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The Predicate Modal Logic of Provability

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Introduction The propositional modal logic of provability *GL* and its arithmetical interpretations have been studied by various logicians: among others, Boolos, De Jongh, Magari and his school, including the author, Smorynski, Solovay, and Visser. As observed by Boolos in the introduction of [4], the arithmetical interpretation of *GL* can be extended to its extension to the language of modal predicate calculus, denoted by *QGL*. By this observation, one might reasonably expect that *QGL* can offer a more complete description of the logic of provability. Unfortunately, however, many desirable properties of *GL* do not extend to its predicate version; for example, in [1], Avron shows that the most natural sequential formulation of *QGL* does not admit cut-elimination (where a similar sequent calculus for *GL* does: see [8], [14], and [19]). In this paper, we show that other important results about *GL* fail to hold for *QGL*; for example, *QGL* is not complete with respect to any class of Kripke frames; moreover, *QGL* is not arithmetically complete, and does not enjoy the fixed point property.

In spite of these negative results, we believe that many aspects of the predicate logic of provability are worthy of further investigation; in particular, since *QGL* is not arithmetically complete, one could try to find new significant provability principles which are arithmetically valid, but not provable in *QGL*. In any case, even if most important problems on *QGL* have a negative solution, there are also positive results: for example, in [1], Avron shows that *QGL* enjoys some closure properties, and that the notion of *PA* validity satisfies

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some kind of disjunction property. In this paper, other positive results, concerning self-reference, are shown; for example, fixed points, when they exist, are unique up to provable equivalence; moreover, the theorem on uniqueness of fixed points for formulas of PA arising from modal formulas $A(p)$ in which p is modalized can be extended in its full generality to the predicate case.

1 ***Preliminaries*** QGL is the predicate version of the modal logic GL; the formulas of QGL are those of the modal predicate calculus (for simplicity, we assume that our language does not contain constants or function symbols); the axioms are those of the predicate calculus for all formulas of QGL together with the schemata

$$\Box A \wedge \Box(A \to B) \to \Box B \quad \text{and} \quad \Box(\Box A \to A) \to \Box A.$$

The rules are those of predicate calculus plus the rule MN: $\dfrac{A}{\Box A}$, where all assumptions on which A depends are axioms of QGL. As in the propositional case, one can show that QGL contains the schema $\Box A \to \Box\Box A$ and is closed under Löb's rule: if $\vdash_{QGL} \Box A \to A$, then $\vdash_{QGL} A$.

Let T be an r.e. extension of PA, and let us associate with every atomic formula P of QGL a formula fP of T whose free variables are exactly those occurring in P; assume that f commutes with the operation of substitution of variables; that is, if $f(P(u_1, \ldots, u_n)) = A(u_1, \ldots, u_n)$ then $f(P(v_1, \ldots, v_n)) = A(v_1, \ldots, v_n)$. (We assume, without loss of generality, that no occurrence of $v_1, \ldots, v_n$ in A is bound; if it is not the case, we replace each bound variable in A by a variable distinct from $v_1, \ldots, v_n$). Then, we define a function $\bar{f}$ from QGL formulas into PA formulas in the following inductive manner:

1. $\bar{f}P = fP$ if P is atomic
2. $\bar{f}$ commutes with all logical connectives and quantifiers
3. $\bar{f}\Box B(v_1, \ldots, v_n) = Pr_T(\overline{\bar{f}B}(\bar{v}_1, \ldots, \bar{v}_n))$, where Pr_T is the usual provability predicate for T, and, for every formula $C(v_1, \ldots, v_n)$, $\overline{C}(\bar{v}_1, \ldots, \bar{v}_n)$ denotes a term for the primitive recursive function $\lambda v_1, \ldots, v_n.\ulcorner C(\bar{v}_1, \ldots, \bar{v}_n)\urcorner$.

$\bar{f}A$ is called *"the value of A under the interpretation f"*. A is called *T-valid* iff, for every interpretation f, $\bar{f}A$ is a theorem of T; it is easily seen that every theorem of QGL is T-valid. If QGL' is an extension of QGL, we say that QGL' is *T-complete* iff the converse holds too, that is if the theorems of QGL' are exactly the T-valid formulas.

The *Barcan Schema* (in short: BS) is the schema $\forall u \Box A(u, \vec{v}) \to \Box \forall u A(u, \vec{v})$, where $\vec{v}$ denotes a (possibly empty) sequence of variables. BS is not PA-valid, because, for every PA sentence ϕ, one has: $\vdash_{PA} \forall x\, Pr_{PA}(\overline{Prf_{PA}(x, \bar{\phi}) \to \phi})$, but if $\nvdash_{PA} \phi$, then $\vdash_{PA} Pr_{PA}(\overline{\forall x\, Prf_{PA}(x, \bar{\phi}) \to \phi})$ (see [6]). Therefore, we do not add BS to QGL.

2 ***Some results on model theory for QGL***

Definition 1 *A* (predicate) *Kripke frame* is a system $\langle X, R, \{W_x\}_{x \in X}\rangle$ such that X is a nonempty set (called "the set of worlds"), R is a binary relation on

X (called "the accessibility relation") and $\{W_x\}_{x \in X}$ is a sequence, indexed with the elements of X, of nonempty sets such that, if xRy, then $W_x \subseteq W_y$.

Definition 2 If $\mathcal{F} = \langle X, R, \{W_x\}_{x \in X}\rangle$ is a Kripke frame, a (predicate) *Kripke model on* $\mathcal{F}$ is a system $\langle X, R, \{W_x\}_{x \in X}, \Vdash\rangle$, where $\Vdash$ is a mapping from the set $\{\langle x, A\rangle$: A is a closed formula with parameters in $W_x\}$ into $\{\top, \bot\}$ such that, writing $x \Vdash A$ for $\Vdash\langle x, A\rangle = \top$, the following conditions hold[1]:

a. $x \Vdash A \vee B$ iff $\Vdash$ is defined on both $\langle x, A\rangle$ and $\langle x, B\rangle$ and either $x \Vdash A$ or $x \Vdash B$
b. $x \Vdash \neg A$ iff $\Vdash$ is defined on $\langle x, A\rangle$ and $x \nVdash A$
c. $x \Vdash \exists u A(u)$ iff for some $\bar{u} \in W_x$ $x \Vdash A(\bar{u})$
d. $x \Vdash \Box A$ iff for every $y \in X$, if xRy, then $y \Vdash A$.

Definition 3 We say that A *is valid in the model* $\langle X, R, \{W_x\}_{x \in X}, \Vdash\rangle$ iff, for every $x \in X$, $x \Vdash A$; we say that A *is valid in the frame* $\mathcal{F}$ iff A is valid in all Kripke models on $\mathcal{F}$; we say that *a set G of sentences is valid in the model* $\mathcal{M}$ (respectively: in the frame $\mathcal{F}$) iff every sentence in G is.

Definition 4 We say that a predicate modal logic L is *complete with respect to the class* **F** of frames (respectively: to the class **M** of models) iff the theorems of L are exactly the formulas which are valid in all frames in **F** (respectively: in all models in **M**).

We can easily prove the following facts (see also [4]):

Proposition 1 *QGL is valid in the frame* $\langle X, R, \{W_x\}_{x \in X}\rangle$ *iff R is transitive and* R^{-1} *is well-founded.*

Proposition 2 *If R is a transitive relation on X, then QGL is valid in the Kripke model* $\langle X, R, \{W_x\}_{x \in X}, \Vdash\rangle$ *iff the following condition holds:*

Condition (+) For every closed formula A with parameters, if there is an $x \in X$ such that $x \Vdash A$, then there is a y such that $y \Vdash A$ and for every z, if yRz, then $z \nVdash A$.

One can prove the following completeness theorem:

Theorem 1 *QGL is complete with respect to the class of all transitive Kripke models satisfying Condition* (+).

Proof: The proof is quite similar to that given in [7] (pp. 174-176) for the modal predicate calculus, therefore we only sketch it. Assume $\nvdash_{\overline{QGL}} A$; we want to construct a model $\mathcal{M} = \langle X, R, \{W_x\}_{x \in X}, \Vdash\rangle$ of QGL such that, for some $x_0 \in X$, $x_0 \Vdash \neg A$. In this model, X will be a subset of the set $\omega^{<\omega}$ of all finite sequences of natural numbers; with every $s \in X$, we associate a Henkin-complete extension T_s of QGL by stages, as follows:

Stage 0: we put the empty sequence ϕ in X and we define T_ϕ to be a Henkin-complete extension of $QGL_+ \neg A$ (that such an extension exists can be shown as in [7]).

Stage n+1: assume that, at Stage n, we have decided which sequences of length $\leqslant n$ belong to X, and we have associated to every such sequence s a suitable

Henkin-complete extension T_s, e.g., QGL. Let $s \in X$ be a sequence of length n. If for no closed formula A in the language of T_s we have $\vdash_{T_s} \Diamond A$, no sequence extending s is placed in X; otherwise, let U_s be the set of all formulas B such that $\vdash_{T_s} \Diamond B$, and let us write U_s as $U_s = \{B_i: i < k\}$ $(1 \leqslant k \leqslant \omega)$. Then, for all $i < k$, we put the sequence[2] $s * i$ in X, and we associate to $s * i$ a Henkin-complete extension T_{s*i} of $QGL \cup \{B_i\} \cup \{B: \vdash_{T_s} \Box B\}$. Now we define: sRt iff t properly extends s; $W_s = \{a: a$ is a constant in the language of $T_s\}$; if $B(v_1, \ldots, v_n)$ is an atomic formula and $a_1, \ldots, a_n$ belong to W_s, we define $s \Vdash B(a_1, \ldots, a_n)$ iff $\vdash_{T_s} B(a_1, \ldots, a_n)$. (Clearly, this requirement completely determines $\Vdash$.)

As in [6], we can show that for every $s \in X$ and for every formula B with parameters in W_s, we have $s \Vdash A$ iff $\vdash_{T_s} A$; then, QGL is valid in $\mathcal{M}$ and $\phi \Vdash \neg A$. Clearly, R is transitive; that Condition (+) holds in $\mathcal{M}$ follows from Proposition 2.

Since Condition (+) involves the forcing relation on all formulas, it is, in general, hard to verify; therefore Theorem 1, which is based on it, does not constitute a satisfactory completeness result; on the other hand, no completeness theorem for QGL can be expressed only in terms of the accessibility relation; indeed, we can prove the following incompleteness result.

Theorem 2 *QGL is not complete with respect to the class of all transitive reversely well-founded frames; therefore by Proposition 1 it is not complete with respect to any class of frames.*

Proof: Let A be the formula $\exists u \Diamond P(u) \wedge \forall v \exists w \Box (P(v) \rightarrow \Diamond P(w))$ where P is a unary predicate letter; we wish to show that $\nvdash_{QGL} \neg A$, but $\neg A$ is valid in all transitive reversely well-founded frames; this claim is contained in the following lemmas:

Lemma 1 *Let $\mathcal{M} = \langle X, R, \{W_x\}_{x \in X}, \Vdash \rangle$ be a transitive Kripke model such that, for some $x_0 \in X$, $x_0 \Vdash A$; then, there is a sequence $\{x_n\}_{n \in \omega}$ of elements of X such that, for every n, $x_n R x_{n+1}$.*

Proof: Assume $x_0 \Vdash A$; since $x_0 \Vdash \exists u \Diamond P(u)$, there is an $a_0 \in W_{x_0}$ such that $x_0 \Vdash \Diamond P(a_0)$; therefore, there is an x_1 such that $x_0 R x_1$, $x_1 \Vdash P(a_0)$. Now, we argue by induction: assume that, for some n, we have defined two finite sequences $x_0, x_1, \ldots, x_{n+1}$ and $a_0, a_1, \ldots, a_n$ such that, for all $i \leqslant n$, $a_i \in W_{x_0}$, $x_i R x_{i+1}$, and $x_{i+1} \Vdash P(a_i)$; since R is transitive, $x_0 \Vdash \exists w \Box (P(a_n) \rightarrow \Diamond P(w))$ and $x_{n+1} \Vdash P(a_n)$, there is an $a_{n+1} \in W_{x_0}$ such that $x_{n+1} \Vdash \Diamond P(a_{n+1})$, therefore, there is an x_{n+2} such that $x_{n+1} R x_{n+2}$, $x_{n+2} \Vdash P(a_{n+1})$. This completes the inductive step; then, Lemma 1 follows.

Lemma 2 *There is a model $\mathcal{M} = \langle X, R, \{W_x\}_{x \in X}, \Vdash \rangle$ of QGL such that, for some $x_0 \in X$, $x_0 \Vdash A$.*

Proof: Let $\mathcal{N} = \langle N, +, \cdot, s, 0 \rangle$ be a nonstandard model of PA; we define the desired model $\mathcal{M}$ as follows:

a. $X = N \cup \{x_0\}$ (where $x_0 \notin N$)
b. For $x, y \in X$, we define xRy iff either $x = x_0$ and $y \neq x_0$ or $x, y \neq x_0$ and $\mathcal{N} \models y < x$

c. $W_x = \begin{cases} N \text{ if } x \neq x_0 \\ \omega \text{ if } x = x_0 \end{cases}$

d. If P^k is any k-ary predicate symbol different from P, we define $x \Vdash P^k(a_1, \ldots, a_k)$ for every $x \in X$ and for every $a_1, \ldots, a_k \in W_x$; the forcing relation on x_0 for formulas of the form $P(a)(a \in W_{x_0})$ can be defined quite arbitrarily; now, let us consider a nonstandard element b of N; for $x \in X$, $x \neq x_0$ and for $a \in W_x$ we define $x \Vdash P(a)$ iff $\mathcal{N} \models x \cdot a > b$.

The forcing relation on elements $x \neq x_0$ is definable in $\mathcal{N}$; more precisely, we can associate with every QGL formula $B(v_0, v_2, \ldots, v_{2n})$ containing no variables indexed with odd numbers a formula[3] $fB(v_1, v_0, v_2, \ldots, v_{2n})$ of PA with parameters in N such that, for all $x, a_1, \ldots, a_n \in N$, we have $x \Vdash B(a_1, \ldots, a_n)$ iff $\mathcal{N} \models fB(x, a_1, \ldots, a_n)$. ($fB$ is defined in an obvious manner if B is atomic; moreover, we require that f commutes with all connectives and quantifiers; lastly, we define $f \Box B(v_0, v_2, \ldots, v_{2n})$ to be the formula $\forall w[w < v_1 \rightarrow fB(w, v_1, v_0, \ldots, v_{2n})]$, where w is a variable, indexed with an odd number, which does not occur in fB.)

Since the induction schema holds in $\mathcal{N}$, if for some formula of the form $fB(v_1, v_0, v_2, \ldots, v_{2n})$ and for some $a_1, \ldots, a_n \in N$ there is an $x \in N$ such that $\mathcal{N} \models fB(x, a_1, \ldots, a_n)$, there is a $y \in N$ such that $\mathcal{N} \models fB(y, a_1, \ldots, a_n)$ and for all $z \in N$, if $\mathcal{N} \models z < y$, then $\mathcal{N} \models \neg fB(z, a_1, \ldots, a_n)$. Since, for all $x, y \neq x_0$, xRy iff $\mathcal{N} \models y < x$, Condition (+) is satisfied; since R is transitive, $\mathcal{M}$ is a model of QGL.

Now, let us prove that $x_0 \Vdash A$; let x be an element of N such that $\mathcal{N} \models x > b$; since $\mathcal{N} \models x \cdot 1 > b$, $x \Vdash P(1)$, therefore $x_0 \Vdash \exists v \Diamond P(v)$; furthermore, if y is an arbitrary element of X such that x_0Ry and, for some $n \in W_{x_0}$, $y \Vdash P(n)$, then $\mathcal{N} \models y \cdot n > b$ therefore y must be nonstandard; so, $\mathcal{N} \models y > n + 1$, whence $\mathcal{N} \models (y - 1)(n + 1) = y \cdot n + y - n - 1 > y\ n > b$; this implies that $y - 1 \Vdash P(n + 1)$. Since $yRy - 1$, we can conclude $y \Vdash \Diamond P(n + 1)$; the arbitrariness of y and n yields $x_0 \Vdash \forall v \exists w \Box(P(v) \rightarrow \Diamond P(w))$, therefore the claim follows.

3 *Arithmetical incompleteness*

In [1], Avron asks whether QGL is PA complete. The analogous problem for GL has been solved affirmatively by Solovay in [18]; on the contrary, we show that, in our case, the problem has a negative solution.

Theorem 3 *QGL is not PA complete.*

Proof: Let T be a finitely axiomatizable consistent theory such that $\vdash_{\overline{PA}} Con_T \rightarrow Con_{PA+Con_{PA}}$. (For example, we can choose $T = NGB$.) Let $[T]$ be the conjunction of all axioms of T, and let A denote the formula $\Diamond[T] \rightarrow \Diamond\Diamond\top$. (We can assume, without loss of generality, that QGL contains the language of T.) We claim that A is PA valid, but is not provable in QGL; to see this, let f be any interpretation, and let $\bar{T}$ be the theory whose axioms are exactly those of the form $\bar{f}B$: B an axiom of T; then, $\bar{f}A$ is provably equivalent to the formula $Con_{PA+\bar{T}} \rightarrow Con_{PA+Con_{PA}}$. In order to show that $\bar{f}A$ is a theorem of PA, we first prove the following lemma.

Lemma 3 *If $B_1, \ldots, B_n$, is a proof of B_n in T, then $\bar{f}B_1, \ldots, \bar{f}B_n$, is a proof of $\bar{f}B_n$ in $\bar{T}$.*

Proof: We argue by induction on the length of the proof: if B_i is a logical axiom, also $\bar{f}B_i$ is; if B_i is an axiom of T, $\bar{f}B_i$ is an axiom of $\bar{T}$; lastly, every application of any rule of predicate calculus yields an application of the rule itself.

Formalizing the proof of Lemma 3 in PA we obtain, for every $\Box$-free formula C of QGL:

$$\vdash_{\overline{PA}} Pr_T\bar{C} \to Pr_{\bar{T}}\overline{\bar{f}C}.$$

Therefore, $\vdash_{\overline{PA}} Con_{\bar{T}} \to Con_T$. Then, we can deduce

$$\begin{aligned}\vdash_{\overline{PA}} Con_{PA+\bar{T}} &\to Con_{\bar{T}}\\ &\to Con_T\\ &\to Con_{PA+Con_{PA}}.\end{aligned}$$

So, A is PA valid.

In order to show that A is not provable in QGL, we consider a model $\mathcal{N}$ of T, and we define a Kripke model $\mathcal{M} = \langle X, R, \{W_x\}_{x\in X}, \Vdash\rangle$ as follows:

a. $X = \{0, 1\}$
b. xRy iff $x = 0$ and $y = 1$
c. $W_x = W_y = N$
d. if $B(v_1, \ldots, v_n)$ is an atomic formula in language of T, and $a_1, \ldots, a_n \in N$, we define $i \Vdash B(a_i, \ldots, a_n)$ iff $\mathcal{N} \vDash B(a_1, \ldots, a_n)$ $(i = 0, 1)$;

(the forcing relation can be quite arbitrary on the other atomic formulas). Since $1 \Vdash [T]$ and $0R1$, $0 \Vdash \Diamond[T]$; on the other hand, $0 \Vdash \Box\Box\bot$, therefore $0 \Vdash \neg A$. Clearly, R is transitive and reversely well founded, whence, by Proposition 1, QGL is valid in $\mathcal{M}$.

Remark: In [18], Solovay shows that the propositional provability logics of all Σ_1 sound r.e. extensions of PA coincide (indeed, all these logics coincide with GL). Also this result cannot be extended to the predicate case. To see this, let us consider a finitely axiomatizable subtheory T of ZF, which is strong enough to construct the structure $\mathcal{N} = \langle \omega, +, \cdot, s, 0\rangle$ and to prove that $\mathcal{N}$ is a model of $PA + Con_{PA}$; then $\vdash_{\overline{PA}} Con_T \to Con_{PA+Con_{PA}}$, therefore the above proof shows that the formula $A = \Diamond[T] \to \Diamond\Diamond\top$ is PA valid. However, A is not ZF valid; indeed, if f is an interpretation such that $fB = B$ for every atomic formula B in the language of T, the ZF value of A under f is $Con_{ZF+T} \to Con_{ZF+Con_{ZF}}$; since T is a subtheory of ZF, the above formula is provably equivalent to $Con_{ZF} \to Con_{ZF+Con_{ZF}}$, which is not provable in ZF, by Gödel's Second Incompleteness Theorem.

The results proved in this section give only negative information about the provability logic of PA, or of related theories; we now formulate two problems whose solution should give also positive information.

1. Is the set of all PA valid formulas recursively enumerable? If it is, find an axiomatization of it.

2. Describe the set of all QGL formulas which are T valid, for every Σ_1 sound r.e. extension T of PA (Conjecture: this set coincides with the set of all theorems of QGL).

*4 **Fixed points*** For technical purposes, throughout this section, we consider a conservative extension QGL' of QGL, obtained by adding a countable set $\{p_0, \ldots, p_n, \ldots\}$ of variables for formulas, and by extending the axioms of QGL to the formulas of the new language; in the following, p denotes a generic variable for formulas, and $A(p_0, \ldots, p_n)$, $B(p_0, \ldots, p_n)$, ... etc. denote QGL' formulas, whose variables for formulas are exactly $p_0, \ldots, p_n$. We say that *p is modalized in* the QGL' formula $A(p)$ iff every occurrence of p in $A(p)$ is under the scope of $\Box$; if the only variable for formulas in A is p, we say that *A is modalized* iff p is modalized in A. The fixed point problem for QGL can be formulated as follows:

Let $A(p)$ be a modalized QGL' formula; is there a QGL formula B, whose free (individual) variables are exactly those of A, such that $\vdash_{QGL} B \longleftrightarrow A(B)$?

A positive answer to this problem would imply that QGL is strong enough to prove the modal translation of Gödel's Diagonalization Lemma. On the other hand, by the De Jongh-Sambin Theorem (see [13]), we know that the fixed point problem for GL has an affirmative solution. Unfortunately, the problem has a negative answer in the predicate case.

Theorem 4 *There is no QGL sentence B such that $\vdash_{QGL} B \longleftrightarrow \forall u \exists v \Box(B \rightarrow P(u, v))$, where P is any binary predicate letter.*

Proof: Let us consider the model $\mathcal{M} = \langle x, R, \{W_x\}_{x \in X}, \Vdash\rangle$ defined as follows:

a. $X = \omega$
b. xRy iff $y < x$
c. $W_x = \{y \in \omega: y \geqslant x\}$
d. If P^k is any k-ary predicate letter different from P, we define $x \Vdash P^k(a_1, \ldots, a_k)$ for all $x \in X$, $a_1, \ldots, a_k \in W_x$. Moreover, for $a, b \in W_x$, we define $x \Vdash P(a, b)$ iff either $b = x + 1$ and $a \neq x + 1$, or $a, b \neq x + 1$ and $a < b$; the accessibility relation, and the total orderings induced by $P(u, v)$ in each W_x are illustrated in Figure 1.

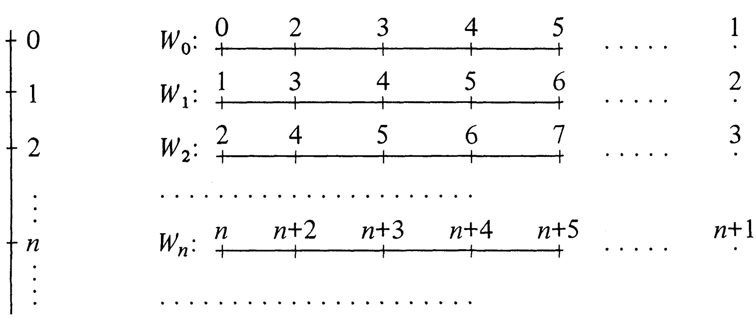

Figure 1.

Since R is transitive and R^{-1} is well founded, QGL is valid in $\mathcal{M}$. Moreover, we can prove that the forcing relation $\Vdash$ is definable in the structure $\mathcal{J} = \langle\omega, <, =\rangle$; more precisely, to every QGL formula $B(v_0, v_2, \ldots, v_{2n})$ containing only variables indexed with even numbers, we associate a formula $fB(v_1, v_0, v_2, \ldots, v_{2n})$ in the language of $\mathcal{J}$ such that, for all $x \in \omega$ and for all $a_1, \ldots, a_n \in W_x$, we have: $x \Vdash B(a_1, \ldots, a_n)$ iff $\mathcal{J} \models fB(x, a_1, \ldots, a_n)$. fB is defined by induction on the complexity of B: if B is of the form $P^k(v_0, v_2, \ldots, v_{2k})$, we put $fB(v_1, v_0, \ldots, v_{2k}) = \bigwedge_{i \leq k} v_1 \leq v_{2i}$; if $B = P(v_0, v_2)$, we define $fB(v_1, v_0, v_2)$ to be the formula

$$v_1 \leq v_0 \wedge v_1 \leq v_2 \wedge [(v_2 = v_1 + 1 \wedge \neg v_0 = v_1 + 1) \vee (\neg v_2 = v_1 + 1 \wedge \neg v_0 = v_1 + 1 \wedge v_0 < v_2)]$$

(where $v_j = v_i + 1$ is an abbreviation for $\exists w[w > v_i \wedge \forall z(z > v_i \rightarrow (w < z \vee v = z) \wedge v_j = w)$; moreover, we require that f commutes with $\vee$ and $\exists$ and we define $f\neg B(v_1, v_0, \ldots, v_{2n}) = \left(\bigwedge_{i \leq n} v_1 \leq v_{2i}\right) \wedge \neg fB(v_1, v_0, \ldots, v_{2n})$. Lastly, we define $f(\Box B(v_0, \ldots, v_{2n})) = \forall w(w < v_1 \rightarrow fB(v_1, v_0, \ldots, v_{2n}))$, where w is a variable, indexed with an odd number, which does not occur in fB.

Then, for every QGL sentence B, the set $\{x \in \omega: x \Vdash B\}$ is definable in $\mathcal{J}$; by a suitable quantifier elimination method, one can prove that every definable subset of $\mathcal{J}$ is either finite or cofinite (see [12]).

Now, assume, for reductio, that B is a QGL sentence such that $\vdash_{QGL} B \longleftrightarrow \forall u \exists v \Box(B \rightarrow P(u, v))$; then, since QGL is valid in $\mathcal{M}$, for all $x \in X$, $x \Vdash B \longleftrightarrow \forall u \exists v \Box(B \rightarrow P(u, v))$. Since $0 \Vdash \Box\bot$, $0 \Vdash B$; moreover, for all $v \in W_1$, $0 \Vdash \neg P(1, v)$; so, $1 \Vdash \exists u \forall v \Diamond(B \wedge \neg P(u, v))$, whence $1 \Vdash \neg B$; now, we argue by induction: assume that, for every $i \leq n$, $2i \Vdash B$ and $2i + 1 \Vdash \neg B$; then $2n + 1 \Vdash \neg B$, and, for all $u \in W_{2n+2}$ and for all $j \leq 2n$, $j \Vdash P(u, u + 1)$; so, for every $u \in W_{2n+2}$, there is a $v \in W_{2n+2}$ such that for all j for which $2n + 2Rj$ either $j \Vdash \neg B$ or $j \Vdash P(u, v)$. Then, $2n + 2 \Vdash \forall u \exists v \Box(B \rightarrow P(u, v))$, whence $2n + 2 \Vdash B$. Furthermore, for all $v \in W_{2n+3}$, $2n + 2 \Vdash B \wedge \neg P(2n + 3, v)$, therefore $2n + 3 \Vdash \exists u \forall v \Diamond(B \wedge \neg P(u, v))$; this implies $2n + 3 \Vdash \neg B$. So, the set $\{x \in X: x \Vdash B\}$ coincides with the set of even numbers, which is neither finite nor cofinite, a contradiction.

At this point, a natural question is: which steps of the De Jongh-Sambin proof do not extend to QGL? As observed by Valentini, *each* proof of the fixed point theorem for GL is based on a lemma, called "Substitution Lemma" (SL) which fails to hold in the predicate case. SL can be stated as follows:

Let $A(p)$, B, C, D, be GL formulas; if $\vdash_{GL} D \rightarrow (B \longleftrightarrow C)$, then

(a) $\vdash_{GL} D \wedge \Box D \rightarrow [A(B) \longleftrightarrow A(C)]$
(b) *If, in addition, p is modalized in Ap, then* $\vdash_{GL} \Box D \rightarrow [A(B) \longleftrightarrow A(C)]$.

That SL does not extend to QGL can be verified as follows: let $A(p) = \forall u \Box p$, $B = D = P(u)$ (where P is any unary predicate letter), and $C = \top$; then $\vdash_{QGL} D \rightarrow (B \longleftrightarrow C)$, but it is easily seen that $\nvdash_{QGL} \Box D \rightarrow (A(B) \longleftrightarrow A(C))$. We can prove, however, that a weaker version of SL does extend to QGL: let us say that the QGL' formula $A(p)$ and the QGL formula B obey the *variable restriction* (*in* *short: VR*) iff D does not contain free occurrences of (individual) variables having bound occurrences in $A(p)$.

Then, we have:

Lemma 4 *An analogue of SL holds for QGL, provided that $A(p)$ and D obey the VR.*

Let us first prove (a). We argue by induction on the complexity of $A(p)$: if $A(p)$ is atomic, the claim is obvious; the induction steps corresponding to $\vee, \neg, \Box$, are handled as in the propositional case; now, let $A(p) = \forall u E(u, p)$ and assume that $\vdash_{\overline{QGL}} D \wedge \Box D \rightarrow [E(u, B) \longleftrightarrow E(u, C)]$. By *GEN*, we obtain $\vdash_{\overline{QGL}} \forall u [D \wedge \Box D \rightarrow [E(u, B) \longleftrightarrow E(u, C)]]$. Since, by the *VR*, u has no free occurrences in D, we can easily deduce

$$\vdash_{\overline{QGL}} D \wedge \Box D \rightarrow [\forall u E(u, B) \longleftrightarrow \forall u E(u, C)],$$

that is

$$\vdash_{\overline{QGL}} D \wedge \Box D \rightarrow [A(B) \longleftrightarrow A(C)].$$

Now, let us prove (b): we first write $A(p)$ as $F(\Box E_1 p, \ldots, \Box E_n p)$ where $F(p_1, \ldots, p_n)$ is a suitable QGL' formula, and $E_1(p), \ldots, E_n(p)$ are $\Box$-free; a simple inductive argument shows that for all $i \leqslant n$ $\vdash_{\overline{QGL}} D \rightarrow [E_i(B) \longleftrightarrow E_i(C)]$. So, we obtain

$$\vdash_{\overline{QGL}} \Box D \rightarrow \Box(E_i(B) \longleftrightarrow E_i(C)) \rightarrow [\Box E_i(B) \longleftrightarrow \Box E_i(C)].$$

From this, we can deduce, by induction on the complexity of F, $\vdash_{\overline{QGL}} \Box D \rightarrow [F(\Box E_1(B), \ldots, \Box E_n(B)) \longleftrightarrow F(\Box E_1(C), \ldots, \Box E_n(C))]$ (again, the atomic case is trivial, the induction steps corresponding to $\vee, \neg, \Box$ are handled as in the propositional case, and the step corresponding to $\forall$ is handled as in (a)).

Lemma 4 allows us to prove the following results on uniqueness of fixed points:

Theorem 5 *For every modalized QGL' formula $A(p)$, and for all QGL formulas B and C, if $\vdash_{\overline{QGL}} A(B) \longleftrightarrow B$, $\vdash_{\overline{QGL}} A(C) \longleftrightarrow C$, then $\vdash_{\overline{QGL}} B \longleftrightarrow C$.*

Proof: Throughout this proof, for every formula A, $\forall A$ denotes the universal closure of A. Since $\vdash_{\overline{QGL}} \forall(B \longleftrightarrow C) \rightarrow (B \longleftrightarrow C)$ and Ap, $\forall(B \longleftrightarrow C)$ obey the *VR*, we can apply Lemma 4, getting:

$$\begin{aligned}\vdash_{\overline{QGL}} \Box \forall(B \longleftrightarrow C) &\rightarrow A(B) \longleftrightarrow A(C)\\ &\rightarrow B \longleftrightarrow C.\end{aligned}$$

By *GEN*, we deduce $\vdash_{\overline{QGL}} \forall[\Box \forall(B \longleftrightarrow C) \rightarrow (B \longleftrightarrow C)]$. Since $\Box \forall(B \longleftrightarrow C)$ is closed, it follows that $\vdash_{\overline{QGL}} \Box \forall(B \longleftrightarrow C) \rightarrow \forall(B \longleftrightarrow C)$. By Löb's rule, we conclude $\vdash_{\overline{QGL}} \forall(B \longleftrightarrow C)$, whence the claim follows.

Corollary 1 *Fixed points for PA formulas arising from modalized QGL' formulas are unique up to provable equivalence.*

Proof: Since *QGL* is *PA* valid and *PA* is closed under (the arithmetical translation of) Löb's rule, the previous proof works. Clearly, in *PA*, fixed points for such formulas always exist.

Problems

1. Find a procedure for deciding if a modalized formula $A(p)$ has a fixed point in QGL (or show that this is impossible).

2. Find a procedure for calculating the possible fixed points of any given modalized formula $A(p)$.

3. By Theorem 4, PA has, roughly speaking, more fixed points than QGL; does this fact imply some new provability principles, which are not provable in QGL?

4. Does the fixed point theorem hold for $QGL + BS$?

NOTES

1. In the following, $\top$ denotes any tautology and $\perp$ denotes the negation of $\top$.

2. If l denotes the length of s, $s * i$ defined by: Dom $s * i = l + 1$

$$(s * i)j = \begin{cases} sj \text{ if } j < 1 \\ i = \text{if } j = 1. \end{cases}$$

3. We wish to distinguish variables for worlds from variables for elements of $\bigcup_{x \in X} W_x$; so we only consider formulas of QGL without occurrences of variables indexed with odd numbers; these variables are used in the formulas of the form fB as variables for worlds.

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