Notre Dame Journal of Formal Logic
Volume 24, Number 3, July 1983

Homogeneous Boolean Algebras with Very Nonsymmetric Subalgebras

SABINE KOPPELBERG and J. DONALD MONK

We prove the following theorems.

Theorem 1 *For every Boolean algebra A there are extensions $C \supseteq B \supseteq A$ such that B and C are homogeneous, every endomorphism or automorphism of A extends to an endomorphism or automorphism of B, and no nontrivial one-one endomorphism of B extends to an endomorphism of C.*

Theorem 2 *Assume ($\diamond$). There is an ω_1-Souslin tree T such that the regular open algebra B of T is homogeneous and has a complete subalgebra A onto which no nontrivial automorphism of B restricts.*

These theorems were motivated by the following question raised by Štěpánek: Does there exist a complete homogeneous Boolean algebra B with a complete homogeneous subalgebra A such that no nontrivial automorphism of A extends to B? Here a Boolean algebra B is called homogeneous if every principal ideal $B \upharpoonright b = \{x \in B \mid x \leqslant b\}$ for $b \neq 0$ is isomorphic to B; because of $B \cong B \upharpoonright b \times B \upharpoonright -b$, $B \upharpoonright b$ is also called a factor of B. B is said to be rigid if it has no nontrivial automorphism.

Štěpánek's question arose from the following facts. Every Boolean algebra A can be embedded into a homogeneous complete algebra B such that every automorphism of A extends to B (see [4] and [5]). Every A can be embedded into a complete rigid B—of course, no nontrivial automorphism of A extends to B (see [7]). Every A can be embedded into a complete B without homogeneous or rigid factors such that either every or no nontrivial automorphism of A extends to B (see [8] and [9]).

We assume acquaintance with [6] for the proof of Theorem 1 and with [1] or [3] for Theorem 2.

Received September 9, 1981; revised October 7, 1982

Proof of Theorem 1: Let A be given. Choose an ordinal α with $cf\,\alpha = \omega$ such that $|A| \leqslant \beth_\alpha$. Let $\kappa = \beth_\alpha$ and $\lambda = 2^\kappa$, so $\kappa^\omega = \lambda$.

Next, let A' be a Boolean algebra such that $|A'| = \kappa$ and each $A' \upharpoonright a$ where $a > 0$ contains a disjoint subset of power κ. By [4], there is a Boolean algebra A'' with $|A''| = \kappa$ such that the free product

$$B = A * A' * A''$$

is homogeneous; clearly $|B| = \kappa$ and each $B \upharpoonright b$ where $b > 0$ has a disjoint subset of power κ. B satisfies the conditions on A in the proof of Theorem 12 in [6]. Hence there is an atomless κ-complicated Boolean algebra B' such that

$$B \subseteq B' \subseteq (B * F)^{compl},$$

where F is the free Boolean algebra on λ free generators and D^{compl} denotes the completion of D. Note that the embedding from B to B' preserves all meets and joins existing in B. By [2], choose E such that

$$C = B' * E$$

is homogeneous.

Now let f be a nontrivial one-one endomorphism of B and assume that $\bar{f}$ is an endomorphism of C extending f. Choose $b \in B$ such that $b > 0$ and $b \cdot f(b) = 0$ and let $(a_\alpha)_{\alpha<\kappa}$ be a disjoint family in $B \upharpoonright b \setminus \{0\}$. By κ-complicatedness of B', there is an $S \subseteq \kappa$ satisfying:

(1) There is some $x \in B'$ such that $a_\alpha \leqslant x$ for $\alpha \in S$ and $a_\alpha \cdot x = 0$ for $\alpha \in \kappa \setminus S$

(2) There is no $y \in B'$ such that $f(a_\alpha) \leqslant y$ for $\alpha \in S$ and $f(a_\alpha) \cdot y = 0$ for $\alpha \in \kappa \setminus S$.

Write, since $\bar{f}(x) \in C = B' * E$,

$$\bar{f}(x) = \sum_{i<n} b_i \cdot e_i$$

where $b_i \in B'$, $e_i \in E$. But then $y = \sum_{i<n} b_i$ is an element of B' contradicting (2): for $\alpha \in S$, we have $a_\alpha \leqslant x$, $f(a_\alpha) \leqslant \bar{f}(x)$, so $f(a_\alpha) \leqslant \sum_{i<n} b_i$. For $\alpha \in \kappa \setminus S$, we have $a_\alpha \cdot x = 0$, $f(a_\alpha) \cdot \bar{f}(x) = 0$, so $f(a_\alpha) \cdot \sum_{i<n} b_i = 0$.

Proof of Theorem 2: Let $(S_\alpha)_{\alpha<\omega_1}$ be a sequence for $(\diamond)$. It is sufficient to construct a normal Souslin tree T of length ω_1 with levels U_α and objects $g_{\alpha uv}$, $\tilde{\alpha}$ such that the following claims (1) to (4) are satisfied.

(1) (a) For $\beta < \alpha < \omega_1$ and $u, v \in U_\beta$, $g_{\alpha uv}$ is an automorphism of $T_\alpha = \bigcup_{\gamma<\alpha} U_\gamma$ such that $g_{\alpha uv}(u) = v$

(b) $g_{\alpha uv} \subseteq g_{\alpha' uv}$ for $\beta < \alpha \leqslant \alpha' < \omega_1$

(c) $g_{\lambda uv} = \bigcup \{g_{\alpha uv} | \beta < \alpha < \lambda\}$ if λ is a limit ordinal such that $\beta < \lambda < \omega_1$.

For $u, v \in U_\beta$, $\bigcup_{\beta<\alpha} g_{\alpha uv}$ is then an automorphism of T mapping u to v. The regular open algebra B of T will then be homogeneous.

(2) (a) For $\alpha < \omega_1$, $\underset{\alpha}{\sim}$ is an equivalence relation on U_α

(b) if $\beta < \alpha < \omega_1$, $y \underset{\alpha}{\sim} y'$ and $x,x' \in U_\beta$ are such that $x < y$, $x' < y'$, then $x \underset{\beta}{\sim} x'$

(c) if $\beta < \alpha < \omega_1$, $x \underset{\beta}{\sim} x'$, $y \in U_\alpha$ and $x < y$, then there are infinitely many $y' \in U_\alpha$ such that $x' < y'$ and $y \underset{\alpha}{\sim} y'$

(d) if $\beta < \alpha < \omega_1$, $x \in U_\beta$, then there are $y,y' \in U_\alpha$ such that $x < y, y'$ and $y \underset{\alpha}{\not\sim} y'$.

The sequence $(\underset{\alpha}{\sim})_{\alpha<\omega_1}$ then gives rise to a complete subalgebra A of B (see [3]).

(3) $(S_\alpha)_{\alpha<\omega_1}$ diagonalizes each possible uncountable antichain of T and each possible nontrivial automorphism of B restricting to A.

The most complicated case to consider is: S_λ codes a maximal antichain a of T_λ plus a nontrivial automorphism ϕ of $T_\lambda \restriction c$, where $c \subseteq \lambda$ is closed unbounded in λ. For two branches b,b' of length λ in T_λ, let $b \approx b'$ mean that for each $\alpha < \lambda$, $x_\alpha \underset{\alpha}{\sim} x'_\alpha$ where x_α (respectively, x'_α) is the unique element of $b \cap U_\alpha$ (respectively, $b' \cap U_\alpha$). Then choose U_λ such that the set Z of λ-branches in T_λ corresponding to points in U_λ satisfies:

(a) $\bigcup Z = T_\lambda$

(b) $b \cap a \neq \phi$ for $b \in Z$

(c) Z is closed under the obvious action of each $g_{\lambda uv}$ (where $u,v \in U_\beta$, $\beta < \lambda$) and of ϕ on the λ-branches of T_λ

(d) if $b \in Z$, $x \in b \cap U_\alpha$, $x \underset{\alpha}{\sim} x'$, then there is some $b' \in Z$ such that $x' \in b'$ and $b \approx b'$.

The existence of Z satisfying this countable list of requirements is most easily seen by a forcing style argument.

(4) If S_λ codes (a,ϕ) and U_λ is chosen as in (3), then there are $u,u', v,w \in U_\lambda$ such that $\phi(u) = v$, $\phi(u') = w$ under the obvious action of ϕ on U_λ and such that $u \underset{\lambda}{\sim} u'$ but $v \underset{\lambda}{\not\sim} w$.

This guarantees that (the automorphism of B induced by) ϕ does not restrict to A.

REFERENCES

[1] Devlin, K. and H. Johnsbraten, *The Souslin Problem*, Lecture Notes in Mathematics 405 (1974).

[2] Grätzer, G., "Homogeneous Boolean algebras," *Notices of the American Mathematical Society*, vol. 20 (1973), A, p. 565.

[3] Jech, T., "Simple complete Boolean algebras," *Israel Journal of Mathematics*, vol. 18 (1974), pp. 1-10.

[4] Koppelberg, S., "A lattice structure on the isomorphism types of complete Boolean algebras," pp. 98-126 in *Set Theory and Model Theory*, Lecture Notes in Mathematics 872 (1981).

[5] Kripke, S., "An extension of a theorem of Gaifman-Hales-Solovay," *Fundamenta Mathematicae*, vol. 61 (1967), pp. 29-32.

[6] Monk, J. D., "A very rigid Boolean algebra," *Israel Journal of Mathematics*, vol. 35, 1-2 (1980), pp. 135-150.

[7] Štěpánek, P. and B. Balcar, "Embedding theorems for Boolean algebras and consistency results on ordinal definable sets," *The Journal of Symbolic Logic*, vol. 42 (1977), pp. 64-75.

[8] Štěpánek, P., "Boolean algebras with no rigid or homogeneous factors," *Transactions of the American Mathematical Society*, vol. 270 (1982), pp. 131-147.

[9] Štěpánek, P., "Embeddings of Boolean algebras and automorphisms," *Abstracts of the American Mathematical Society*, vol. 3, no. 1 (1982), p. 131.

Sabine Koppelberg
II. Mathematisches Institut
Freie Universität Berlin
Königin-Luise-Str. 24-26
D-Berlin 33, West Germany

J. Donald Monk
Department of Mathematics
University of Colorado
Campus Box 426
Boulder, Colorado 80309