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The Simple Consistency of a Set Theory Based on the Logic CSQ

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This paper proves the simple consistency of the set theory *CST*. *CST* has the Generalized Comprehension Axiom (*GCA*), $(\exists y)(\forall x)(x \in y \longleftrightarrow A)$, and the Extensionality Rule, $x = y \Rightarrow x \in w \longleftrightarrow y \in w$, where $x = y =_{df} (\forall z)(z \in x \longleftrightarrow z \in y)$. *CST* is based on a logic *CSQ*, which is semantically described below.

CSQ Primitives

1. $\sim, \&, \rightarrow, \forall$ (connectives and quantifier)
2. $f, g, h, f', \ldots$ (predicate constants)
3. $x, y, z, x', \ldots$ (individual variables)
4. $a_1, a_2, a_3, a_4, \ldots$ (individual constants).

CSQ Formulas

1. An individual variable or constant is a term.
2. If $t_1, \ldots, t_n$ are terms and f is a predicate constant, then $ft_1 \ldots t_n$ is an atomic formula.
3. If A and B are formulas and x is an individual variable then $\sim A$, $A \& B$, $A \rightarrow B$ and $(\forall x)A$ are formulas.

A sentence is a formula with no free variables.

A *CSQ model structure* (*CSQ m.s.*) consists of ordered triples $\langle T, K, R \rangle$, such that K is a set, T is a member of K, and R is a two-place relation on K, with the following postulates holding: For $\alpha \in K$,

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p1 $R T\alpha$.
p2 If $R\alpha\alpha$ then $\alpha = T$.
p3 If $J \subseteq K$ and $J \neq \phi$ then, for some $\alpha \in J$, for all $\beta \in J$, if $\alpha \neq \beta$ then $R\beta\alpha$ and not $R\alpha\beta$.

A *valuation* V *on the CSQ m.s. M* is a function assigning one value, 1, $\frac{1}{2}$, or 0, to each atomic sentence, for each member α of K. [In symbols, $V(fa_1 \ldots a_n, \alpha) = 1, \frac{1}{2}$, or 0.]

Each valuation is inductively extended to all sentences, as follows:

For $\alpha \in K$,

(i) $V(\sim A, \alpha) = 1$ iff $V(A, \alpha) = 0$.
$V(\sim A, \alpha) = 0$ iff $V(A, \alpha) = 1$.

(ii) $V(A \,\&\, B, \alpha) = 1$ iff $V(A, \alpha) = 1$ and $V(B, \alpha) = 1$.
$V(A \,\&\, B, \alpha) = 0$ iff $V(A, \alpha) = 0$ or $V(B, \alpha) = 0$.

(iii) $V(A \rightarrow B, \alpha) = 1$ iff, for all $\beta \in K$, if $R\alpha\beta$ then $V(A \underset{L}{\rightarrow} B, \beta) = 1$.
$V(A \rightarrow B, \alpha) = 0$ iff, for some $\beta \in K$, $R\alpha\beta$ and $V(A \underset{L}{\rightarrow} B, \beta) = 0$.

(iv) $V((\forall x)A, \alpha) = 1$ iff, for all individual constants a, $V(A\, {}^a\!/\!_x, \alpha) = 1$.
$V((\forall x)A, \alpha) = 0$ iff, for some individual constant a, $V(A\, {}^a\!/\!_x, \alpha) = 0$.

The connective '$\underset{L}{\rightarrow}$' is introduced here to simplify (iii), and is evaluated as follows:

(v) $V(A \underset{L}{\rightarrow} B, \alpha) = 1$ iff $V(A, \alpha) \leqslant V(B, \alpha)$.
$V(A \underset{L}{\rightarrow} B, \alpha) = 0$ iff $V(A, \alpha) = 1$ and $V(B, \alpha) = 0$.

A formula A with free variables $x_1, \ldots, x_n$ is *valid in a CSQ m.s. M* iff, for each valuation V on M, for all individual constants $a_1, \ldots, a_n$, $V(A\, {}^{a_1}\!/\!_{x_1} \ldots {}^{a_n}\!/\!_{x_n}, T) = 1$.

A formula A is *valid in this CSQ semantics* iff, A is valid in all the *CSQ* model structures.

As can be seen, *CSQ* is an intensionalized Łukasiewicz three-valued predicate logic, with substitutional quantification. The above semantics is abstracted from the model structure of *MC* of Section 2, which is used to prove the simple consistency of the set theory *CST*. The semantics is used in Section 1 to establish axioms and rules of logics for which the simple consistency result applies.

Let us fit this result into the context of the previous work on the consistency of the axioms of comprehension and extensionality. Let us consider the Comprehension Axiom in the form:

(*) $(\exists y)(\forall x)(x \in y \longleftrightarrow A(x))$, *where y is not free in* $A(x)$.

Previous work has concentrated on two logics, the Łukasiewicz infinitely valued predicate logic, $L_\infty Q$, and the Łukasiewicz three-valued predicate logic, $L_3 Q$.

Using $L_\infty Q$, Skolem [9] showed the consistency of (*), where $A(x)$ contains no quantifiers. Then, using the same logic, Chang [6] showed the consistency of (*), where $A(x)$ has at most the variable x free, and also of (*), where each bound variable u in $A(x)$ is restricted to occur only in the second place in atomic formulas of the form $v \in w$. This latter result is a strengthening of Skolem's result. Again, using $L_\infty Q$, Fenstad [8] showed the consistency of

(*), with the free variable x of $A(x)$ being allowed to occur only in the first place in atomic formulas of the form $v \in w$ in $A(x)$. Chang [7] points out that the restriction on (*) that y not occur free in $A(x)$ can be removed for these four consistency results. Then, more recently, White [12] strengthened these four results by showing the consistency of (*) using $L_\infty Q$, retaining the restriction that y not occur free in $A(x)$. White also shows that the Axiom of Extensionality in the form $(\forall z)(z \in x \longleftrightarrow z \in y) \to (\forall w)(x \in w \longleftrightarrow y \in w)$ cannot be consistently added to (*), and suggests replacing the main connective '$\to$' by a new connective '$\Rightarrow$' of L_∞.

Using L_3Q, Skolem, in [10] and [11], showed the consistency of (*), with $A(x)$ not containing any quantifier nor any occurrence of '$\to$'. He also showed that the Axiom of Extensionality in the form $(\forall z)(z \in x \longleftrightarrow z \in y) \overset{2}{\to} (\forall w)(x \in w \longleftrightarrow y \in w)$ could be consistently added, $A \overset{2}{\to} B$ being defined as $A \to. A \to B$. In [2], the present author, using L_3Q, strengthened Skolem's result by showing the consistency of (*), with $A(x)$ not containing any occurrence of '$\to$', and also showing that the above form of the Axiom of Extensionality can be consistently added.

The result of the present paper strengthens the Comprehension Axiom (*) to the generalized form, *GCA*, where $A(x)$ may have y occurring free, includes the Extensionality Rule in place of an axiom, and replaces the logic by *CSQ*. Moreover, it should be possible to convert the Extensionality Rule to an axiom of the form $(\forall z)(z \in x \longleftrightarrow z \in y) \underset{T}{\to} (\forall w)(x \in w \longleftrightarrow y \in w)$, where '$\underset{T}{\to}$' is a suitable new connective, similar to White's '$\Rightarrow$' of L_∞, but I leave this for now.

The method of proof is similar to the method adopted in [4] for the proof of the nontriviality of a dialectical set theory. The three-valued logic *RM3Q* is replaced by L_3Q, and a transfinite sequence of transfinite sequences of L_3Q model structures is employed:

$$M_{\tau,0} \leqslant M_{\tau,1} \leqslant \ldots \leqslant M_{\tau,\lambda_\tau} \leqslant \ldots, \text{ for countable ordinals } \tau.$$

It is shown that $\{M_{0,\lambda_0}, \ldots, M_{\kappa,\lambda_\kappa}\}$, with κ satisfying $M_{\kappa,\lambda_\kappa} = M_{\kappa+1,\lambda_{\kappa+1}}$, is a model structure for *CST*.

1 The logic CTQ and the set theory CST I show that the following system *CTQ* is a subsystem of *CSQ*, in order to establish a system of axioms and rules, for which the ensuing consistency proof works.

CTQ Primitives: As for *CSQ*.

CTQ Definitions: $A \vee B =_{df} \sim(\sim A \,\&\, \sim B)$; $A \longleftrightarrow B =_{df} (A \to B) \,\&\, (B \to A)$; $\Box A =_{df} (A \to A) \to A$; $\Diamond A =_{df} \sim\Box\sim A$; $(\exists x)A =_{df} \sim(\forall x)\sim A$.

Axioms

1. $A \to A$.
2. $\Box(A \to B) \to. B \to C \to. A \to C$.
3. $\Box(A \to B) \to. C \to A \to. C \to B$.
4. $A \,\&\, B \to A$.
5. $A \,\&\, B \to B$.
6. $(A \to B) \,\&\, (A \to C) \to. A \to B \,\&\, C$.
7. $A \,\&\, (B \vee C) \to (A \,\&\, B) \vee (A \,\&\, C)$.

8. $A \to {\sim}B \to . B \to {\sim}A$.
9. ${\sim\sim}A \to A$.
10. $A \to B \to . C \to . A \to B$.
11. $\Box A \to . B \to A$.
12. $\Box\Box A \to . A \to B \to \Box B$.
13. $\Box A \,\&\, \Box B \to \Box(A \,\&\, B)$.
14. $\Box(A \to B) \to . \Box A \to \Box B$.
15. $(\forall x)A \to A\, {}^{y}/x$, *where y is free for x in A.*
16. $(\forall x)(A \to B) \to . A \to (\forall x)B$, *where x is not free in A.*
17. $(\forall x)(A \vee B) \to A \vee (\forall x)B$, *where x is not free in A.*

Rules

1. $A, A \to B \Rightarrow B$.
2. $A, B \Rightarrow A \,\&\, B$.
3. $C \vee A, C \vee (A \to B) \Rightarrow C \vee B$.
4. $C \vee A \Rightarrow C \vee \Diamond A$.
5. $A \Rightarrow (\forall x)A$.

In showing that the axioms of *CTQ* are valid in the *CSQ* semantics and the rules of *CTQ* preserve validity in the semantics, the following five notes are of assistance.

(1) It suffices, in order to show the validity of a formula A, to consider only its sentential substitution instances. It also suffices, in order to show the preservation of validity of a rule, to consider only sentential substitution instances, except for Rule 5, where x is considered free in A.

(2) For sentences A and B, for $\alpha \in K$,
$V(A \to B, \alpha) = 1$ iff, for all $\beta \in K$ such that $R\alpha\beta$, if $V(A, \beta) = 1$ then $V(B, \beta) = 1$, and if $V(A, \beta) = 1$ or $\frac{1}{2}$ then $V(B, \beta) = 1$ or $\frac{1}{2}$.
$V(A \to B, \alpha) = 1$ or $\frac{1}{2}$ iff, for all $\beta \in K$ such that $R\alpha\beta$, if $V(A, \beta) = 1$ then $V(B, \beta) = 1$ or $\frac{1}{2}$.
$V(A \to B, T) = 1$ iff, for all $\beta \in K$, if $V(A, \beta) = 1$ then $V(B, \beta) = 1$, and $V(A, \beta) = 1$ or $\frac{1}{2}$ then $V(B, \beta) = 1$ or $\frac{1}{2}$.
$V(A \to B, T) = 1$ or $\frac{1}{2}$ iff, for all $\beta \in K$, if $V(A, \beta) = 1$ then $V(B, \beta) = 1$ or $\frac{1}{2}$ [due to $p1$].

(3) For a sentence A, for $\alpha \in K$,
$V(\Box A, \alpha) = 1$ iff, for all $\beta \in K$ such that $R\alpha\beta$, $V(A, \beta) = 1$.
$V(\Box A, \alpha) = 1$ or $\frac{1}{2}$ iff, for all $\beta \in K$ such that $R\alpha\beta$, $V(A, \beta) = 1$ or $\frac{1}{2}$.
$V(\Diamond A, \alpha) = 1$ iff, for some $\beta \in K$ such that $R\alpha\beta$, $V(A, \beta) = 1$.
$V(\Diamond A, \alpha) = 1$ or $\frac{1}{2}$ iff, for some $\beta \in K$ such that $R\alpha\beta$, $V(A, \beta) = 1$ or $\frac{1}{2}$.

(4) For sentences A and B, for $\alpha \in K$,
$V(A \vee B, \alpha) = 1$ iff $V(A, \alpha) = 1$ or $V(B, \alpha) = 1$.
$V(A \vee B, \alpha) = 0$ iff $V(A, \alpha) = 0$ and $V(B, \alpha) = 0$.

(5) For $\alpha, \beta, \gamma \in K$, if $R\alpha\beta$ and $R\beta\gamma$ then $R\alpha\gamma$ [due to p3].

Theorem 1 *For all formulas A, if A is a theorem of CTQ then A is valid in the semantics for CSQ.*

Proof: Let A, B, and C be sentences. Let $\alpha \in K$. We test the following sample of axioms and rules of *CTQ*. The remainder follow by the same techniques.

(A2) (i) Let $V(\Box(A \to B), \alpha) = 1$. Then, for all $\beta \in K$ such that $R\alpha\beta$, $V(A \to B, \beta) = 1$.

(a) Let $V(B \to C, \beta) = 1$ and $R\alpha\beta$. Then, for all γ such that $R\beta\gamma$, if $V(B, \gamma) = 1$ then $V(C, \gamma) = 1$, and if $V(B, \gamma) = 1$ or $\frac{1}{2}$ then $V(C, \gamma) = 1$ or $\frac{1}{2}$. Since $V(A \to B, \beta) = 1$, if $V(A, \gamma) = 1$ then $V(C, \gamma) = 1$, and if $V(A, \gamma) = 1$ or $\frac{1}{2}$ then $V(C, \gamma) = 1$ or $\frac{1}{2}$. Hence, $V(A \to C, \beta) = 1$.

(b) Let $V(B \to C, \beta) = 1$ or $\frac{1}{2}$ and $R\alpha\beta$. Then, for all γ such that $R\beta\gamma$, if $V(B, \gamma) = 1$ then $V(C, \gamma) = 1$ or $\frac{1}{2}$. Since $V(A \to B, \beta) = 1$, if $V(A, \gamma) = 1$ then $V(C, \gamma) = 1$ or $\frac{1}{2}$. Hence, $V(A \to C, \beta) = 1$ or $\frac{1}{2}$. By (a) and (b), $V(B \to C \to . A \to C, \alpha) = 1$.

(ii) Let $V(\Box(A \to B), \alpha) = 1$ or $\frac{1}{2}$. Then, for all β such that $R\alpha\beta$, $V(A \to B, \beta) = 1$ or $\frac{1}{2}$. Let $V(B \to C, \beta) = 1$ and $R\alpha\beta$. Then, for all γ such that $R\beta\gamma$, if $V(B, \gamma) = 1$ or $\frac{1}{2}$ then $V(C, \gamma) = 1$ or $\frac{1}{2}$. Hence, if $V(A, \gamma) = 1$ then $V(C, \gamma) = 1$ or $\frac{1}{2}$, and $V(A \to C, \beta) = 1$ or $\frac{1}{2}$. So, $V(B \to C \to . A \to C, \alpha) = 1$ or $\frac{1}{2}$.

(A8) (i) Let $V(A \to {\sim}B, \alpha) = 1$. Then, for all β such that $R\alpha\beta$, if $V(A, \beta) = 1$ then $V({\sim}B, \beta) = 1$, and if $V(A, \beta) = 1$ or $\frac{1}{2}$ then $V({\sim}B, \beta) = 1$ or $\frac{1}{2}$. Hence, if $V(B, \beta) = 1$ then $V({\sim}A, \beta) = 1$, and if $V(B, \beta) = 1$ or $\frac{1}{2}$ then $V({\sim}A, \beta) = 1$ or $\frac{1}{2}$. So, $V(B \to {\sim}A, \alpha) = 1$.

(ii) Let $V(A \to {\sim}B, \alpha) = 1$ or $\frac{1}{2}$. Then, for all β such that $R\alpha\beta$, if $V(A, \beta) = 1$ then $V({\sim}B, \beta) = 1$ or $\frac{1}{2}$. Hence, if $V(B, \beta) = 1$ then $V({\sim}A, \beta) = 1$ or $\frac{1}{2}$. So, $V(B \to {\sim}A, \alpha) = 1$ or $\frac{1}{2}$.

(A10) (i) Let $V(A \to B, \alpha) = 1$. Then, for all γ such that $R\alpha\gamma$, if $V(A, \gamma) = 1$ then $V(B, \gamma) = 1$, and if $V(A, \gamma) = 1$ or $\frac{1}{2}$ then $V(B, \gamma) = 1$ or $\frac{1}{2}$.

(a) Let $V(C, \beta) = 1$ and $R\alpha\beta$. Let $R\beta\gamma$. Then, since $R\alpha\gamma$, $V(A \to B, \beta) = 1$.

(b) Let $V(C, \beta) = 1$ or $\frac{1}{2}$ and $R\alpha\beta$. As for (a), $V(A \to B, \beta) = 1$ or $\frac{1}{2}$. By (a) and (b), $(C \to . A \to B, \alpha) = 1$.

(ii) Let $V(A \to B, \alpha) = 1$ or $\frac{1}{2}$. Then, for all γ such that $R\alpha\gamma$, if $V(A, \gamma) = 1$ then $V(B, \gamma) = 1$ or $\frac{1}{2}$. Let $V(C, \beta) = 1$ and $R\alpha\gamma$. Let $R\beta\gamma$. Then, since $R\alpha\gamma$, $V(A \to B, \beta) = 1$ or $\frac{1}{2}$. So, $V(C \to . A \to B, \alpha) = 1$ or $\frac{1}{2}$.

(R1) Let $V(A, T) = 1$ and $V(A \to B, T) = 1$. Then if $V(A, \alpha) = 1$ then $V(B, \alpha) = 1$, for all $\alpha \in K$. Hence, if $V(A, T) = 1$ then $V(B, T) = 1$, and so $V(B, T) = 1$.

(R4) Let $V(C \vee A, T) = 1$. Then $V(C, T) = 1$ or $V(A, T) = 1$. If $V(A, T) = 1$ then, since RTT, $V(\Diamond A, T) = 1$. Hence, $V(C, T) = 1$ or $V(\Diamond A, T) = 1$, and so $V(C \vee \Diamond A, T) = 1$.

(R5) Let A be a formula with at most the variable x free. Let $V(A\,{}^{a}\!/\!_{x}, T) = 1$ for all constants a. Then $V((\forall x)A, T) = 1$.

Some Theorems and Derived Rules of CTQ

$A \to B \to \Box(A \to B)$
$A \to B \to.\ B \to C \to.\ A \to C$
$A \to B \to.\ C \to A \to.\ C \to B$
$\Box A \to \Box\Box A.$

$A \Rightarrow \Diamond A$
$\Box A \Rightarrow A$
$C \vee \Box A \Rightarrow C \vee A.$

It should be noted that all theorems of *CTQ* are theorems of the Łukasiewicz infinitely valued logic $L_\infty Q$, and that all rules of *CTQ* are derived rules of $L_\infty Q$.

A quantified relevant subsystem *RCTQ* of *CTQ* can be obtained as follows:

$$RCTQ = CTQ - \text{Ax.10} - \text{Ax.11} + \text{Ax.18, where Ax.18 is:}$$

18. $A \to B \to \Box(A \to B)$.

RCTQ is a subsystem of the quantified relevant logic *RWQ* + Rule 3, and *RCTQ* properly contains the system *TWQ*. In fact,

$$TWQ = RCTQ - \text{Ax.2} - \text{Ax.3} - \text{Ax.12} - \text{Ax.13} - \text{Ax.14} - \text{Rule 3} - \text{Rule 4}$$
$$+ \text{Ax.19} + \text{Ax.20, where the additional axioms are:}$$

19. $A \to B \to.\ B \to C \to.\ A \to C.$
20. $A \to B \to.\ C \to A \to.\ C \to B.$

TWQ is singled out as being reasonably "deductively stable", as it does not contain the rather incongruous Axioms 2, 3, 12, 13, and 14 and Rules 3 and 4. The system *TWQ* is in fact obtained by dropping $(A \to.\ A \to B) \to.\ A \to B$ and $A \to {\sim}A \to {\sim}A$ from the system *T* and by adding the usual quantificational axioms and rule. Note that $A \to {\sim}A \to {\sim}A$ is dropped as for the system *RW*, where it is deductively equivalent to $(A \to.\ A \to B) \to.\ A \to B$.

The system *CST* of set theory is formally set out as follows:

CST Primitives

1. $\sim$, &, $\to$, $\underset{L}{\to}$, $\forall$ (connectives and quantifier)
2. t, f, n (sentential constants)
3. $x, y, z, x', \ldots$ (set variables)
4. ϵ (membership relation).

CST Formation Rules

1. Where t_1 and t_2 are set variables or terms, $t_1 \in t_2$ is an initial formula, and so also a formula.
2. A sentential constant, t, f, or n, is an initial formula, and so also a formula.
3. If A and B are initial or admissible formulas and x is a set variable, then ${\sim}A$, $(A \,\&\, B)$, and $(\forall x)A$ are admissible formulas, and so also formulas. Thus all initial formulas are also admissible formulas.
4. If A is an admissible formula and x and y are distinct set variables then $\{xy\colon A\}$ is a term.

5. If A and B are admissible formulas then $A \rightarrow B$ is an initial formula, and so also an admissible formula and a formula.
6. If A and B are formulas and x is a set variable then $\sim A$, $(A \mathbin{\&} B)$, $(A \underset{L}{\rightarrow} B)$, and $(\forall x)A$ are formulas.

As usual, a sentence is a formula with no free variables. Admissible and initial formulas are distinguished for the same purposes as in [3] and [4]. That is, admissible formulas are those that can be put on the right hand side of the *GCA* and hence can be used to form the terms $\{xy\colon A\}$, and initial formulas are those that cannot be evaluated using the valuation conditions of the connectives $\sim$, &, $\underset{L}{\rightarrow}$, and the quantifier, $\forall$, of the Łukasiewicz three-valued logic, L_3Q.

CST Definitions

$A \vee B =_{df} \sim(\sim A \mathbin{\&} \sim B)$, $A \longleftrightarrow B =_{df} (A \rightarrow B) \mathbin{\&} (B \rightarrow A)$, $A \underset{L}{\longleftrightarrow} B =_{df} (A \underset{L}{\rightarrow} B) \mathbin{\&} (B \underset{L}{\rightarrow} A)$, $(\exists x)A =_{df} \sim(\forall x)\sim A$, $\{x\colon A\} =_{df} \{xy\colon A\}$, where y is not free in A, $x = y =_{df} (\forall z)(z \in x \longleftrightarrow z \in y)$.

The set of logically valid formulas are those of CSQ' and the rules are those that preserve validity in each model structure of CSQ', where CSQ' is *CSQ* as set out in the introduction, but with the following adjustments which are needed to deal with the above set-theoretic formulas:

1. The connective '$\underset{L}{\rightarrow}$' is formally introduced, for the reason given in the introduction.
2. The sentential constants, t, f, and n are added.
3. The predicate constants are replaced by the single membership predicate, 'ϵ'.
4. The individual variables are construed as set variables.
5. The individual constants are construed as constant terms of the form $\{xy\colon A\}$, where A has at most the variables x and y free. (Use the symbols: $a, b, c, \ldots$, to refer to these constant terms.)
6. The above formation rules apply.
7. Each valuation V assigns 1 to t, $\frac{1}{2}$ to n, and 0 to f, for each $\alpha \in K$.

CST has the following set-theoretical axiom and rule:

(GCA′) (Generalized Comprehension Axiom) $z \in \{xy\colon A\} \longleftrightarrow A\, {}^{z}/_{x}{}^{\{xy:A\}}/_{y}$, *where y and z are distinct variables.*

(ER) (Extensionality Rule) $x = y \Rightarrow x \in w \longleftrightarrow y \in w$.

The existential form of the Generalized Comprehension Axiom, i.e.,

(GCA) $(\exists y)(\forall x)(x \in y \longleftrightarrow A)$, can be derived from GCA'.

Substitution of Identity takes the form of a rule:

(SI) $x = y \Rightarrow A \longleftrightarrow B$, where B is A with any number of free occurrences of x in A replaced by y (where y is free for x in A).

The GCA' provides the motivation and interpretation for the two-variable set abstract $\{xy\colon A\}$. The extra variable y in A gives those argument places in A where the set $\{xy\colon A\}$ is substituted on the right hand side of the GCA', thus enabling the GCA to be established in cases where y is free in A. Clearly $\{xy: A\}$ reduces to $\{x: A\}$, when y is not free in A. $\{xy: A\}$ is then a set of all elements x such that $A^{\{xy:A\}}/_y$. Note that such a set need not be unique, as can be shown by putting $x \in y$ for A.

2 The determination of the model structure MC for the set theory CST The model structure MC consists of a transfinite sequence of L_3Q model structures $\{M_{0,\lambda_0}, M_{1,\lambda_1}, \ldots, M_{\kappa,\lambda_\kappa}\}$, where $M_{\kappa,\lambda_\kappa} = M_{\eta,\lambda_\eta}$ for all $\eta > \kappa$. Each of these L_3Q model structures M_{τ,λ_τ} is obtained as a fixed point of L_3Q model structures, $M_{\tau,0}, M_{\tau,1}, \ldots, M_{\tau,\lambda_\tau}, \ldots$, in the same manner as in [4]. All admissible sentences of the form $A \rightarrow B$ are given a fixed L_3Q value over each τ-sequence, $M_{\tau,0}, \ldots, M_{\tau,\lambda_\tau}, \ldots$. That MC is a model structure for the set theory CST is proved in a similar manner to the proof that MD is a model structure for DST in [4].

Let us proceed with the determination of MC. Similarly to [3] and [4], L_3Q model structures, called *structures*, are introduced. Each structure M is specified by a valuation $\vee[M]$ which assigns the L_3Q values 1 to t, $\frac{1}{2}$ to n, and 0 to f, and exactly one of the values 1, $\frac{1}{2}$, and 0 to every other initial sentence.

Each valuation $\vee[M]$ is extended from initial sentences to all sentences in accordance with the matrices for L_3 and the valuation rules for the quantifier $\forall$ as follows:

$\sim$	
*1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

&	1	$\frac{1}{2}$	0
*1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\vec{L}$	1	$\frac{1}{2}$	0
*1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

$\vee[M]((\forall x)A) = 1$ iff $\vee[M](A\ {}^a/_x) = 1$ for all constant terms a.
$\vee[M]((\forall x)A) = 0$ iff $\vee[M](A\ {}^a/_x) = 0$ for some constant term a.
$\vee[M]((\forall x)A) = \frac{1}{2}$ otherwise.

The matrices of the defined connectives 'v' and '$\underset{L}{\leftrightarrow}$' are:

v	1	$\frac{1}{2}$	0
*1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\overleftrightarrow{L}$	1	$\frac{1}{2}$	0
*1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
0	0	$\frac{1}{2}$	1

The valuation rules for $\exists$ are:

$\vee[M]((\exists x)A) = 1$ iff $\vee[M](A\ {}^a/_x) = 1$ for some constant term a.
$\vee[M]((\exists x)A) = 0$ iff $\vee[M](A\ {}^a/_x) = 0$ for all constant terms a.
$\vee[M]((\exists x)A) = \frac{1}{2}$ otherwise.

A formula A with free variables $x_1, \ldots, x_n$ is *valid in a structure M* iff, for all constant terms $a_1, \ldots, a_n$, the valuation $\vee[M]$ specifying the structure M is such that $\vee[M](A\ {}^{a_1}/_{x_1}\ {}^{a_2}/_{x_2} \ldots {}^{a_n}/_{x_n}) = 1$, i.e., $A\ {}^{a_1}/_{x_1} \ldots {}^{a_n}/_{x_n}$ takes the designated value. Such a formula A is *invalid in a structure M* otherwise, i.e.,

$\mathsf{V}[M](A\ ^{a_1}/x_1 \ldots ^{a_n}/x_n) = \frac{1}{2}$ or 0, for some constant terms $a_1, \ldots, a_n$. In particular, a sentence A is valid in M iff $\mathsf{V}[M](A) = 1$, and is invalid in M iff $\mathsf{V}[M](A) = \frac{1}{2}$ or 0.

The following ordering relation is defined for structures, as in [3] and [4]. For structures M_1 and M_2, $M_1 \leqslant M_2 =_{df}$ for all initial sentences A, if $\mathsf{V}[M_1](A) = 1$ then $\mathsf{V}[M_2](A) = 1$, and if $\mathsf{V}[M_1](A) = 0$ then $\mathsf{V}[M_2](A) = 0$. $\leqslant$ is a partial ordering on structures, identity for structures being defined as follows: $M_1 = M_2 =_{df}$ for all initial sentences A, $\mathsf{V}[M_1](A) = \mathsf{V}[M_2](A)$.

Similarly to [3] and [4], we can show that the ordering preserves the valuations 1 and 0 for admissible sentences.

Lemma 1 *Let $M_1 \leqslant M_2$ for structures M_1 and M_2. Then, for all admissible sentences A, if $\mathsf{V}[M_1](A) = 1$ then $\mathsf{V}[M_2](A) = 1$, and if $\mathsf{V}[M_1](A) = 0$ then $\mathsf{V}[M_2](A) = 0$.*

Proof: The proof is as for Lemma 2 of [3] with 1 replacing t and 0 replacing f.

We next determine the following transfinite sequence of transfinite sequences of structures, which are used to establish the model structure MC for CST:

$$M_{0,0} \leqslant M_{0,1} \leqslant \ldots \leqslant M_{0,\nu} \leqslant \ldots \leqslant M_{0,\lambda_0} \leqslant \ldots$$
$$M_{1,0} \leqslant M_{1,1} \leqslant \ldots \leqslant M_{1,\nu} \leqslant \ldots \leqslant M_{1,\lambda_1} \leqslant \ldots$$
$$\vdots$$
$$M_{\tau,0} \leqslant M_{\tau,1} \leqslant \ldots \leqslant M_{\tau,\nu} \leqslant \ldots \leqslant M_{\tau,\lambda_\tau} \leqslant \ldots$$
$$\vdots$$

__________ (S)

This well-ordering of the sequences in (S) is established by Lemma 2. The ordinals λ_τ, for each ordinal τ, are established by Lemma 3. The sequence of sequences, (S), is defined by double transfinite induction, in conjunction with the proofs of Lemmas 2 and 3 to follow. It is required then to define $\mathsf{V}[M_{\tau,\nu}](A)$, for all initial sentences A other than sentential constants, for all τ, for all ν. The double induction is divided into four cases:

Case 1. For the structure $M_{0,0}$, there are two subcases:

a. For all admissible sentences A and B, $\mathsf{V}[M_{0,0}](A \rightarrow B) = 1$.

b. For all admissible formulas A, with at most the variables x and y free, and all constant terms a, $\mathsf{V}[M_{0,0}](a \in \{xy: A\}) = \frac{1}{2}$.

Case 2. For the structure $M_{\tau,0}$, for $\tau > 0$, there are two subcases:

a. For all admissible sentences A and B,
$\mathsf{V}[M_{\tau,0}](A \rightarrow B) = 1$ iff $\mathsf{V}[M_{\rho,\lambda_\rho}](A \underset{L}{\rightarrow} B) = 1$ for all $\rho < \tau$,
$\mathsf{V}[M_{\tau,0}](A \rightarrow B) = 0$ iff $\mathsf{V}[M_{\rho,\lambda_\rho}](A \underset{L}{\rightarrow} B) = 0$ for some $\rho < \tau$.

b. For all admissible formulas A, with at most the variables x and y free, and all constant terms a, $\mathsf{V}[M_{\tau,0}](a \in \{xy: A\}) = \frac{1}{2}$.

Case 3. For the structures $M_{\tau,\nu}$, for any τ and for ν a successor ordinal, there are two subcases:

a. For all admissible sentences A and B,
$\mathsf{V}[M_{\tau,\nu}](A \to B) = \mathsf{V}[M_{\tau,0}](A \to B)$.

b. For all admissible formulas A, with at most the variables x and y free, and all constant terms a,
$\mathsf{V}[M_{\tau,\nu}](a \in \{xy: A\}) = [M_{\tau,\nu-1}](A\ {}^{a}/_{x}{}^{\{xy:A\}}/_{y})$.

Case 4. For the structures $M_{\tau,\nu}$, for any τ and for ν a limit ordinal, there are two subcases:

a. For all admissible sentences A and B,
$\mathsf{V}[M_{\tau,\nu}](A \to B) = \mathsf{V}[M_{\tau,0}](A \to B)$.

b. For all admissible formulas A, with at most the variables x and y free, and all constant terms a,
$\mathsf{V}[M_{\tau,\nu}](a \in \{xy: A\}) = 1$ iff $\mathsf{V}[M_{\tau,\rho}](a \in \{xy: A\}) = 1$ for some $\rho < \nu$,
$\mathsf{V}[M_{\tau,\nu}](a \in \{xy: A\}) = 0$ iff $\mathsf{V}[M_{\tau,\rho}](a \in \{xy: A\}) = 0$ for some $\rho < \nu$.

In the above definition, Lemma 2 was assumed for earlier pairs of structures in Subcase 4b, and Lemma 3 was assumed for earlier sequences of structures in Subcase 2a.

Let A be an admissible formula, with at most the variables x and y free, and let a be a constant term. Define the *corresponding admissible sentence*, $C(a \in \{xy: A\})$, of $a \in \{xy: A\}$ as $A\ {}^{a}/_{x}{}^{\{xy:A\}}/_{y}$. Note that for successor ordinals ν, for any τ, $\mathsf{V}[M_{\tau,\nu}](a \in \{xy: A\}) = \mathsf{V}[M_{\tau,\nu-1}](C(a \in \{xy: A\}))$.

Lemma 2 *For all ordinals* τ, $M_{\tau,\nu} \leqslant M_{\tau,\nu'}$, *whenever* $\nu \leqslant \nu'$.

Proof: The proof is similar to that of Lemma 3 of [3], given that initial sentences of the form $A \to B$ have a fixed value for each τ-sequence.

Lemma 3 (Fixed Point Lemma) *For each* τ *sequence,* $M_{\tau,0} \leqslant M_{\tau,1} \leqslant \ldots \leqslant M_{\tau,\nu} \leqslant \ldots$, *there is a countable ordinal* λ_τ, *such that:* (i) *for all ordinals* $\mu \geqslant \lambda_\tau$, $M_{\tau,\mu} = M_{\tau,\lambda_\tau}$, *and* (ii) *for all ordinals* $\rho < \lambda_\tau$, $M_{\tau,\rho} \neq M_{\tau,\rho+1}$.

Proof: The proof is as given for Lemma 3 of [4], which in turn follows the proof of Lemma 4 of [3].

Corollary *For each ordinal* τ, *the ordinal* λ_τ *of Lemma 3 is a denumerable limit ordinal.*

Proof: The proof is similar to that of the corollary to Lemma 4 of [3].

Lemma 4 *There is a countable ordinal* κ *such that*: (i) $M_{\kappa,\lambda_\kappa} = M_{\kappa+1,\lambda_{\kappa+1}}$, *and* (ii) *for all ordinals* $\rho < \kappa$, $M_{\rho,\lambda_\rho} \neq M_{\rho+1,\lambda_{\rho+1}}$.

Proof: As can be seen from the construction (S) of each τ-sequence, $M_{\tau,0} \leqslant M_{\tau,1} \leqslant \ldots \leqslant M_{\tau,\nu} \ldots$, the τ-sequences are distinguished entirely by their valuations of the initial sentences of the form, $A \to B$, where A and B are admissible, in the structure $M_{\tau,0}$. Hence,

(a) $M_{\rho,\lambda_\rho} = M_{\eta,\lambda_\eta}$, if $\mathsf{V}[M_{\rho,0}](A \to B) = [M_{\eta,0}](A \to B)$, for all admissible sentences A and B. It is also clear from the construction (S) that if $\rho \leqslant \eta$ then,

(b) if $\mathsf{V}[M_{\rho,0}](A \to B) = 0$ then $\mathsf{V}[M_{\eta,0}](A \to B) = 0$, and

(c) if $\vee[M_{\rho,0}](A \to B) = 0$ or $\frac{1}{2}$ then $\vee[M_{\eta,0}](A \to B) = 0$ or $\frac{1}{2}$,

for all admissible sentences A and B.

Let $S_\rho^0(S_\rho^{0,1/2})$ be the set of initial sentences of the form $A \to B$ (A and B admissible) such that $\vee[M_{\rho,0}](A \to B) = 0$ (= 0 or $\frac{1}{2}$). Then if $\rho \leqslant \eta$ then, by (b), $S_\rho^0 \subseteq S_\eta^0$ and, by (c), $S_\rho^{0,1/2} \subseteq S_\eta^{0,1/2}$. Also, by (a), it is clear that if $S_\rho^0 = S_\eta^0$ and $S_\rho^{0,1/2} = S_\eta^{0,1/2}$ then $M_{\rho,\lambda_\rho} = M_{\eta,\lambda_\eta}$. Since there are denumerably many sentences of the form $A \to B$ (A and B admissible), there are only countably many distinct sets S_ρ^0 and only countably many distinct sets $S_\rho^{0,1/2}$. Hence, there are countably many distinct pairs, $\langle S_\rho^0, S_\rho^{0,1/2}\rangle$, for ordinals ρ. So, there are ordinals ρ and η such that $\rho < \eta$ and $\langle S_\rho^0, S_\rho^{0,1/2}\rangle = \langle S_\eta^0, S_\eta^{0,1/2}\rangle$, and hence $S_\rho^0 = S_\eta^0$ and $S_\rho^{0,1/2} = S_\eta^{0,1/2}$. For such ρ and η, let $\rho \leqslant \nu \leqslant \eta$. Then, since $S_\rho^0 \subseteq S_\nu^0 \subseteq S_\eta^0$ and $S_\rho^{0,1/2} \subseteq S_\nu^{0,1/2} \subseteq S_\eta^{0,1/2}$, $S_\rho^0 = S_\nu^0 = S_\eta^0$ and $S_\rho^{0,1/2} = S_\nu^{0,1/2} = S_\eta^{0,1/2}$. So, there is an ordinal ρ such that $\langle S_\rho^0, S_\rho^{0,1/2}\rangle = \langle S_{\rho+1}^0, S_{\rho+1}^{0,1/2}\rangle$, and hence $M_{\rho,\lambda_\rho} = M_{\rho+1,\lambda_{\rho+1}}$. Let κ be the least ordinal such that $M_{\kappa,\lambda_\kappa} = M_{\kappa+1,\lambda_{\kappa+1}}$. Then, for all ordinals $\rho < \kappa$, $M_{\rho,\lambda_\rho} \neq M_{\rho+1,\lambda_{\rho+1}}$, establishing (ii). Further, for all ordinals $\rho < \kappa$, $\langle S_\rho^0, S_\rho^{0,1/2}\rangle \neq \langle S_{\rho+1}^0, S_{\rho+1}^{0,1/2}\rangle$, and hence, for all distinct ordinals $\rho < \kappa$ and $\eta < \kappa$, $\langle S_\rho^0, S_\rho^{0,1/2}\rangle \neq \langle S_\eta^0, S_\eta^{0,1/2}\rangle$. Since there are only countably many distinct pairs $\langle S_\rho^0, S_\rho^{0,1/2}\rangle$, for ordinals ρ, κ is a countable ordinal.

The following corollaries of Lemma 4 are not needed in the proof of consistency of *CST* but are added as a matter of interest.

Corollary 1 *For ordinal κ satisfying Lemma 4, $M_{\kappa,\lambda_\kappa} = M_{\rho,\lambda_\rho}$, for all ordinals $\rho \geqslant \kappa$.*

Proof: The proof is by transfinite induction on all ordinals ρ such that $\kappa < \rho$. The case for $\rho = \kappa + 1$ is trivial. Let ρ be greater than $\kappa + 1$ and let the corollary hold for all ordinals ν such that $\kappa < \nu < \rho$. It suffices to show that $\vee[M_{\rho,0}](A \to B) = \vee[M_{\kappa+1,0}](A \to B)$, for all admissible sentences A and B.

$$\begin{aligned}\vee[M_{\rho,0}](A \to B) = 1 \quad &\Longleftrightarrow \vee[M_{\sigma,\lambda_\sigma}](A \underset{L}{\to} B) = 1 \text{ for all } \sigma < \rho.\\ &\Longleftrightarrow \vee[M_{\sigma,\lambda_\sigma}](A \underset{L}{\to} B) = 1 \text{ for all } \sigma \leqslant \kappa.\\ &\Longleftrightarrow \vee[M_{\kappa+1,0}](A \to B) = 1.\end{aligned}$$

$$\begin{aligned}\vee[M_{\rho,0}](A \to B) = 0 \quad &\Longleftrightarrow \vee[M_{\sigma,\lambda_\sigma}](A \underset{L}{\to} B) = 0 \text{ for some } \sigma < \rho.\\ &\Longleftrightarrow \vee[M_{\sigma,\lambda_\sigma}](A \underset{L}{\to} B) = 0 \text{ for some } \sigma \leqslant \kappa.\\ &\Longleftrightarrow \vee[M_{\kappa+1,0}](A \to B) = 0.\end{aligned}$$

Hence, $\vee[M_{\rho,0}](A \to B) = \vee[M_{\kappa+1,0}](A \to B)$, $M_{\rho,\lambda_\rho} = M_{\kappa+1,\lambda_{\kappa+1}}$, and $M_{\kappa,\lambda_\kappa} = M_{\rho,\lambda_\rho}$.

Corollary 2 *Let κ be an ordinal satisfying Lemma 4. Then κ is a limit ordinal.*

Proof: Let $\kappa = 0$. Then $M_{0,\lambda_0} = M_{1,\lambda_1}$, by Lemma 4. However, $\vee[M_{0,\lambda_0}](t \underset{L}{\to} f) = 0$ and hence $\vee[M_{1,\lambda_1}](t \to f) = 0$. Then $\vee[M_{0,\lambda_0}](t \to f) = 0$, which is a contradiction. Hence $\kappa > 0$.

Let κ be a successor ordinal. It suffices to show that $\vee[M_{\kappa-1,0}](A \to B) = \vee[M_{\kappa,0}](A \to B)$, for all admissible sentences A and B.

$\mathsf{V}[M_{\kappa-1,0}](A \to B) = 1 \Rightarrow \mathsf{V}[M_{\rho,\lambda_\rho}](A \underset{L}{\to} B) = 1$ for all $\rho < \kappa - 1$.
$\Rightarrow \mathsf{V}[M_{\tau,0}](A \to B) = 1$ for all $\tau \leqslant \kappa - 1$.
$\Rightarrow \mathsf{V}[M_{\tau,\lambda_\tau}](A \to B) = 1$ for all $\tau \leqslant \kappa - 1$.
$\Rightarrow \mathsf{V}[M_{\tau,\lambda_\tau}](A \to A \underset{L}{\to}. A \to B) = 1$ for all $\tau \leqslant \kappa - 1$, since $\mathsf{V}[M_{\tau,\lambda_\tau}](A \to A) = 1$ for all $\tau \leqslant \kappa - 1$.
$\Rightarrow \mathsf{V}[M_{\kappa,0}](A \to A \to. A \to B) = 1$.
$\Rightarrow \mathsf{V}[M_{\kappa+1,0}](A \to A \to. A \to B) = 1$, since $M_{\kappa,\lambda_\kappa} = M_{\kappa+1,\lambda_{\kappa+1}}$.
$\Rightarrow \mathsf{V}[M_{\kappa,\lambda_\kappa}](A \to A \underset{L}{\to}. A \to B) = 1$.
$\Rightarrow \mathsf{V}[M_{\kappa,0}](A \to B) = 1$.

$\mathsf{V}[M_{\kappa,0}](A \to B) = 1 \Rightarrow \mathsf{V}[M_{\rho,\lambda_\rho}](A \underset{L}{\to} B) = 1$ for all $\rho < \kappa$.
$\Rightarrow \mathsf{V}[M_{\rho,\lambda_\rho}](A \underset{L}{\to} B) = 1$ for all $\rho < \kappa - 1$.
$\Rightarrow \mathsf{V}[M_{\kappa-1,0}](A \to B) = 1$.

$\mathsf{V}[M_{\kappa,0}](A \to B) = 0 \Rightarrow \mathsf{V}[M_{\kappa,\lambda_\kappa}](A \to A \underset{L}{\to}. A \to B) = 0$.
$\Rightarrow \mathsf{V}[M_{\kappa+1,\lambda_{\kappa+1}}](A \to A \to. A \to B) = 0$.
$\Rightarrow \mathsf{V}[M_{\kappa,\lambda_\kappa}](A \to A \to. A \to B) = 0$, since $M_{\kappa,\lambda_\kappa} = M_{\kappa+1,\lambda_{\kappa+1}}$.
$\Rightarrow \mathsf{V}[M_{\rho,\lambda_\rho}](A \to A \underset{L}{\to}. A \to B) = 0$ for some $\rho < \kappa$.
$\Rightarrow \mathsf{V}[M_{\rho,\lambda_\rho}](A \to B) = 0$ for some $\rho < \kappa$.
$\Rightarrow \mathsf{V}[M_{\mu,\lambda_\mu}](A \underset{L}{\to} B) = 0$ for some $\mu < \kappa - 1$.
$\Rightarrow \mathsf{V}[M_{\kappa-1,0}](A \to B) = 0$.

$\mathsf{V}[M_{\kappa-1,0}](A \to B) = 0 \Rightarrow \mathsf{V}[M_{\rho,\lambda_\rho}](A \underset{L}{\to} B) = 0$ for some $\rho < \kappa - 1$.
$\Rightarrow \mathsf{V}[M_{\rho,\lambda_\rho}](A \underset{L}{\to} B) = 0$ for some $\rho < \kappa$.
$\Rightarrow \mathsf{V}[M_{\kappa,0}](A \to B) = 0$.

Hence, $\mathsf{V}[M_{\kappa-1,0}](A \to B) = \mathsf{V}[M_{\kappa,0}](A \to B)$ and $M_{\kappa-1,\lambda_{\kappa-1}} = M_{\kappa,\lambda_\kappa}$, contradicting Lemma 4. Hence κ is not a successor ordinal and must be a limit ordinal.

Thus, the sequence of structures $\{M_{0,\lambda_0}, M_{1,\lambda_1}, \ldots, M_{\kappa,\lambda_\kappa}\}$, obtained by the above construction of (S), furnishes the model structure *MC* for *CST*.

A formula A with free variables $x_1, \ldots, x_n$ is *valid in MC* iff A is valid in $M_{\kappa,\lambda_\kappa}$, i.e., for all constant terms $a_1, \ldots, a_n$, $\mathsf{V}[M_{\kappa,\lambda_\kappa}](A\ {}^{a_1}\!/x_1 \ldots {}^{a_n}\!/x_n) = 1$. Such a formula A is *invalid in MC* iff $\mathsf{V}[M_{\kappa,\lambda_\kappa}](A\ {}^{a_1}\!/x_1 \ldots {}^{a_n}\!/x_n) = \frac{1}{2}$ or 0 for some constant terms $a_1, \ldots, a_n$. In particular, a sentence A is valid in *MC* iff $\mathsf{V}[M_{\kappa,\lambda_\kappa}](A) = 1$, and is invalid in *MC* iff $\mathsf{V}[M_{\kappa,\lambda_\kappa}](A) = \frac{1}{2}$ or 0. Validity in *MC* can be determined in terms of the structures in *MC* in accordance with the construction of (S). Those structures of (S) which are outside *MC* are essentially used as a means of obtaining the structures in *MC* and are not part of the valuation procedure for showing validity of formulas in *MC*.

It remains to show that the valid formulas of *CSQ'* and the axiom *GCA'* are valid in *MC*, and that the rules of *CTQ* and the rule *ER* preserve validity in *MC*. The consistency of *CST* can then easily be shown.

3 The soundness of CST with respect to MC In order to show that the valid formulas of *CSQ'* are valid in *MC*, I show that *MC* is a model structure of the *CSQ'* semantics and that its valuations V on its structures constitute a valuation on such a model structure. *MC* will consist of the ordered triple $\langle T, K, R\rangle$ where $K = \{M_{0,\lambda_0}, M_{1,\lambda_1}, \ldots, M_{\kappa,\lambda_\kappa}\}$, $T = M_{\kappa,\lambda_\kappa}$ and R consists of all the ordered pairs $\langle M_{\tau,\lambda_\tau}, M_{\nu,\lambda_\nu}\rangle$, where $\tau > 0$ and $\nu < \tau$, or $\tau = \nu = \kappa$. This definition of R satisfies $p1$, $p2$, and $p3$. The valuations $\mathsf{V}[M_{\tau,\lambda_\tau}]$ on the structures M_{τ,λ_τ}, for $0 \leqslant \tau \leqslant \kappa$, comprise a valuation V on the above *CSQ* m.s. Note that the relation R determines the precise set of structures needed for the evaluation of $A \rightarrow B$ at M_{0,λ_0}, $M_{\tau,\lambda_\tau} (0 < \tau < \kappa)$ and $M_{\kappa,\lambda_\kappa}$. Note also that the effect of the identity $M_{\kappa,\lambda_\kappa} = M_{\kappa+1,\lambda_{\kappa+1}}$, is built into the evaluation of $A \rightarrow B$ at $M_{\kappa,\lambda_\kappa}$ by requiring that:

$$\mathsf{V}[M_{\kappa,\lambda_\kappa}](A \rightarrow B) = 1 \text{ iff } \mathsf{V}[M_{\rho,\lambda_\rho}](A \underset{L}{\rightarrow} B) = 1 \text{ for all } \rho \leqslant \kappa.$$
$$\mathsf{V}[M_{\kappa,\lambda_\kappa}](A \rightarrow B) = 0 \text{ iff } \mathsf{V}[M_{\rho,\lambda_\rho}](A \underset{L}{\rightarrow} B) = 0 \text{ for some } p \leqslant \kappa.$$

This avoids the need for reference to $M_{\kappa+1,\lambda_{\kappa+1}}$, and allows K to be restricted to $\{M_{0,\lambda_0}, \ldots, M_{\kappa,\lambda_\kappa}\}$.

The rules that preserve validity in each *CSQ' m.s.* preserve validity in *MC* since *MC* is a *CSQ' m.s.* It is also clear that the axioms of *CTQ* are valid in *MC*, since again *MC* is a *CSQ' m.s.*

The following lemma enables the validity in *MC* of *GCA'* and hence *GCA*, to be shown.

Lemma 5 *For all admissible formulas A, with at most the variables x and y free, and for all constant terms a,* $a \in \{xy: A\} \underset{L}{\leftrightarrow} A\, {}^{a}\!/_{x}{}^{\{xy:A\}}\!/_{y}$ *is valid in* M_{τ,λ_τ}, *for all* τ.

Proof: $\mathsf{V}[M_{\tau,\lambda_\tau}](a \in \{xy: A\}) = \mathsf{V}[M_{\tau,\lambda_\tau+1}](a \in \{xy: A\}) = \mathsf{V}[M_{\tau,\lambda_\tau}](A\, {}^{a}\!/_{x}{}^{\{xy:A\}}\!/_{y})$, and hence $\mathsf{V}[M_{\tau,\lambda_\tau}](a \in \{xy: A\} \underset{L}{\leftrightarrow} A\, {}^{a}\!/_{x}{}^{\{xy:A\}}\!/_{y}) = 1$.

Theorem 2 *The GCA' and GCA are valid in MC.*

Proof: By Lemma 5, for all admissible formulas A, with at most x and y free, for all constant terms a, $\mathsf{V}[M_{\tau,\lambda_\tau}](a \in \{xy: A\} \underset{L}{\leftrightarrow} A\, {}^{a}\!/_{x}{}^{\{xy:A\}}\!/_{y}) = 1$, for all $\tau < \kappa$. Hence, $\mathsf{V}[M_{\kappa,\lambda_\kappa}](a \in \{xy: A\} \longleftrightarrow A\, {}^{a}\!/_{x}{}^{\{xy:A\}}\!/_{y}) = 1$ and the *GCA'*, $z \in \{xy: A\} \longleftrightarrow A\, {}^{z}\!/_{x}{}^{\{xy:A\}}\!/_{y}$, is valid in *MC*. Also, $\mathsf{V}[M_{\kappa,\lambda_\kappa}]((\forall x)(x \in \{xy: A\} \longleftrightarrow A\, {}^{\{xy:A\}}\!/_{y})) = 1$ and hence $\mathsf{V}[M_{\kappa,\lambda_\kappa}]((\exists y)(\forall x)(x \in y \longleftrightarrow A)) = 1$. Thus the *GCA* is valid in *MC*.

We now proceed to show that the *ER* preserves validity in *MC*. We show that if $\mathsf{V}[M_{\kappa,\lambda_\kappa}]((\forall z)(z \in a \longleftrightarrow z \in b)) = 1$ then $\mathsf{V}[M_{\kappa,\lambda_\kappa}](a \in d \longleftrightarrow b \in d) = 1$, for all constant terms a, b, and d. We let $\mathsf{V}[M_{\tau,\lambda_\tau}](c \in a) = \mathsf{V}[M_{\tau,\lambda_\tau}](c \in b)$, for constant terms c, for all $\tau < \kappa$, for the above constant terms a and b. We prove $\mathsf{V}[M_{\tau,\lambda_\tau}](a \in d) = \mathsf{V}[M_{\tau,\lambda_\tau}](b \in d)$, for all $\tau < \kappa$, by transfinite induction, with the induction hypothesis, $\mathsf{V}[M_{\nu,\lambda_\nu}](a \in d) = \mathsf{V}[M_{\nu,\lambda_\nu}](b \in d)$, for all $\nu < \tau$, for $\tau > 0$.

Hence, we will generally consider a τ-sequence, $M_{\tau,0}, M_{\tau,1}, \ldots, M_{\tau,\lambda_\tau}, \ldots$, for some $\tau < \kappa$, and, as in [3] and [4], analyse the process by which admissible sentences obtain a value 1 or 0 in M_{τ,λ_τ}. As in [3] and [4], we also require the concepts of determining set and the general determining set for admissible sentences.

Let A be an admissible sentence such that $\mathsf{V}[M_{\tau,\lambda_\tau}](A) = 1$ or 0. Let $\nu(\tau, A)$ be defined to be the least ordinal σ such that $\mathsf{V}[M_{\tau,\sigma}](A) = 1$ or 0. Consider the set $I(A)$ of all maximal initial subformulas of A, the maximality being with respect to subformula containment. Note that A can be constructed from its maximal initial subformulas by using $\sim$, &, and $\forall$. Form the set $S(A)$ of all sentential substitution instances of the initial formulas in $I(A)$, obtained by substituting constant terms for each of the free variables in the initial formulas. Call the set $D(A)$ of all sentential substitution instances in $S(A)$ which take the value 1 or 0 in $M_{\tau,\nu(\tau,A)}$, *the determining set of A*.

For any admissible sentence A and any constant term a, we can give special consideration to a subset of the occurrences of a in A. Let such a subset of occurrences of a in A be called *the set of designated occurrences of a in A*. For an admissible sentence A with a set of designated occurrences of the constant term a, we use the symbolization '$A(a)$'. If the constant term b is substituted for each of the designated occurrences of a in $A(a)$, we obtain $A(^b/_a)$ with a set of designated occurrences of b which occur at exactly the same argument places as the designated occurrences of a in $A(a)$.

As in [3] and [4], the point of introducing designated occurrences of constant terms is to trace these specified occurrences through the determining sets and general determining sets for admissible sentences and corresponding admissible sentences for initial sentences, thus enabling one to trace these occurrences through the process by which admissible sentences obtain a value 1 or 0 in M_{τ,λ_τ}.

The method of tracing designated occurrences of $A(a)$ through to the members of the determining set $D(A(a))$ of $A(a)$ is as follows: Let $\vee[M_{\tau,\lambda_\tau}](A(a)) = 1$ or 0. Then, for any initial sentences $B(a)$ of $D(A(a))$, the designated occurrences of a in $B(a)$ are those occurrences of a which are in the maximal initial subformula of $I(A(a))$ of which $B(a)$ is a sentential substitution instance, and which are designated occurrences of $A(a)$.

The following lemma gives the basic property of determining sets of admissible sentences:

Lemma 6 *Let $A(a)$ be an admissible sentence with a set of designated occurrences of a, such that $\vee[M_{\tau,\lambda_\tau}](A(a)) = 1$ or 0. If, for all $B(a) \in D(A(a))$, $\vee[M_{\tau,\lambda_\tau}](B(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](B(a))$, then $\vee[M_{\tau,\lambda_\tau}](A(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](A(a))$.*

Proof: The proof is as for Lemma 5 of [3], with M_{τ,λ_τ} replacing M_λ, and 1 and 0 replacing t and f.

Let $A(a)$ be an initial sentence with a set of designated occurrences of a, such that $A(a)$ is of the form $b \in \{xy: B\}(a)$, with B having at most the variables x and y free, and with $\{xy: B\}$ not being a designated occurrence of a.

Define *the corresponding admissible sentence of* $A(a)$, $C(A(a))$, as $B\ {}^b/_x{}^{\{xy:B\}}/_y$.

Note that, for successor ordinals ν, $\vee[M_{\tau,\nu}](A(a)) = \vee[M_{\tau,\nu-1}](C(A(a)))$. Hence, if $\vee[M_{\tau,\lambda_\tau}](A(a)) = 1$ or 0 then $\vee[M_{\tau,\lambda_\tau+1}](A(a)) = 1$ or 0 and $\vee[M_{\tau,\lambda_\tau}](C(A(a))) = 1$ or 0.

For all admissible sentences $A(a)$ with a set of designated occurrences of a, such that $\vee[M_{\tau,\lambda_\tau}](A(a)) = 1$ or 0, we define *the general determining set* $G(A(a))$ for each of these admissible sentences $A(a)$ by recursion on $\nu(\tau, A(a))$ as follows:

$$G(A(a)) = (D(A(a) - D'(A(a))) \cup \bigcup_{B(a)\in D'(A(a))} G(C(B(a))),$$

where $D'(A(a)) = \{B(a) \in D(A(a))$: $B(a)$ contains a designated occurrence of a and is of the form $c \in \{xy\colon C\}(a)$, with C having at most the variables x and y free, and with $\{xy\colon C\}$ not being a designated occurrence of $a\}$.

Note that all members of $G(A(a))$ are initial sentences, as can be shown by induction on $\nu(\tau, A(a))$.

In order to show that $G(A(a))$ is well-defined, we need to show that, for all $B(a) \in D'(A(a))$, $\vee[M_{\tau,\lambda_\tau}](C(B(a))) = 1$ or 0 and $\nu(\tau, C(B(a))) < \nu(\tau, A(a))$. Lemma 7 gives these results.

Lemma 7 *Let $A(a)$ be an admissible sentence with a set of designated occurrences of a, such that $\vee[M_{\tau,\lambda_\tau}](A(a)) = 1$ or 0. Let $B(a) \in D'(A(a))$. Then $\vee[M_{\tau,\lambda_\tau}](C(B(a))) = 1$ or 0 and $\nu(\tau, C(B(a))) < \nu(\tau, A(a))$.*

Proof: Note that $B(a)$ is an initial sentence of the correct form for $C(B(a))$ to be defined. Since $B(a) \in D(A(a))$, $\vee[M_{\tau,\nu(\tau,A(a))}](B(a)) = 1$ or 0, and hence $\vee[M_{\tau,\lambda_\tau}](B(a)) = 1$ or 0. Then $\vee[M_{\tau,\lambda_\tau}](C(B(a))) = 1$ or 0, as required. Also, by the construction (S), it is clear that $\nu(\tau, B(a))$ is a successor ordinal and that $\nu(\tau, C(B(a))) = \nu(\tau, B(a)) - 1$. Since $\nu(\tau, B(a)) \leqslant \nu(\tau, A(a))$, $\nu(\tau, C(B(a))) < \nu(\tau, A(a))$.

The following lemma provides a property of members of general determining sets that is needed for Lemma 12.

Lemma 8 *Let $A(a)$ be an admissible sentence with a set of designated occurrences of a, such that $\vee[M_{\tau,\lambda_\tau}](A(a)) = 1$ or 0. Then, for all $C(a) \in G(A(a))$, $\vee[M_{\tau,\lambda_\tau}](C(a)) = 1$ or 0 and $\nu(\tau, C(a)) \leqslant \nu(\tau, A(a))$.*

Proof: The proof is by transfinite induction on $\nu(\tau, A(a))$. (i) Let $C(a) \in D(A(a)) - D'(A(a))$. Then $C(a) \in D(A(a))$ and the lemma holds by definition of $D(A(a))$. (ii) Let $C(a) \in G(C(B(a)))$, for some $B(a) \in D'(A(a))$. By Lemma 7, $\nu(\tau, C(B(a))) < \nu(\tau, A(a))$, and, hence, by the induction hypothesis, $\vee[M_{\tau,\lambda_\tau}](C(a)) = 1$ or 0 and $\nu(\tau, C(a)) \leqslant \nu(\tau, C(B(a)))$. So $\nu(\tau, C(a)) \leqslant \nu(\tau, A(a))$, as required.

The following lemma shows for general determining sets, the basic property given in Lemma 6 for the determining sets of admissible sentences.

Lemma 9 *Let $A(a)$ be an admissible sentence with a set of designated occurrences of a, such that $\vee[M_{\tau,\lambda_\tau}](A(a)) = 1$ or 0. Then, if for all $B(a) \in G(A(a))$, $\vee[M_{\tau,\lambda_\tau}](B(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](B(a))$, then $\vee[M_{\tau,\lambda_\tau}](A(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](A(a))$.*

Proof: The proof is by transfinite induction on the ordinals $\nu(\tau, A(a))$. Let $\vee[M_{\tau,\lambda_\tau}](B(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](B(a))$, for all $B(a) \in G(A(a))$. Let $B'(a) \in D'(A(a))$. Then $\vee[M_{\tau,\lambda_\tau}](B(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](B(a))$, for all $B(a) \in G(C(B'(a)))$. By Lemma 7 and the induction hypothesis, the lemma holds for all admissible sentences $C(B'(a))$ such that $B'(a) \in D'(A(a))$, and therefore $\vee[M_{\tau,\lambda_\tau}](C(B'(a))(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](C(B'(a)))$, for each $B'(a) \in D'(A(a))$. Since $B'(a)$ has the form $c \in \{xy\colon C\}(a)$, with at most the variables x and y free, $C(B'(a))(^b/_a)$ is $C\,{}^c/_x{}^{\{xy:C\}}/_y(^b/_a)$, with b substituted for the designated occurrences of a in c and C. $C(B'(^b/_a))$ is also $C\,{}^c/_x{}^{\{xy:C\}}/_y(^b/_a)$, with same substitution, and hence $C(B'(a))(^b/_a) = C(B'(^b/_a))$. Then $\vee[M_{\tau,\lambda_\tau}](C(B'(^b/_a))) = \vee[M_{\tau,\lambda_\tau}](C(B'(a)))$, and by the definition of $C(B'(a))$, $V[M_{\tau,\lambda_\tau+1}](B'(^b/_a)) = V[M_{\tau,\lambda_\tau+1}](B'(a))$ and hence $V[M_{\tau,\lambda_\tau}](B'(^b/_a)) = V[M_{\tau,\lambda_\tau}](B'(a))$. Since $V[M_{\tau,\lambda_\tau}](B(^b/_a)) = V[M_{\tau,\lambda_\tau}](B(a))$, for all $B(a) \in D(A(a)) - D'(A(a))$, it follows that $V[M_{\tau,\lambda_\tau}](B(^b/_a)) = V[M_{\tau,\lambda_\tau}](B(a))$, for all $B(a) \in D(A(a))$. By Lemma 6, $V[M_{\tau,\lambda_\tau}](A(^b/_a)) = V[M_{\tau,\lambda_\tau}](A(a))$.

The next lemma gives the essential existential property for general determining sets, which enables $V[M_{\tau,\lambda_\tau}](a \in d) = V[M_{\tau,\lambda_\tau}](b \in d)$ to be shown by succeeding lemmas.

Lemma 10 *Let $A(a)$ be an admissible sentence with a set of designated occurrences of a, such that $V[M_{\tau,\lambda_\tau}](A(a)) = 1$ or 0. Then all the initial sentences of $G(A(a))$ that contain a designated occurrence of a are of one of the two forms:*

(I) $(c \in a)(a)$, *where the displayed occurrence of a is designated*

or

(II) $B(a) \rightarrow C(a)$, *where B and C are admissible sentences.*

Proof: The proof is by transfinite induction on $\nu(\tau, A(a))$.

(i) Let $B(a) \in D(A(a)) - D'(A(a))$ with $B(a)$ containing a designated occurrence of a. Then, by the definition of $D'(A(a))$, $B(a)$ is *not* of the form $c \in \{xy: C\}(a)$, with C having at most the variables x and y free, and with $\{xy: C\}$ not being a designated occurrence of a. Such an initial sentence $B(a)$ can only take one of the two forms (I) and (II) above, thus satisfying the lemma.

(ii) Let $B(a) \in G(C(B'(a)))$, for some $B'(a) \in D'(A(a))$. By Lemma 7 and the induction hypothesis, the lemma holds for $C(B'(a))$, and hence $B(a)$ takes one of the forms (I) and (II), thus satisfying the lemma.

We proceed to deal with the substitution of b for a into initial sentences of form (II) of Lemma 10.

Lemma 11 *Let $A(a)$ be an initial sentence with a set of designated occurrences of a, such that $A(a)$ is of the form $B(a) \rightarrow C(a)$ and $V[M_{\tau,\lambda_\tau}](A(a)) = 1$ or 0. If $\tau > 0$, let $V[M_{\nu,\lambda_\nu}](a \in d) = V[M_{\nu,\lambda_\nu}](b \in d)$, for all constant terms d, for all $\nu < \tau$. Then, $V[M_{\tau,\lambda_\tau}](A(^b/_a)) = V[M_{\tau,\lambda_\tau}](A(a))$.*

Proof: Let $\tau = 0$. Then $V[M_{\tau,\lambda_\tau}](B(a) \rightarrow C(a)) = 1$ and $V[M_{\tau,\lambda_\tau}](B(^b/_a) \rightarrow C(^b/_a)) = 1$. Hence, $V[M_{\tau,\lambda_\tau}](A(^b/_a)) = V[M_{\tau,\lambda_\tau}](A(a))$. So let $\tau > 0$. We first prove the following:

(+) For all admissible sentences $B(a)$, for all $\nu < \tau$, $V[M_{\nu,\lambda_\nu}](B(^b/_a)) = V[M_{\nu,\lambda_\nu}](B(a))$.

Since $V[M_{\nu,\lambda_\nu}](a \in d) = V[M_{\nu,\lambda_\nu}](b \in d)$, for all constant terms d, $V[M_{\nu,\lambda_\nu}](a \in \{z: B(^z/_a)\}) = V[M_{\nu,\lambda_\nu}](b \in \{z: B(^z/_a)\})$, where $B(a)$ is an admissible sentence with a set of designated occurrences of a, and z does not occur in $B(a)$. Hence, $V[M_{\nu,\lambda_\nu+1}](a \in \{z: B(^z/_a)\}) = V[M_{\nu,\lambda_\nu+1}](b \in \{z: B(^z/_a)\})$ and $V[M_{\nu,\lambda_\nu}](B(a)) = V[M_{\nu,\lambda_\nu}](B(^b/_a))$.

(i) Let $\vee[M_{\tau,\lambda_\tau}](B(a) \rightarrow C(a)) = 1$. Then, for all $\nu < \tau$, if $\vee[M_{\nu,\lambda_\nu}](B(a)) = 1$ then $\vee[M_{\nu,\lambda_\nu}](C(a)) = 1$, and if $\vee[M_{\nu,\lambda_\nu}](B(a)) = 1$ or $\frac{1}{2}$ then $\vee[M_{\nu,\lambda_\nu}](C(a)) = 1$ or $\frac{1}{2}$. By (+), for all $\nu < \tau$, if $\vee[M_{\nu,\lambda_\nu}](B(^b/_a)) = 1$ then $\vee[M_{\nu,\lambda_\nu}](C(^b/_a)) = 1$, and if $\vee[M_{\nu,\lambda_\nu}](B(^b/_a)) = 1$ or $\frac{1}{2}$ then $\vee[M_{\nu,\lambda_\nu}](C(^b/_a)) = 1$ or $\frac{1}{2}$. Hence, $\vee[M_{\tau,\lambda_\tau}](B(^b/_a) \rightarrow C(^b/_a)) = 1$ and $\vee[M_{\tau,\lambda_\tau}](A(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](A(a))$.

(ii) Let $\vee[M_{\tau,\lambda_\tau}](B(a) \rightarrow C(a)) = 0$. Then, for some $\nu < \tau$, $\vee[M_{\nu,\lambda_\nu}](B(a)) = 1$ and $\vee[M_{\nu,\lambda_\nu}](C(a)) = 0$. By (+), for some $\nu < \tau$, $\vee[M_{\nu,\lambda_\nu}](B(^b/_a)) = 1$ and $\vee[M_{\nu,\lambda_\nu}](C(^b/_a)) = 0$. Hence, $\vee[M_{\tau,\lambda_\tau}](B(^b/_a) \rightarrow C(^b/_a)) = 0$ and $\vee[M_{\tau,\lambda_\tau}](A(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](A(a))$.

We can now proceed to deal with the substitution of b for a into initial sentences of form (I) of Lemma 10.

Lemma 12 *Let $A(a)$ be an initial sentence with a set of designated occurrences of a, such that $A(a)$ is of the form $(c \in a)(a)$, with the displayed occurrence of a designated, and such that $\vee[M_{\tau,\lambda_\tau}](A(a)) = 1$ or 0. If $\tau > 0$, let $\vee[M_{\nu,\lambda_\nu}](a \in d) = \vee[M_{\nu,\lambda_\nu}](b \in d)$, for all constant terms d, for all $\nu < \tau$. Also, let $\vee[M_{\tau,\lambda_\tau}](e \in a) = \vee[M_{\tau,\lambda_\tau}](e \in b)$, for all constant terms e. Then $\vee[M_{\tau,\lambda_\tau}](A(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](A(a))$.*

Proof: The proof follows that of Lemma 12 of [4] and is by transfinite induction on $\nu(\tau, A(a))$. It is clear from the construction (S) that $\nu(\tau, A(a))$ is a successor ordinal. For the purposes of the applications of Lemmas 8, 9, 10, and 11 in this proof, we consider the set of designated occurrences of a in A as just those in c. We call A with this set of designated occurrences, $A(a)'$. By Lemma 10, all the initial sentences of $G(A(a)')$ that contain a designated occurrence of a are of one of the two forms: (I) $(c_1 \in a)(a)$, with the displayed occurrence of a designated, or (II) $B(a) \rightarrow C(a)$, where B and C are admissible sentences. For initial sentences $D(a)$ of form (II), by Lemma 11, $\vee[M_{\tau,\lambda_\tau}](D(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](D(a))$. We now consider initial sentences $(c_1 \in a)(a)$ of form (I). Since $A(a)' \in D'(A(a)')$,

$$G(A(a)') = \bigcup_{B(a) \in D'(A(a)')} G(C(B(a)))$$

and

$$(c_1 \in a)(a) \in G(C(B(a)))$$

for some $B(a) \in D'(A(a)')$. By Lemma 8, $\nu(\tau, (c_1 \in a)(a)) \leqslant \nu(\tau, C(B(a)))$, for some $B(a) \in D'(A(a)')$. By Lemma 7, $\nu(\tau, C(B(a))) < \nu(\tau, A(a)')$ and hence $\nu(\tau, (c_1 \in a)(a)) < \nu(\tau, A(a)')$. By the induction hypothesis, $\vee[M_{\tau,\lambda_\tau}]((c_1 \in a)(^b/_a)) = \vee[M_{\tau,\lambda_\tau}]((c_1 \in a)(a))$, for the initial sentences $(c_1 \in a)(a)$ of form (I). Hence, for all $D(a) \in G(A(a)')$, $\vee[M_{\tau,\lambda_\tau}](D(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](D(a))$. By Lemma 9, $\vee[M_{\tau,\lambda_\tau}](A(^b/_a)') = \vee[M_{\tau,\lambda_\tau}](A(a)')$, i.e., $\vee[M_{\tau,\lambda_\tau}](c(^b/_a) \in a) = \vee[M_{\tau,\lambda_\tau}](c(a) \in a)$, where $c(a)$ indicates that the constant term c has a set of designated occurrences of a, and $c(^b/_a)$ is c with b substituted for the designated occurrences of a in $c(a)$. By the condition of the lemma, $\vee[M_{\tau,\lambda_\tau}](c(^b/_a) \in a) = \vee[M_{\tau,\lambda_\tau}](c(^b/_a) \in b)$, and hence $\vee[M_{\tau,\lambda_\tau}](c(^b/_a) \in b) = \vee[M_{\tau,\lambda_\tau}](c(a) \in a)$, i.e., $\vee[M_{\tau,\lambda_\tau}](A(^b/_a)) = \vee[M_{\tau,\lambda_\tau}](A(a))$, as required.

The following lemma sets out what is required for the *ER* to preserve validity in *MC*.

Lemma 13 *Let a and b be constant terms. Then, if* $V[M_{\tau,\lambda_\tau}](e \in a) = V[M_{\tau,\lambda_\tau}](e \in b)$, *for all constant terms e, for all* $\tau < \kappa$, *then* $V[M_{\tau,\lambda_\tau}](a \in d) = V[M_{\tau,\lambda_\tau}](b \in d)$, *for all constant terms d, for all* $\tau < \kappa$.

Proof: We prove $V[M_{\tau,\lambda_\tau}](a \in d) = V[M_{\tau,\lambda_\tau}](b \in d)$, for all d, by transfinite induction on τ. For $\tau > 0$, as induction assumption, let $V[M_{\nu,\lambda_\nu}](a \in d) = V[M_{\nu,\lambda_\nu}](b \in d)$, for all $\nu < \tau$, for all d.

(α) Let $V[M_{\tau,\lambda_\tau}](a \in d) = 1$ or 0. Let $(a \in d)(a)$ have the set of designated occurrences of a consisting of just that occurrence on the left of '$\in$'. By Lemma 10, all the initial sentences of $G((a \in d)(a))$ which contain a designated occurrence of a, are of one of the two forms:

(I) $(c \in a)(a)$, with the displayed occurrence of a designated

or

(II) $B(a) \rightarrow C(a)$.

By Lemma 8, $V[M_{\tau,\lambda_\tau}](B(a) \rightarrow C(a)) = 1$ or 0. Hence, by Lemma 11, $V[M_{\tau,\lambda_\tau}](B(^b/_a) \rightarrow C(^b/_a)) = V[M_{\tau,\lambda_\tau}](B(a) \rightarrow C(a))$, for all these initial sentences of form (II). By Lemma 8, $V[M_{\tau,\lambda_\tau}]((c \in a)(a)) = 1$ or 0. Hence, by Lemma 12, $V[M_{\tau,\lambda_\tau}]((c \in a)(^b/_a)) = V[M_{\tau,\lambda_\tau}]((c \in a)(a))$, for all these initial sentences of form (I). Hence, for all $B(a) \in G((a \in d)(a))$, $V[M_{\tau,\lambda_\tau}](B(^b/_a)) = V[M_{\tau,\lambda_\tau}](B(a))$. Then, by Lemma 9, $V[M_{\tau,\lambda_\tau}]((a \in d)(^b/_a)) = V[M_{\tau,\lambda_\tau}]((a \in d)(a))$, i.e., $V[M_{\tau,\lambda_\tau}](b \in d) = V[M_{\tau,\lambda_\tau}](a \in d) = 1$ or 0.

(β) Let $V[M_{\tau,\lambda_\tau}](b \in d) = 1$ or 0. Let $(b \in d)(b)$ have the set of designated occurrences of b consisting of just that occurrence on the left of '$\in$'. Apply Lemmas 10, 8, 11, 12, and 9, as in (α), but substitute a for the designated occurrences of b, rather than vice versa. Then, $V[M_{\tau,\lambda_\tau}](a \in d) = V[M_{\tau,\lambda_\tau}](b \in d) = 1$ or 0.

By (α) and (β), we have shown that $V[M_{\tau,\lambda_\tau}](a \in d) = V[M_{\tau,\lambda_\tau}](b \in d)$, for all constant terms d. Thus, by transfinite induction, $V[M_{\tau,\lambda_\tau}](a \in d) = V[M_{\tau,\lambda_\tau}](b \in d)$, for all d, for all $\tau < \kappa$.

Theorem 3 *The Extensionality Rule ER preserves validity in MC.*

Proof: Let $V[M_{\kappa,\lambda_\kappa}]((\forall z)(z \in a \longleftrightarrow z \in b)) = 1$. Then $V[M_{\kappa,\lambda_\kappa}](e \in a \longleftrightarrow e \in b) = 1$, for all constant terms e, and hence $V[M_{\tau,\lambda_\tau}](e \in a) = V[M_{\tau,\lambda_\tau}](e \in b)$, for all e, for all $\tau < \kappa$. By Lemma 13, $V[M_{\tau,\lambda_\tau}](a \in d) = V[M_{\tau,\lambda_\tau}](b \in d)$, for all d, for all $\tau < \kappa$. Hence, $V[M_{\kappa,\lambda_\kappa}](a \in d \longleftrightarrow b \in d) = 1$.

Theorems 1, 2, and 3 enable simple consistency to be established for *CST*, as given by the following theorem.

Theorem 4 *CST is simply consistent.*

Proof: In the beginning of Section 3, it was shown that the valid formulas of *CSQ'* are valid in *MC* and that the rules that preserve validity in each *CSQ' m.s.*

preserve validity in *MC*. By Theorem 2, the *GCA'* is valid in *MC* and by Theorem 3, the rule *ER* preserves validity in *MC*. Hence *MC* is a model structure of *CST*, in which all the theorems of *CST* are valid. Let A be a theorem of *CST*. Then, $V[M_{\kappa,\lambda_\kappa}](A\ ^{a_1}/x_1, \ldots, {}^{a_n}/x_n) = 1$, for all constant terms $a_1, \ldots, a_n$, and $V[M_{\kappa,\lambda_\kappa}](\sim A\ ^{a_1}/x_1 \ldots {}^{a_n}/x_n) = 0$, for all constant terms $a_1, \ldots, a_n$. Hence, $\sim A$ is invalid in *MC* and is not a theorem of *CST*. So, *CST* is simply consistent, as required.

It is worth noting that if the logic *CSQ'* is replaced by *CTQ* or *TWQ*, with the same formulas as for *CSQ'*, then the resulting set theory is also simply consistent.

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