Notre Dame Journal of Formal Logic
Volume XX, Number 1, January 1979
NDJFAM

THE COMPLETENESS OF STOIC PROPOSITIONAL LOGIC

IAN MUELLER

In his comprehensive study [8] of Stoic logic Mates described a system of propositional logic based on five *anapodeiktoi* or undemonstrated arguments and four *themata* or rules for transforming valid arguments into valid arguments. His formulation of the *anapodeiktoi* was based on ancient schematic representations of them, such as the following:

> [The mood or schema] of the first *anapodeiktos* is 'If the first, the second; the first; therefore the second.' Of the second: 'If the first, the second; not the second; therefore not the first.' Of the third: 'Not both the first and the second; the first; therefore not the second.' (Sextus Empiricus [9] 227)
>
> A fourth *anapodeiktos* is . . . for example: 'Either the first or the second; but the first; therefore not the second.' (Diogenes Laertius [4] 81)
>
> This is the fifth: 'Either the first or the second; but not the first; therefore the second.' (Martianus Capella [7] 420)

There survives one statement of the first *thema*:

> There is another method of proof common to all, even the undemonstrateds, which is called *ad absurdum* and, by the Stoics, the first *constitutio* or *expositum* (apparently attempts to render *thema* in Latin). They state it thus: 'If from two [propositions] some third is deduced, either of them with the opposite of the conclusion implies the opposite of the other.' (Apuleius [2] 191)

There are two different but roughly equivalent statements of the third *thema*:

> The compass of the so-called third *thema* is this: 'When from two [propositions] some third is inferred and one of the two is taken from external premisses, then the same [conclusion] can be inferred from the other [proposition] and the external premisses of the first.' (Alexander [1] 278)
>
> The formulation [of the third *thema*] according to the earlier [Stoics] is this: 'If from two [propositions] some third is inferred and the conclusion with some external [proposition] implies something, this same thing can be inferred from the first two with the addition of the external [proposition].' (Simplicius [11] 236)

Received January 5, 1976

About the second and fourth *themata* we have no information, except that Alexander, a consistently negative critic of Stoic logic, considered them both to be alternative versions of the third.

Mates suggested that the Stoics claimed completeness for the system of five *anapodeiktoi* and four *themata*, but concluded, "Whether Stoic logic was in fact complete cannot be decided until we know all four of the meta-rules (*themata*) for analyzing arguments." (Mates [8], p. 82) In 1957 Becker [3] published an argument for the completeness of a system he called the Stoic propositional logic. In fact he tried to show that all truth-functionally valid arguments with formulas containing only conjunction and negation as connectives are derivable in the system. He then used the definability of material implication and exclusive disjunction in terms of conjunction and negation to extend the result to all the basic Stoic connectives. In 1964 the Kneales described a somewhat different system of which they said, with a reference to Becker's work, "It can be shown that the Stoic system as we have presented it . . . is complete in a strict modern sense with respect to conjunction and negation . . . The result is not very important if, as we have argued, the early Stoics did not think of conditional and disjunctive statements as definable by reference to conjunctions and negations." ([6], p. 174)

In this paper I wish to pursue in more detail the question of the completeness of Stoic propositional logic. I shall bring out certain anomalies in Becker's argument which obscure the precise sense in which his system is complete. The Kneales' system will be shown to be complete in a stronger sense than Becker's but not to be as historically plausible a reconstruction of the Stoic theory. In conclusion I shall suggest a modification of both systems which is historically more plausible than either and also complete in the stronger sense. In the course of the paper I will also discuss other logical and historical points about the systems. I shall take for granted the truth-functionality of the Stoic propositional connectives but disregard interdefinability relationships. I will also formulate the systems of Becker and the Kneales in ways which diverge slightly but unproblematically from their own presentations.

Becker's argument is based on Gentzen's calculus of sequents. He formulates the *anapodeiktoi* and *themata* in Gentzen's notation and then reduces his own system to an unspecified version of Gentzen's calculus known to be complete. In this paper I presuppose familiarity with Gentzen's notation and ideas. (See Gentzen [5].) However, for the sake of specificity I shall reproduce the essential inference schemata of a simplified Gentzen-style system **G** which seems appropriate for comparison with Stoic propositional logic. The schemata in question are those for the introduction and elimination of the propositional connectives. The schemata for 'or' are left out because Gentzen rules are based on inclusive rather than Stoic exclusive disjunction.

Introduction	Elimination
$\dfrac{A, \Gamma \rightarrow B}{\Gamma \rightarrow A \supset B}$	$\dfrac{\Gamma \rightarrow A \qquad \Delta \rightarrow A \supset B}{\Gamma, \Delta \rightarrow B}$

& $\dfrac{\Gamma \rightarrow A \qquad \Delta \rightarrow B}{\Gamma, \Delta \rightarrow A \,\&\, B}$ $\qquad$ $\dfrac{\Gamma \rightarrow A \,\&\, B}{\Gamma \rightarrow A}$ $\quad$ $\dfrac{\Gamma \rightarrow A \,\&\, B}{\Gamma \rightarrow B}$

$\neg$ $\dfrac{A, \Gamma \rightarrow B \qquad A, \Delta \rightarrow \neg B}{\Gamma, \Delta \rightarrow \neg A}$ $\qquad$ $\dfrac{\Gamma \rightarrow \neg\neg A}{\Gamma \rightarrow A}$

In addition to these schemata **G** includes "structural" schemata for rearranging premisses, introducing additional premisses, and eliminating redundant ones. In the sequel I take these schemata and their application for granted. A derivation in **G** is a system of sequents in tree form with uppermost or initial sequents of the form $A \rightarrow A$ and subsequent sequents derived from immediate predecessors using the inference schemata. It can be shown that any truth-functionally valid sequent containing only $\neg$ and & or $\neg$ and $\supset$ is derivable using only initial sequents, structural schemata, and the inference schemata for those connectives. A valid sequent containing all three connectives is derivable using the full strength of **G**.

The Stoic *anapodeiktoi* correspond to initial sequents. In Mates's formulation they would run

S1M $A \supset B, A \rightarrow B$
S2M $A \supset B, \neg B \rightarrow \neg A$
S3M $\neg(A \,\&\, B), A \rightarrow \neg B$
S4M $A \veebar B, A \rightarrow \neg B$
S5M $A \veebar B, \neg A \rightarrow B$.

I use $\veebar$ for exclusive disjunction, $\vee$ for inclusive. The *themata* correspond to inference schemata, but whereas the Gentzen schemata allow sequents with any number of premisses, including 0, the *themata* seem to have been formulated in terms of pairs of premisses. However, we know the Stoics considered arguments with more than two premisses, and such arguments must be permitted to make possible a claim to full completeness. Hence we formulate Mates's first and third *themata* as

SIM $\dfrac{A, \Gamma \rightarrow B}{\neg B, \Gamma \rightarrow \neg A}$

SIIIM $\dfrac{\Gamma \rightarrow A \qquad A, \Delta \rightarrow B}{\Gamma, \Delta \rightarrow B}$

SIIIM corresponds to the redundant cut schema in Gentzen systems. Since we incorporate the structural rules into representations of the Stoic system, there are in the representation of Mates's system derivations of sequents with one premiss, e.g.

$$\neg P \supset P, \neg P \rightarrow P \qquad \text{(S1M)}$$
$$\neg P \rightarrow \neg(\neg P \supset P) \qquad \text{(SIM)}$$

The Kneales ([6], p. 174) object to the derivability of such sequents on the grounds that the Stoic Chrysippus "explicitly excluded" arguments with one premiss. However, the only one-premissed arguments to which we know Chrysippus objected were non-truth-functional ones such as 'You breathe;

therefore you are alive.' A more serious objection would be to the inclusion of the schemata permitting the introduction of redundant premisses. For, according to Sextus Empiricus ([10], 146 ff.), arguments with a redundant premiss were considered invalid by the Stoics. Mates ([8], p. 83) expresses disbelief in this assertion. Clearly there could be no question of completeness in the modern sense unless such arguments were permitted.

With the inclusion of structural rules, the system under consideration, call it **M**, permits the derivation of sequents with any positive number of premisses. However, since application of the inference and structural schemata to sequents with premisses never eliminates those premisses entirely, there are no derivations in **M** of sequents without premisses. Nor is the sequent $P \to P$ derivable. To show this we show that there are no derivations in **M** of sequents $\Gamma \to A$ in which Γ contains nothing other than occurrences of P. Imagine the derivation of such a sequent with the least possible number of applications of SIIIM. Since the sequent is not an instance of one of S1M-S5M nor inferrable by SIM, the last inference of the derivation can be assumed to look like this:

$$\frac{\theta \to B \qquad B, \Delta \to A}{\Gamma \to A,} \qquad \text{(SIIIM)}$$

where θ and Δ contain nothing but occurrences of P. But then $\theta \to B$ is a sequent of the kind in question derivable with fewer applications of SIIIM.

It is relatively clear that there is no plausible way of filling out the *themata* to make $P \to P$ derivable. Becker simply adds $A \to A$ to the Stoic initial sequents, calling it the "self-evident rule." ([3], p. 41) In his reconstruction of the Stoic *anapodeiktoi* Becker relies not on ancient schematic representations but on verbal representations such as the following:

> They envisage many *anapodeiktoi* but mostly set out five to which all the rest seem to be referred. The first is the one which infers the consequent from a conditional and its antecedent, . . . the second the one which infers the opposite of the antecedent from a conditional and the opposite of its consequent, . . . the third the one which infers from the negation of a conjunction and one of its conjuncts the opposite of the other, . . . the fourth the one which infers from a disjunction and one of its disjuncts the opposite of the other, . . . the fifth the one which infers from a disjunction and the opposite of one of its disjuncts the other disjunct. (Sextus Empiricus [10] 157-58)

Becker insists on the significance of the word 'opposite' (*antikeimenon*, *contrarium*) in these formulations and in Apuleius's formulation of the first *thema* quoted above. To represent Becker's account of this word I introduce the symbol $\ulcorner$, to be read 'opposite.' Becker does not give a precise definition of opposition and may have in mind either

D1. $\ulcorner A$ is B if A is of the form $\neg B$; otherwise it is $\neg A$,

or

D2. $\ulcorner A$ is either $\neg\neg B$ or B if A is of the form $\neg B$; otherwise it is $\neg A$.

I shall show shortly that the choice of definitions does not affect the strength of Becker's system. However, it is generally convenient for me to make the second definition the basis of my discussion because with it $\ulcorner\ulcorner A$ can always be taken to be A and Becker's five *anapodeiktoi*,

$$\begin{array}{lll} \text{S1B} & A \supset B, A \rightarrow B & \\ \text{S2B} & A \supset B, \ulcorner B \rightarrow \ulcorner A & \\ \text{S3B} & \neg(A \,\&\, B), A \rightarrow \ulcorner B & (\neg(A \,\&\, B), B \rightarrow \ulcorner A) \\ \text{S4B} & A \veebar B, A \rightarrow \ulcorner B & (A \veebar B, B \rightarrow \ulcorner A) \\ \text{S5B} & A \veebar B, \ulcorner A \rightarrow B & (A \veebar B, \ulcorner B \rightarrow A), \end{array}$$

include S1M-S5M. On the first definition if A is $\neg P$, $\ulcorner A$ is P, and $\neg P \supset Q$, $\neg Q \rightarrow \neg\neg P$ is not an instance of S2B although an instance of S2M. The sequents in parentheses are justified by the ancient verbal formulations but not included by Becker. In fact they are equivalent to their counterparts, given Becker's formulation of the first *thema*:

$$\text{SIB} \ \frac{A, \Gamma \rightarrow B}{\ulcorner B, \Gamma \rightarrow \ulcorner A.}$$

The first and the second *anapodeiktoi* are likewise equivalent, but the second would seem to be derivable from the first on any reasonable rendering of them and the first *thema*.

Becker identifies the second *thema* with a "dialectical theorem for the analysis of syllogisms" enunciated by Sextus:

> When we have the premisses which imply some conclusion, we have that conclusion potentially in the premisses even if it is not explicitly stated. ([9] 231)

Sextus's illustrations of the application of this theorem show that it amounts to SIIIM. Unlike the ancient statements of the third *thema*, Sextus's theorem and his applications of it are not restricted to arguments with pairs of premisses. Thus Sextus provides some justification for representing the third *thema* as SIIIM. However, he does not provide any basis for treating the second and third *themata* as logically independent rules. Alexander's lumping together of them suggests that they may not have been independent. In any case Becker treats them as essentially equivalent. In representing his system I shall take SIIIB to be SIIIM and not formulate a second *thema*.

For the fourth *thema* Becker proposes

$$\text{SIVB} \ \frac{A, B, \Gamma \rightarrow C}{A \,\&\, B, \Gamma \rightarrow C},$$

although admitting as a possibility

$$\text{SIVB}' \ \frac{A, \Gamma \rightarrow B}{\Gamma \rightarrow A \supset B},$$

the schema for $\supset$-introduction. Becker acknowledges the absence of any kind of ancient formulation of SIVB but explains this absence on the grounds that a Peripatetic like Alexander would not have understood SIVB because of the tendency to slur over the difference between a string of premisses and their conjunction. Becker also cites the important role of SIVB in his completeness proof. S1B-S5B, SIB, SIIIB can reasonably be ascribed to the Stoics. SIVB is a conjecture where little more than conjecture is possible. To render a formal completeness proof even minimally possible it is necessary to add the structural rules and, as further initial sequents, Becker's "self-evident"

$$\text{S6B} \quad A \to A.$$

As already remarked, Becker's strategy is to show that all truth-functionally valid sequents containing only $\neg$ and & are provable in his system. Because such sequents are provable in **G** using only the inference schemata for $\neg$ and &, it suffices to show that derivations of the upper sequent or sequents of these rules can be extended in Becker's system to a derivation of the lower sequent. SIVB is in fact simply an alternative formulation of &-elimination. For, given a derivation of

$$\Gamma \to A \,\&\, B,$$

one can add

$$\begin{array}{rcr} & A, B \to A & \text{(S6B)} \\ & A \,\&\, B \to A & \text{(SIVB)} \\ \Gamma \to A. & & \text{(SIIIB)} \end{array}$$

Clearly it is also possible to extend a derivation of $\Gamma \to A \,\&\, B$ to one of $\Gamma \to B$. To see that SIVB would be obtainable if &-elimination had been taken as the fourth *thema*, one need only note that to a derivation of

$$A, B, \Gamma \to C$$

one could add

$$\begin{array}{ccr} A \,\&\, B \to A \,\&\, B & A \,\&\, B \to A \,\&\, B & \text{(S6B)} \\ A \,\&\, B \to A & A \,\&\, B \to B & \text{(\&-elimination)} \\ \multicolumn{2}{c}{A \,\&\, B, B, \Gamma \to C} & \text{(SIIIB)} \\ \multicolumn{2}{c}{A \,\&\, B, \Gamma \to C.} & \text{(SIIIB)} \end{array}$$

&-introduction is equally simple with Becker's formulation of the first *thema*. For to derivations of

$$\Gamma \to A \qquad \Delta \to B$$

one can add

$$\begin{array}{cr} \neg(A \,\&\, B), A \to \neg B & \text{(S3B)} \\ A, B \to A \,\&\, B & \text{(SIB)} \\ B, \Gamma \to A \,\&\, B & \text{(SIIIB)} \\ \Gamma, \Delta \to A \,\&\, B. & \text{(SIIIB)} \end{array}$$

An application of SIM to $\neg(A \,\&\, B), A \to \neg B$ would yield only

$$A, \neg\neg B \to \neg\neg(A \,\&\, B),$$

and Becker's proof would not go through. It would go through if one could establish

$$\text{T1} \;\; \neg\neg A \rightarrow A$$
$$\text{T2} \;\; A \rightarrow \neg\neg A.$$

The first of these is obviously equivalent to $\neg$-elimination in Becker's system. Becker supports its inclusion with the words, "This rule corresponds to the concept of opposite; it expresses the elimination of a negation through a second (the Stoic rule of hypernegation)." ([3], p. 48) The one ancient text illustrating this rule is somewhat obscure:

> A hypernegation is a negation of a negation, 'Not it is not day.' It posits (*tithemi*) 'It is day.' (Diogenes Laertius [4] 69)

Even if this statement is accepted as establishing that the Stoics acknowledged elimination of a double negation as a logical principle, Becker's treatment of the principle seems to amount to little more than adding T1 to the initial sequents of his system. In fact it is derivable by two applications of SIB:

$$\neg\neg A, \neg\neg\neg\neg A \rightarrow \neg\neg A \qquad \text{(S6B)}$$
$$\neg\neg A, \neg A \rightarrow \neg\neg\neg A \qquad \text{(SIB)}$$
$$\neg\neg A \rightarrow A. \qquad \text{(SIB)}$$

Since T2 follows from $\neg\neg\neg A \rightarrow \neg A$, an instance of T1, by an application of SIB, T2 is also derivable in Becker's system. The derivations of T1 and T2 given here are compatible with either definition of $\ulcorner$. They are also the only crucial uses of the distinction between $\ulcorner$ and $\neg$. For once one has these two sequents, initial double negations can be added and dropped at will using SIIIB. Thus it does not matter which definition of $\ulcorner$ is chosen, and the strength of Becker's system would not be altered if one formulated the *anapodeiktoi* and the first *thema* as in **M** and added T1 and T2 as initial sequents. However, for the purpose of comparing Becker's and the Kneales' interpretations of Stoic logic the difference between SIB and SIM must be borne in mind.

It remains to consider $\neg$-introduction. Here a problem in construing Becker's presentation arises. The schema for $\neg$-introduction in **G** is not contained in Becker's system. For if it were, the sequent $\rightarrow \neg(A \,\&\, \neg A)$ would be derivable as follows:

$$A, \neg A \rightarrow A \qquad A, \neg A \rightarrow \neg A \qquad \text{(S6B)}$$
$$A \,\&\, \neg A \rightarrow A \qquad A \,\&\, \neg A \rightarrow \neg A \qquad \text{(SIVB)}$$
$$\rightarrow \neg(A \,\&\, \neg A). \qquad (\neg\text{-introduction})$$

However, since the structural rules and the inference schemata of Becker's system when applied to sequents with at least one premiss yield such sequents, no sequent without premisses is derivable and

I. *The system* **B** *consisting of* S1B-S6B *and* SIB, SIIIB, SIVB *is incomplete in the sense that if A is a valid formula* $\rightarrow A$ *is not a derivable sequent.*

I am not sure exactly what Becker has in mind by $\neg$-introduction, but the following weaker form of the rule is contained in **B**:

-introduction' $\dfrac{A, B, \Gamma \to C \qquad A, B, \Delta \to \neg C}{B, \Gamma, \Delta \to \neg A.}$

For to derivations of

$$A, B, \Gamma \to C \qquad A, B, \Delta \to \neg C$$

one can add

$$\begin{array}{cr} C, A, \Delta \to \neg B & \text{(SIB)} \\ A, B, \Gamma, \Delta \to \neg B & \text{(SIIIB)} \\ B, \Gamma, \Delta \to \neg A. & \text{(SIB)} \end{array}$$

As a result it can be shown that

II **B** *is complete in the sense that every valid sequent with at least one premiss and containing only* $\neg$ *and* & *as connectives is derivable.*

I give a brief sketch of the proof. First one establishes

IIa *Let A be a formula containing only $\neg$, &, and no propositional letters other than the distinct $P_1, \ldots, P_n$; let there be an assignment of truth values to these letters, and let P_i^* be P_i if the assignment to P is truth; otherwise let it be $\neg P_i$; similarly let A^* be A if A is true under the assignment, otherwise $\neg A$. Then $P_1^*, \ldots, P_n^* \to A^*$ is derivable in* **B**.

The proof of **IIa** is standard and does not depend on the full strength of $\neg$-introduction.

IIb *If $A, B, \Gamma \to C$ and $\neg A, B, \Gamma \to C$ are derivable, so is $B, \Gamma \to C$.*

For to derivations of

$$A, B, \Gamma \to C \qquad \neg A, B, \Gamma \to C$$

one can add

$$\begin{array}{cr} \neg C, B, \Gamma \to \neg A \qquad \neg C, B, \Gamma \to A & \text{(SIB)} \\ B, \Gamma \to C. & (\neg\text{-introduction', T1, SIIIB}) \end{array}$$

From **IIa** and **IIb** it follows that

IIc *If A is valid, $B \to A$ is derivable.*

For if A is valid and contains all and only the distinct propositional letters $P_1, \ldots, P_n$, then by **IIa** $B, P_1^*, \ldots, P_n^* \to A$ is derivable for every assignment of truth values to $P_1, \ldots, P_n$. n applications of **IIb** yield the derivability of $B \to A$. Thus although for valid A, $\to A$ is not derivable in **B**, $B \to A$ is, for any formula B.

To complete the proof of **II**, suppose that the sequent $A_1, \ldots, A_n \to A$ is valid. Then so is the formula $\neg((\ldots(A_1 \& A_2) \& \ldots \& A_n) \& \neg A)$. Hence by **IIc** there is a derivation of

$A_1 \to \neg((\ldots(A_1 \,\&\, A_2) \,\&\, \ldots \,\&\, A_n) \,\&\, \neg A)$.

To this may be added

$$\neg((\ldots(A_1 \,\&\, A_2) \,\&\, \ldots \,\&\, A_n) \,\&\, \neg A),\ (\ldots(A_1 \,\&\, A_2) \,\&\, \ldots \,\&\, A_n) \to A \quad \text{(S3B)}$$
$$A_1,\ (\ldots(A_1 \,\&\, A_2) \,\&\, \ldots \,\&\, A_n) \to A. \quad \text{(SIIIB)}$$

The desired sequent follows by SIIIB from this last one and

$$A_1, \ldots, A_n \to (\ldots(A_1 \,\&\, A_2) \,\&\, \ldots \,\&\, A_n);$$

to show that this sequent is derivable I sketch a derivation.

$$\neg(A_1 \,\&\, A_2),\ A_1 \to \neg A_2 \quad \text{(S3B)}$$
$$A_1,\ A_2 \to A_1 \,\&\, A_2 \quad \text{(SIB)}$$
$$\neg((A_1 \,\&\, A_2) \,\&\, A_3),\ A_1 \,\&\, A_2 \to \neg A_3 \quad \text{(S3B)}$$
$$A_1 \,\&\, A_2,\ A_3 \to ((A_1 \,\&\, A_2) \,\&\, A_3) \quad \text{(SIB)}$$
$$A_1, A_2, A_3 \to ((A_1 \,\&\, A_2) \,\&\, A_3) \quad \text{(SIIIB)}$$
$$\vdots$$
$$A_1, A_2, \ldots, A_{n-1} \to (\ldots(A_1 \,\&\, A_2) \,\&\, \ldots \,\&\, A_{n-1}) \quad \text{(SIIIB)}$$
$$\neg((\ldots(A_1 \,\&\, A_2) \,\&\, \ldots \,\&\, A_{n-1}) \,\&\, A_n),$$
$$(\ldots(A_1 \,\&\, A_2) \,\&\, \ldots \,\&\, A_{n-1}) \to \neg A_n \quad \text{(S3B)}$$
$$(\ldots(A_1 \,\&\, A_2) \,\&\, \ldots \,\&\, A_{n-1}),\ A_n \to$$
$$((\ldots(A_1 \,\&\, A_2) \,\&\, \ldots A_{n-1}) \,\&\, A_n) \quad \text{(SIB)}$$
$$A_1, A_2, \ldots, A_n \to (\ldots(A_1 \,\&\, A_2) \,\&\, \ldots \,\&\, A_n). \quad \text{(SIIIB)}$$

Since completeness in the sense of **II** includes the derivability of $B \to A$, for every valid formula A, the failure of completeness in the sense of **I** is unlikely to be an advantage for a reconstruction of Stoic logic. The Kneales' system, as we shall see, is complete in both senses. Another disadvantage of Becker's reconstruction is the extreme arbitrariness of SIVB. The same can be said of SIVB′, which I shall discuss in connection with the Kneales' system. Perhaps the least satisfactory feature of Becker's completeness argument is its failure to use any *anapodeiktos* but the third. Even Becker's extension of the proof to the other connectives *via* truth-functional definitions does not provide any role for the other *anapodeiktoi* unless one places great weight on Becker's cryptic remark ([3], p. 45) that these "define in an implicit way the functors $\veebar$ and (to some extent) $\supset$." (At best these *anapodeiktoi* show the conditions under which a conditional or disjunction is false and hence can be said to specify the conditions for the elimination, but not for the introduction, of $\veebar$ and $\supset$.)

Before considering the Kneals' system I would like to establish some further facts about **B** and related systems.

III *The system* **B**′ *which results from* **B** *by replacing* SIVB *with* SIVB′ *is complete in the sense that every valid sequent containing only* $\neg$ *and* $\supset$ *is derivable.*

Proof: Since SIVB′ is $\supset$-introduction, and $\neg$-elimination is contained in S6B and SIB, it need only be shown that $\supset$-elimination and $\neg$-introduction hold in

B$'$. But any system with the first *anapodeiktos* and Sextus's dialectical theorem includes $\supset$-elimination. For to derivations of

$$\Gamma \rightarrow A \qquad\qquad \Delta \rightarrow A \supset B$$

one can add

$$A \supset B, A \rightarrow B \qquad \text{(S1B)}$$
$$A \supset B, \Gamma \rightarrow B \qquad \text{(SIIIB)}$$
$$\Gamma, \Delta \rightarrow B. \qquad \text{(SIIIB)}$$

For $\neg$-introduction we first derive

$$\text{T3} \quad A \supset B, A \supset \neg B \rightarrow \neg A.$$

$$A \supset \neg B, A \rightarrow \neg B \qquad \text{(S1B)}$$
$$A \supset B, \neg B \rightarrow \neg A \qquad \text{(S2B)}$$
$$A \supset \neg B, A, A \supset B \rightarrow \neg A \qquad \text{(SIIIB)}$$
$$A \supset \neg B, A \rightarrow \neg(A \supset B) \qquad \text{(SIB)}$$
$$A \supset B, A \supset \neg B \rightarrow \neg A. \qquad \text{(SIB)}$$

But then to derivations of

$$A, \Gamma \rightarrow B \qquad A, \Delta \rightarrow \neg B$$

one can add

$$\Gamma \rightarrow A \supset B \qquad \Delta \rightarrow A \supset \neg B, \qquad \text{(SIVB}')$$

from which $\Gamma, \Delta \rightarrow \neg A$ follows by two applications of SIIIB using T3.

Although **B** is complete with respect to $\neg$ and & and **B**$'$ with respect to $\neg$ and $\supset$,

IV *There are valid sequents $A \rightarrow B$ containing only $\neg$ and $\supset$(&) which are not derivable in* **B** (**B**$'$).

Proof: If $A \supset B$ is interpreted as $A \equiv B$, S1B-S5B are truth-functionally valid, and SIB, SIIIB, SIVB preserve validity. But $P \rightarrow Q \supset P$ is not valid under this interpretation of $\supset$. Hence it is not derivable in **B**, and SIVB$'$ is not contained in **B**.

On the other hand, if A & B is read as $A \vee B$, S1B-S5B remain truth-functionally valid, and SIB, SIIIB, SIVB$'$ preserve validity. Since P & $Q \rightarrow P$ is not valid under this interpretation of &, it is not derivable in **B**$'$, and SIVB is not contained in **B**$'$.

V $P \rightarrow P \veebar \neg P$ *is not derivable in* **B** *or* **B**$'$.

Proof: If $A \veebar B$ is taken to be always false, S1B-S5B are truth-functionally valid, and SIB, SIIIB, SIVB, SIVB$'$ preserve validity. Since $P \rightarrow P \veebar \neg P$ is not valid under this interpretation, it is derivable in neither **B** nor **B**$'$.

Thus neither **B** nor **B**$'$ is complete with respect to $\neg$ and $\veebar$. It is perhaps worth remarking that a standard completeness argument will establish

VI **B** (**B′**) *becomes complete with respect to* $\neg$, $\veebar$, *and* & ($\supset$) *in the sense of* **II** (**I**) *with the addition of initial sequents for* $\veebar$*-introduction, namely*

$$A, \neg B \rightarrow A \veebar B \qquad \neg A, B \rightarrow A \veebar B.$$

I turn now to the Kneales' system **K**, which I take to be the following:

S1K $A \supset B, A \rightarrow B$
S2K $A \supset B, \neg B \rightarrow \neg A$
S3K $\neg(A \,\&\, B), A \rightarrow \neg B$
S4K $A \veebar B, A \rightarrow \neg B$
S5K $A \veebar B, \neg B \rightarrow A$
S6K $A \rightarrow A$
S7K $\rightarrow A \veebar \neg A$

SIK $$\frac{A, \Gamma \rightarrow B}{\neg B, \Gamma \rightarrow \neg A}$$

SIIIK $$\frac{\Gamma \rightarrow A \qquad A, \Delta \rightarrow B}{\Gamma, \Delta \rightarrow B}$$

SIVK $$\frac{A_1, \ldots, A_n \rightarrow A}{\rightarrow (\,.\,.\,(A_1 \,\&\, A_2) \,\&\, .\,.\,\&\, A_n) \supset A}$$

plus structural schemata.

It is to be noted that S5K differs not only from S5B in the substitution of $\neg$ for $\ulcorner$ but also from S5M in having $\neg$ applied to the second rather than the first disjunct of the first premiss. The Kneales' formulation of the fifth *anapodeiktos* depends not on ancient schematic representations of it but on an ancient example: "Either it is day or it is night; not it is night; therefore it is day." Although the sources for this example, Diogenes, Sextus, and Galen, are probably more reliable than the sources of the schema, Cicero and Martianus Capella, and although the other ancient examples which correspond to the schema are given by authors less reliable than Sextus or Diogenes, it would seem rash to prefer S5K over S5M on the basis of the example alone. However, S5K is essential to the Kneales' version of Stoic propositional logic and to its completeness. For

VII *Neither* S5K *nor* T1 *is derivable in a system like* **K** *except for having* S5M *in place of* S5K.

The proof is a standard one of independence. Tables are given which assign a value to any formula, given an evaluation of its component letters. A sequent $A_1, \ldots, A_n \rightarrow A$ ($\rightarrow A$) is said to be correct if the formula $(\,.\,.\,(A_1 \,\&\, A_2) \,\&\, .\,.\,\&\, A_n) \supset A$ (A) always has the value 0. The tables are such that the initial sequents of the system under consideration are correct and the inference schemata preserve correctness but the sequent or sequents to be proved underivable are incorrect. In the present case such tables are

A	$\neg A$
0	1
1	0
2	1

A	B	$A \supset B$	$A \mathbin{\&} B$	$A \mathbin{\overset{\vee}{\vee}} B$
0	0	0	0	1
1	0	0	1	0
2	0	0	2	1
0	1	1	1	0
1	1	0	1	1
2	1	1	1	0
0	2	1	2	1
1	2	0	1	1
2	2	0	2	1

These tables give the formulas corresponding to S5K and T1 the value 1 when A is assigned 2 and B 1. With S5K it is easy to derive both T1 and T2 and hence to eliminate the need for the greater strength of SIB over SIK.

T1	$\rightarrow A \mathbin{\overset{\vee}{\vee}} \neg A$			(S7K)
			$A \mathbin{\overset{\vee}{\vee}} \neg A, \neg\neg A \rightarrow A$	(S5K)
		$\neg\neg A \rightarrow A$		(SIIIK)
T2	$\rightarrow A \mathbin{\overset{\vee}{\vee}} \neg A$			(S7K)
			$A \mathbin{\overset{\vee}{\vee}} \neg A, A \rightarrow \neg\neg A$	(S4K)
		$A \rightarrow \neg\neg A$		(SIIIK)

Both of these derivations use S7K, the status of which in the Kneales' reconstruction of Stoic logic is not completely clear. They speak of the law of the excluded middle as a "supplementary premiss which can be excluded without lack of rigour" ([6], p. 173) and say, apparently in justification of including the law in their reconstruction, ". . . we know that [the Stoics] attached great importance to the principle of excluded middle in the form 'Either the first or not the first', and it is easy to see how they could have used it to prove two important theorems." ([6], p. 168); the theorems are T1 and T2.) As an additional justification, one may note that the Kneales' reconstruction, unlike Becker's, gives a role to the fourth and fifth *anapodeiktoi* in the completeness argument. On the other hand, whatever importance the Stoics attached to the law of the excluded middle, there is no direct evidence that they ever formulated it as an independent principle of their propositional logic. Nor is it derivable from the other principles of **K**, as the proof of **V** shows.

The Kneales do not explicitly state a version of the fourth *thema*. However, their description and use of it make clear that SIVK is what they have in mind even though they wrongly imply that SIVK and SIVB′ are equivalent. ([6], p. 170 fn.) A main advantage of SIVK over either SIVB or SIVB′ is that it was explicitly formulated by the Stoics. (See Mates [8], pp. 74-77.) With SIVK, SIVB and hence &-elimination are easily derivable in **K**. For it should be clear that **K** includes the strength of S1B-S6B, SIB, and SIIIB, and hence includes &-introduction and $\neg$-elimination. Therefore to a derivation of

$$A_1, \ldots, A_n \rightarrow A$$

one can add

$$\rightarrow (\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \supset A \qquad \text{(SIVK)}$$
$$(\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \supset A,$$
$$(\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \rightarrow A \qquad \text{(S1K)}$$
$$(\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \rightarrow A. \qquad \text{(SIIIK)}$$

That one can continue this derivation to obtain the desired sequent $A_1 \mathbin{\&} A_2, A_3, \ldots, A_n \rightarrow A$ is shown by the derivability of the sequent $A_1 \mathbin{\&} A_2, A_3, \ldots, A_n \rightarrow (\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n)$. But the derivability of this latter sequent is easily surmised from the derivation sketched at the end of the proof of **II**. Since **K** includes &-introduction, &-elimination, and $\neg$-elimination, to establish

VIII **K** *is complete in the sense of* **I**

it suffices to show that $\neg$-introduction is contained in **K**. I first point out that the derivation of T3 given in the proof of **III** goes through in **K**. Hence to derivations of

$$A, A_1, \ldots, A_k \rightarrow B \qquad\qquad A, A_{k+1}, \ldots, A_n \rightarrow \neg B$$

one can add

$$A_1, \ldots, A_n, A \rightarrow B \qquad\qquad A_1, \ldots, A_n, A \rightarrow \neg B$$

(structural rules)

$$\rightarrow ((\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \mathbin{\&} A) \supset B \qquad \rightarrow ((\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \mathbin{\&} A) \supset \neg B. \qquad \text{(SIVK)}$$

Applying SIIIK to these two sequents and the following instance of T3

$$((\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \mathbin{\&} A) \supset B,\ ((\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \mathbin{\&} A) \supset \neg B \rightarrow \neg((\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \mathbin{\&} A)$$

one gets

$$\rightarrow \neg((\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \mathbin{\&} A).$$

Applying SIIIK to this sequent and the following instance of S3K

$$\neg((\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \mathbin{\&} A),\ (\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \rightarrow \neg A$$

one obtains

$$(\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n) \rightarrow \neg A.$$

Finally, if one applies SIIIK to this sequent and the obviously derivable $A_1, \ldots, A_n \rightarrow (\,.\,.\,(A_1 \mathbin{\&} A_2) \mathbin{\&} \,.\,.\, \mathbin{\&} A_n)$, one obtains the desired lower sequent of $\neg$-introduction:

$$A_1, \ldots, A_n \rightarrow \neg A.$$

The Kneales make no claims concerning completeness with respect to the other connectives. Of course **K** is complete with respect to $\supset$ and $\veebar$ if standard truth-functional equivalences are added as definitions. Without such additions

IX *In* **K** *neither* $P \rightarrow Q \supset P$ *nor* $P \veebar Q \rightarrow Q \veebar P$ *is derivable.*

The proof of underivability is of the standard type and depends on the following tables:

A	$\neg A$
0	1
1	0
2	2

A	B	$A \supset B$	$A \& B$	$A \veebar B$
0	0	0	0	1
1	0	0	1	0
2	0	0	2	1
0	1	1	1	0
1	1	0	1	1
2	1	1	1	1
0	2	1	2	1
1	2	0	1	2
2	2	0	1	0

Given these tables, $P \supset (Q \supset P)$ has the value 1 when P is assigned 2 and Q 0, and $(P \veebar Q) \supset (Q \veebar P)$ has the value 1 when P is assigned 1 and Q 2. A corollary of **IX** and **III** is that SIVB′ is not contained in **K**. Clearly also SIVK is not contained in **B′**, since if it were, **B′** would be complete with respect to $\neg$ and &. Nor is SIVK contained in **B**, since no sequents without premisses are derivable in **B**.

The main merit of the Kneales' reconstruction *vis-à-vis* Becker's is that SIVK is a principle apparently formulated by the Stoics. However, the Kneales' addition of S7K and their formulation of the fifth *anapodeiktos* seem rather arbitrary although needed for completeness. Perhaps the most satisfactory reconstruction of Stoic propositional logic now obtainable results from combining Becker's formulation of the five *anapodeiktoi* and first *thema* with the standard formulation of the third *thema* and the Kneales' formulation of the fourth. Of these principles all but the last can with great probability be said to be a fundamental law of Stoic propositional logic. And the last is at least a principle which the Stoics seem to have acknowledged. For completeness the system requires the addition of Gentzen's initial sequents and structural schemata, but it is hard to imagine any complete reconstruction of Stoic logic which would not include these principles. The resulting system **S** is of course only complete with respect to $\neg$ and &. To make it complete for $\supset$ and $\veebar$ it would suffice to add SIVB′ for $\supset$ and the sequents for $\veebar$-introduction described in **VI** for $\veebar$. Such additions would not, however, seem to have historical justification.

The law of the excluded middle, S7K, is not derivable in **S**. (See the proof of **V**.) The motivation for its inclusion in **K** seems to stem from two argument schemata ascribed to the Stoics, namely T3, or the special case of it when A and B are identical, and

$$\text{T4} \quad A \supset A, \neg A \supset A, A \veebar \neg A \rightarrow A.$$

T3 is of course derivable in **S**. So is T4:

$$\neg A \supset A, \neg A \rightarrow A \qquad \text{(S1)}$$

$$\neg A \supset A, \neg A \rightarrow \neg A \qquad \text{(S6)}$$

$$\neg A \supset A \rightarrow \neg\neg A \qquad (\neg\text{-introduction})$$

$$\neg A \supset A \rightarrow A \qquad (\neg\text{-elimination})$$

$$A \supset A, \neg A \supset A, A \veebar \neg A \rightarrow A \qquad \text{(structural rules)}$$

Indeed all of the theorems derived for **K** by the Kneales are derivable in **S**. There seems, then, to be no reason to add the law of the excluded middle to the fundamental principles of Stoic logic.

REFERENCES

[1] Alexander of Aphrodisias, *In Aristotelis Analyticorum Priorum Librum I Commentarium*, M. Wallies (ed.), Berlin (1883).

[2] Apuleius, *Peri Hermeneias*, in *Opera*, P. Thomas (ed.), Leipzig (1908), vol. III.

[3] Becker, O., *Zwei Studien zur Antiken Logik*, Wiesbaden (1957).

[4] Diogenes Laertius, *Life of Zeno*, in *Vitae Philosophorum*, H. S. Long (ed.), Oxford (1964).

[5] Gentzen, G., "Untersuchungen ueber das logische Schliessen. I," *Mathematische Zeitschrift*, vol. 39 (1935), pp. 176-210.

[6] Kneale, W. and M. Kneale, *The Development of Logic*, Oxford (1962).

[7] Martianus Capella, *De Nuptiis Philologiae et Mercurii*, A. Dick (ed.), Leipzig (1925), book III.

[8] Mates, B., *Stoic Logic*, Berkeley and Los Angeles (1961).

[9] Sextus Empiricus, *Adversus Mathematicos*, book VII, in *Opera*, H. Mutschumann (ed.), Leipzig (1914), vol. II.

[10] Sextus Empiricus, *Pyrroneion Hypotyposeon*, book II, in *Opera*, H. Mutschmann (ed.), with revision by J. Mau, Leipzig (1958), vol. I.

[11] Simplicius, *In Aristotelis de Caelo Commentaria*, J. L. Heiberg (ed.), Berlin (1894).

University of Chicago
Chicago, Illinois