as much in need of clarification: one need only consider the property of "not holding of itself". Also, if we think of classes as genuine properties, then **NBG** does not seem to be reasonable for

(i) why should properties be extensional?
(ii) presumably there is a property $U$ with $x \in U$ corresponding to "$x$ is identical with $x$", so that $U \in U$ would have to hold.

There have been attempts to modify **NBG** to meet the second of these criticisms and we shall consider these in chapter **3**.

The second of the alternative programmes was to consider classes as the extensions of properties, possibly over some given collection. Without the added condition, this view is still open to an obvious modification of (ii). Further, it is not at all obvious that the amended scheme could be carried out as the following situation might well arise. Suppose that we are taking classes as the extensions of properties over $V$, where, as in Ackermann's set theory, $V$ is thought of as the collection of all sets. Then there should be a property $P$ meaning "is a set" and a property $Q$ meaning "is identical to itself" so that although these properties have the same extensions on $V$ we would obviously want $\exists x(x \in Q \wedge \neg x \in P)$ to be true.

Thus none of the above explanations of the intended meaning of classes seem to be convincing. This leads us to consider two weaker alternatives. Firstly, classes could be thought of as virtual objects, in the sense of Quine [32], so that they are identified with first order definable predicates. On this view they become a convenient aid and, although they add nothing to our understanding of the nature of sets, they might make proofs easier to follow. Finally, one could adopt a formalist position and maintain that one is only interested in the usual models of first order **ZF**. Then classes are thought of as (certain) subcollections of the domain of the relevant model. This view, possibly that which is held by a number of people who work with **NBG**, has the disadvantage that it becomes meaningless when applied to the intended Cantorian interpretation of sets. It could still be useful though, if one thinks of formal set theories as picking out certain sets via their natural models etc.

**2.2** *Shoenfield's principle* When introducing **ZF** in set theory courses now it is very popular to use the idea of building up a cumulative type structure as the heuristic guide. A typical treatment of this is given in [37], where we find

> We then form sets in successive stages. At each stage we have already the urelements and the sets formed at earlier stages; and we form into sets all collections of these objects. A collection is said to be a set only if it is formed at some stage in this construction. . . . Since we wish to allow a set to be as arbitrary a collection as possible, we agree that there shall be such a stage [i.e., one following a given collection of stages] whenever possible, i.e. whenever we can visualise a situation in which all the stages of the collection are completed. . . . If a collection consists of an infinite sequence $S_1, S_2, \ldots$ of stages, then we can visualise a situation in which all of these stages are completed, so there is to be a stage after all of the $S_n$ . . . Suppose that we have a set $A$ and that we have assigned a stage $S_a$ to each element $a$ of $A$. Since we can visualise the collection $A$ as a single object (viz. the set $A$), we can also visualise a situation in which all of these stages are completed. This result is called the principle of cofinality.

There are certain problems connected with a literal interpretation of these ideas, such as what indexes the stages and what "assigned" means, but these do not affect what is the intended meaning. Shoenfield goes on to justify all the axioms of **ZF** using this principle. We consider this principle, which is sometimes known as Shoenfield's principle, to be a variant of Cantor's second principle (from the 1883 paper) combined with the power set axiom. Later, we shall show that it follows from considerations of the Absolute so that, in an imprecise sense, it is half way between **ZF** and the Absolute.

A significant problem for Shoenfield's principle is that it is phrased in terms of the notions of building up stages and visualizing situations so that the usual first order semantics do not give an intended model. Thus it only justifies **ZF** if we can jump to the conclusion that the process of visualizing and completing has itself been completed as otherwise it is not obvious that the law of the excluded middle would hold. This is suggested by Kripke's constructive semantics [21] where the law of the excluded middle can fail although, as Kreisel [19] mentions, this only holds for models which are themselves sets. Also, this slightly dubious point (if the building up and visualizing is completed, then why can we not start again?) makes the set concept seem more complex than is necessary (see the next section). This makes some people worry about such building up processes.

The problem of formalizing Shoenfield's principle is considered in [33]. Reinhardt slightly modifies it to

(S) "If $P$ is a property of stages and if we can imagine a situation in which all the stages having $P$ have been built up, then there exists a stage $s$ beyond all of the stages which have $P$."

He introduces a new constant $V$ such that $x \in V$ is to be thought of as "$x$ is a set", and then he produces a set theory $S^+$ which has some similarities with Ackermann's system. $S^+$ has variables for properties and an axiom corresponding to (S). Reinhardt shows that $S^+$ is very much stronger than **ZF** and, although this is very interesting, there are still problems about what $V$ and the properties are intended to be. It is suggested in [33] that the usual semantics are not really adequate for these ideas and it is a significant open problem to introduce a suitable semantics. Perhaps this is where one should start in formalizing classes. In the philosophical remarks at the end of [33], Reinhardt states that

> I have tried to introduce the axioms for properties in such a way that the naive reader will find them natural for naive (or Cantor's) set theory

but, again at the risk of overemphasizing a point, we do not think that it is reasonable to introduce properties as consistent collections whilst maintaining a Cantorian viewpoint.

Finally, we note that Shoenfield's principle could be argued to give answers to some questions which are independent of **ZF**. For instance, it seems much easier to visualize a situation in which there is a scale for ${}^{\omega}\omega$ than one where there is no such scale. Are we then justified in asserting the existence of such a scale?

**2.3 ZF** *from the Absolute* In this section we hope to show that **ZF** can be justified by considerations of the Absolute. The viewpoint which we adopt is an extrapolation from that of [7], but we do not claim that this is an exposition of Cantor's views.

We are thinking in terms of collections of objects where a collection is thought of as a 'bringing together' of the objects under consideration. However, we must firstly ask what the Absolute is. Basically, we think of it in terms of everything which has ultimate existence; we shall not consider its metaphysical overtones. With Cantor, we believe that the Absolute can be recognized (which implies that it is a meaningful notion, of course) but that it can never be known. The latter point means that it is not good enough to imagine some very large set playing the part of the Absolute because the inherent nature of the Absolute ensures that it cannot be thought of as a unity in itself. Our usage of consistent and inconsistent multiplicities will be as in the last chapter and we identify sets and consistent multiplicities. It does not seem to be immediately true that all inconsistent multiplicities have the same "size" as the Absolute, but we shall often assume that they share much of the nature of the Absolute. If we add a new principle saying that all inconsistent multiplicities are of the same "size" (this would be analogous to von Neumann's maximal principle), then many of our arguments would flow more smoothly. We shall not do this as we do not find such a principle completely convincing.

Extensionality is basic for the view of sets which we have adopted and we next indicate how a version of Shoenfield's principle can be justified. The axiom of infinity follows from this by considering the natural numbers. Consider the version of (S) with 'property' replaced by 'collection', and then if we imagine a situation in which all the stages in $P$ have been completed, we can imagine the collection of those stages as a consistent totality. The nature of this collection is not that of the Absolute (or any other inconsistent multiplicity) so that we have a consistent multiplicity and there is a stage beyond all those in the collection $P$. We shall not use Shoenfield's principle to justify the remaining axioms of **ZF** as we believe it overcomplicates matters, but we indicate how they can be got directly from considerations of the Absolute.

The replacement axiom follows from Cantor's statement that "two equipotent multiplicities are both consistent or both inconsistent". This is the same as saying that there cannot be two equipotent collections, one of which is an inconsistent multiplicity and the other of which is a set: this seems a transparent fact from the nature of the Absolute. The comprehension axiom, in the form that every subcollection of a set is a set, similarly follows from the nature of inconsistent multiplicities.

The sum and power set axioms follow as it is inconceivable that an inconsistent multiplicity could be obtained from a set by one of these visualizable operations. This is even clearer if we assume that all inconsistent multiplicities are the same size, for then the power set axiom, for instance, says that there is no set for which the collection of all its subcollections is the same size as the Absolute.

The axiom of foundation does not seem to be evident on this interpretation, although there is no reason why one should not restrict one's attention to well founded sets if it is desired. Of course, the non existence of cycles of sets follows from our basic viewpoint of forming collections by bringing together certain objects. We consider the axiom of choice to be a logical principle for sets so that it is not in need of justification.

Now we consider two other kinds of axioms from this point of view.

(i) Let $\Omega$ be the inconsistent multiplicity consisting of all ordinals, ordered by their natural ordering. We consider certain axioms about "stopping points" in $\Omega$. It is convenient to think in terms of processes for going up $\Omega$ and then the nature of the Absolute shows that there cannot be any definite process, the completion of which is $\Omega$. Thus if $\forall\alpha\exists\beta\,\phi(\alpha,\beta)$ there must be a cardinal $\kappa$ such that from below $\kappa$ this process (i.e., going from $\alpha$ to the least $\beta$ satisfying $\phi(\alpha,\beta)$) does not get beyond $\kappa$. Further, it is reasonable to insist that $\kappa$ is regular as otherwise the process can be continued by taking the union of a shorter cofinal sequence. Consequently, we have the schema

$$\forall\alpha\exists\beta\,\phi(\alpha,\beta) \rightarrow \exists\kappa(\mathrm{Reg}(\kappa) \wedge \forall\alpha\in\kappa\exists\beta\in\kappa\,\phi(\alpha,\beta)),$$

which, together with **ZF**, gives the theory **ZM** (we showed that in [22])

(ii) The existence of a measurable cardinal does not seem to be justified, at the moment, by arguments similar to those which we have already encountered.

(i) shows that **ZM** can be justified from the Absolute and (ii) suggests that one should investigate other ways of justifying axioms from the Absolute. Whether or not measurable cardinals turn out to be reasonable, the latter programme should be very useful. For instance, does it give any new structural information?

**2.4** *Intuitionistic* **ZF** Intuitionistic **ZF** is **ZF** set theory based on intuitionistic logic. Myhill, in a seminar, suggested that such a theory, without the axiom of choice, corresponds to that part of **ZF** which gives effective results. This is a thoroughly reasonable attitude and, like Church's thesis in recursion theory, the conjecture is open to empirical testing.

However, intuitionistic **ZF** is also the end product of Pozsgay [30], and for the remainder of this section we shall be considering this paper. Pozsgay claims to be formalizing a certain intuitive approach to set theory which he thinks represents the basic insights underlying the **ZF** axioms. He thinks of sets as mental constructions and he gives the following principle for set construction.

> Any well defined mental process for constructing sets which has been clearly envisioned without ambiguities or contradictions may be regarded as already completed, regardless of any merely practical difficulties which may prevent one from actually carrying it out.

On the basis of this principle Pozsgay argues that we can justify the axioms of **ZF** and, in particular, the power set axiom. But what mental process is available for constructing the power set of $\omega$? Certainly we cannot give any step by step procedure for doing this as any countable number of countable processes will remain countable. Somehow we need to jump to the uncountable set. Consequently, we feel that this principle does not justify the power set axiom, but that it must be added as a further principle. Then we seem to get Shoenfield's principle, though.

Pozsgay's paper splits into two sections and in the second he turns to the problem of formalizing his principle, where he says

> As far as set theoretic axioms go, the best available seem to be the **ZF** axioms, and the main question is whether the underlying logic should be intuitionistic or classical.

The procedure now seems to have very little to do with the original principle. For example, a first order theory is assumed without any explanation of how this affects the power set operation, although, in justifying the comprehension axiom Pozsgay circumvented the problem of impredicativity by saying that he took all possible subcollections of a set in the power set. Consequently we feel that the reasons for using **ZF** to formalize this work are a little obscure, but the reason for using intuitionistic logic seems even less clear.

Pozsgay states that he wants $\exists x\, B(x)$ only to be provable if there is "at hand a definite construction for producing a set $x$ with the property $B(x)$". Two pages previously he justified the axiom of choice and it remains a complete mystery how we are to give a definite construction for a choice function on infinitely many pairs of socks.

Basically, [30] belongs to those approaches to set theory which can be thought of as "building up in time" and hence we do not see how $\omega_1$ can be thought to exist (unless one adds the power set as an additional basic operation). Hence Powell's approach [29] to such a theory seems more reasonable, if one is not going to allow time to be completed.

In [22] we gave a possible axiomatization of Pozsgay's building up ideas, but we now think that Wang's system of predicative set theory ($\Sigma$, see [39]) is probably a better candidate for such a theory. To really axiomatize $\Sigma$ we should make explicit the principles by which one indexes the types: perhaps we could just allow completions of fundamental sequences for some given system of notations. Section 4 of [22] contains some considerations of the power set axiom and we now believe that the ideas of that section are superseded by that of a mild second order logic, which we introduced in the last chapter.

## CHAPTER 3—SET THEORIES WITH A UNIVERSAL SET

**3.1** *Introduction* In this chapter we shall consider some aspects of set theories in which there is a universal set (i.e., a set $x$ such that for all sets $y$, $y \epsilon x$). Such a set cannot exist from a Cantorian viewpoint so there must be some other motivation for such theories. One possible approach is via properties and such theories are discussed in sections **3.3** and **3.4.** The remaining theories all seem to result from formalist inspiration and the main one of these theories is **NF**, which is considered in section **3.2.**

Another approach to set theories with a universal set has been made by Church [8]. Here the motivation is that the abstraction principle is desirable but (unfortunately?) it turns out to be inconsistent so that we must investigate all (formalistic) ways of approximating to it whilst remaining within the realms of consistency or, at least, relative consistency. This view also seems to be an assumption for the book by Frankel, Bar-Hillel and Levy [10]. We have little sympathy with such ideas as there does not seem to be any *clear* reason why we should have believed the abstraction principle in the first place.

**3.2** *Quine's* **NF** The theory **NF** was introduced in [31] and is formulated with $\epsilon$ as the only predicate. Equality is introduced by definition and there is an axiom of extensionality. The only other axiom is the abstraction principle for those formulae $\phi$ which are stratified (i.e., one can attach numerals to the variables in such a way that whenever $x \epsilon y$ occurs in $\phi$ with $n$ attached to $x$, then $n + 1$ is attached to $y$). The motivation behind this is that stratified formulae correspond, in an obvious way, to those of type theory and that the paradoxes (at least, the old familiar favorites) do not seem to be derivable in the theory. Thus **NF** is a formalist's theory, but it still could be a reasonable set theory as well.

In [10] it is suggested that the unprovability of all instances of induction in **NF**, if this theory is consistent, shows that it is not a reasonable theory, but it would be nicer to have a stronger condemnation. The next section contains some arguments which show that **NF** is not, as it stands, a good set theory, in the sense that it is not adequate to describe certain mathematical notions.

Section 2 of Rosser and Wang [36] claims to show that if **NF** is consistent (we always assume this when discussing its models) then it does not have a standard model. Briefly, the argument is as follows. **NF** is assumed to have a model in which the natural numbers are standard and then, using Rosser [34], one shows that transfinite induction cannot hold for all the formulae of **NF**. Consequently, the order relation of the ordinals in the model is not really well founded and **NF** cannot have a standard model.

The actual arguments which are used in the proof are correct but it is implicit throughout that the definition of ordinal which is used (equivalence classes of 'well ordered classes', in the sense of **NF**-ordinal(**NF**), say) corresponds to the intuitive notion of ordinal (ordinal(**I**), say). There is no attempt in [36] to show that ordinal(**NF**) is a good approximation to ordinal(**I**). Usually, the definition of an ordinal occurs within an environment where we may suppose that all instances of the comprehension axiom hold and when this is not the case the definition of an ordinal is suitably modified (see, for instance, [11]). From page 474 of [35] we know that **NF** does not ensure that the order type of the class of ordinals(**NF**) less than an ordinal(**NF**) $\alpha$ is $\alpha$, so that it is natural to strengthen the definition ordinal(**NF**) to

$$\text{ordinal}'(\mathbf{NF})(x) = \text{ordinal}(\mathbf{NF})(x) \wedge \text{``the order type of the ordinals}(\mathbf{NF})\text{ less than } x \text{ is } x\text{''}.$$

However, we still would not know that ordinal'(**NF**) is a good approximation to ordinal(**I**) in **NF**. Indeed, there might be no formula of **NF** which satisfies this requirement.

On this basis we suggest that Rosser and Wang's result shows that if **NF** has a standard model, then ordinal(**NF**) does not represent the notion ordinal(**I**) in **NF**. This suggests that one should look at the adequacy of the representations of the usual mathematical notions in **NF**, rather than assuming that a formal definition gets its intended meaning. We started this in [23], and, on the basis of that paper, we think it reasonable to claim that **NF** is not a nice set theory as various natural notions, such as equipollence, depend on the way in which ordered pairs are represented. Further, if the theory is extended to take care of these problems, then the resulting system would be extremely complicated and completely unusable.

**3.3** *Properties as properties* Sets can be considered as collections of objects which satisfy a given property, or in other words, as the extensions of properties. This is the usual view from which people argue that the abstraction principle is intuitively plausible, but there seems to be no agreement as to whether the variables are ranging over properties, objects, extensions over some collection, or anything else.

The property of "not satisfying itself" might show that if properties are allowed to apply to properties, then we cannot expect them to be everywhere defined: this is probably the motivation behind Kreisel's following remarks on properties in [20].

> For this notion, with $y \in x$ being interpreted as: the property $y$ has the property $x$, $\exists x \forall y\,(y \in x \leftrightarrow P)$ [i.e., the abstraction principle] is indeed evident, provided that the most general kind of property is considered, including properties which are not everywhere defined.

He goes on to say that we cannot expect the usual logical laws to hold in such a system but we find it unlikely that the logical laws must be altered before we can talk about properties: we consider another way of approaching this problem below. Kreisel also suggests that no property can be defined for itself as argument whilst consideration of the property of "being a property" suggests that sometimes this might be quite harmless.

An earlier suggestion regarding an approach to properties (or concepts—we make no distinction between these notions) was given by Gödel [12], where he says

> It is not impossible that the idea of limited ranges of significance could be carried out without the above restrictive principle [referring to type theory]. It might even turn out that it is possible to assume every concept to be significant everywhere except for certain "singular points" or "limiting points", so that the paradoxes would appear as something analogous to dividing by zero.

We next outline a framework, based on the first order predicate calculus with identity, within which such ideas can be formalized. There are two predicates:

$M(x,y)$ for "it is meaningful to ask if the property $x$ has the property $y$", and

$x\eta y$ for "the property $x$ has the property $y$", if $M(x,y)$ no intended interpretation, if $\neg M(x,y)$.

If $K$ is any $\eta$-formula, then we define a translation giving a formula $K^+$, as follows: every instance of $\forall x\, x\eta y$ is replaced by $\forall x\,(M(x,y) \rightarrow x\eta y)$, of $\exists x\, x\eta y$ by $\exists x\,(M(x,y) \wedge x\eta y)$ etc., in such a way that $x\eta y$ only occurs when we have $M(x,y)$. Then if $K$ is an $\eta$-formula, the abstraction principle takes the form

$$(*) \qquad \exists y \forall x\,(M(x,y) \rightarrow (x\eta y \leftrightarrow K^+)).$$

Thus we have formalized a framework for talking about properties which are not meaningfully defined everywhere, without altering the underlying logic. The paradoxes give us examples of properties for which $\neg M(x,y)$ holds and the main open problem is to say for which properties we have $M(x,y)$. [16] shows that if we have $\forall x \neq y\; M(x,y)$, then $(*)$ is still inconsistent, and if we take $M(x,y)$ as $\exists z\; x\eta z$, then $(*)$ turns into the class existence axiom of **NBG**.

*Question 3.1* Is there any natural way (syntatic, or otherwise) of saying when $M(x,y)$ holds in the above system?

During the above considerations the variables were assumed to be ranging over properties. Given that a system of properties could be produced, it is often suggested that extensional collections can be obtained just by "taking the extensions of the properties". Two possible interpretations of this view are

(i) the extensions are taken over all possible objects, and
(ii) the extensions are taken over some given collection of individuals,

and we suppose that $x, y, \ldots$ range over the resulting extensions. If (i) is assumed and we suppose that the extensions are already objects, then it seems quite possible for two extensions to have the same extensions as members, but to differ over some property. Thus, such a system would only be extensional if there are urelements in the theory: this seems a little surprising. If (ii) is adopted, then it is not at all clear what the membership relation is intended to mean and it certainly cannot be the original $\eta$. Consequently, we suggest that the notion of taking the extensions of properties to get an extensional system is still in need of clarification.

**3.4** *Other views of properties* The approach to properties with which most people are familiar is that of Zermelo [40]; which was refined in [2] and [14]. Basically, this view assumes the existence of a totality of all sets and works with it exactly as if it were a set: we criticized this in chapter **2**.

Zermelo's original motivation seems to be similar to Russell's notion of a propositional function and, although it is not completely clear, one way of viewing this is as a variable ranging over the first order formulae of a given language (cf. a weak second order logic). However, during his later work (see [41]) Zermelo has extended his ideas to arbitrary propositional functions and it might be possible to make some sense of this idea without using proper classes.

One method of extending **NBG** is considered by Powell [28]. Here, properties are identified with their extensions on $V$ and a different predicate is used for "has the property". This is shown to lead to quite a strong theory with other interesting features, but a point which does not seem to have been considered is why two different properties should not have the same extension over $V$. Also, this approach does not allow quantifiers over properties to occur in the main comprehension axiom.

Another extension of Zermelo's approach is [33], where Reinhardt includes an axiom corresponding to Shoenfield's principle (see section **2.2**). The intended semantics of this system has modal overtones and there are some similarities between the systems of [28] and [33].

Despite our doubts about the ontological overtones of systems such as **NBG**, it is still possible to view these theories as ways of delimiting various levels in the cumulative hierarchy by means of their natural models. There seems to be an implicit belief that any reasonable set theory will have such a natural model, but next we attempt to give a counterexample to this. In [24] we suggested that the following is a reasonable axiom of set theory

(C) If $X$ is a class of ordinals such that for some $\beta$, $X$ is a branch of $E(\beta)$, then $X$ is a set.

We suggest that **NBG** + (C) is a suitable theory as it clearly has no natural models (i.e., models of the form $\langle R\alpha + 1, \epsilon\rangle$). The consistency of **NBG** + (C) can be proved in **MK** as follows. Let $\kappa$ be the least cardinal for which $R\kappa \prec V$ and then $R\kappa \models$ **ZF** with the property that (C) is true for $X$ being any subclass of $R\kappa$. The usual relative consistency proof for **NBG** and **ZF** (see [27]) then gives a model $\langle R\kappa \cup A, \epsilon\rangle$ of **NBG** + (C), for some $A \subseteq R\kappa + 1$.

Of course, **NBG** + (C) is not a reasonable set theory from our point of view because of the existence of proper classes, but it might be possible to include the essence of its axioms in a modified version of **ZF** (strong replacement is catered for in a mild second order logic so (C) is the only remaining problem). Also, the fact that **MK** + (C) is inconsistent can be taken as a condemnation of the naive approach to proper classes.

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