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Minimally Incomplete Sets of Łukasiewiczian Truth Functions

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By an *n-valued truth function* we shall understand a function on the set $\{1,2,\ldots,n\}$. If such a function is closed on the set $\{1,n\}$, it will be said to be *pure*. And, if it can be defined by composition from $\neg$ and $\rightarrow$, it will be referred to as *Łukasiewiczian*. Here:

$$\neg p = (n - p) + 1$$

and

$$(p \rightarrow q) = \max[1,(q - p) + 1].$$

In [3] it is proved, for the three-valued case, that the set of Łukasiewiczian functions and the set of pure functions are one and the same. It is also observed, again in the three-valued case, that if f is non-Łukasiewiczian, then $\{\neg, \rightarrow, f\}$ is functionally complete (i.e., all three-valued functions can be defined by composition from $\neg$, $\rightarrow$, and f). The import of the latter result is that although $\neg$ and $\rightarrow$ are together functionally incomplete, their incompleteness is *minimal*. That is, when $\{\neg,\rightarrow\}$ is supplemented with a "new" function, the resulting set is always functionally complete.

It is the purpose of the present essay to establish a more general result from which the previous two can be derived as corollaries:

Theorem 3 *The following are equivalent if* $2 < n$:

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(a) *$n - 1$ is prime*
(b) *The set of $Ł_n$-functions is exactly the set of pure n-valued functions*
(c) *The incompleteness of $Ł_n$ is minimal.*

Therefore:

Corollary 1 *The incompleteness of $Ł_3$ is minimal* (Theorem 1 of [3]).
Corollary 2 *The set of $Ł_3$-functions is exactly the set of pure three-valued functions* (Theorem 2 of [3]).

The theorem is an immediate consequence of two theorems that follow. But first some lemmas will be needed.

Lemma 1 *If the set of $Ł_n$-functions is exactly the set of pure n-valued functions, then $n - 1$ is prime.*

Proof: We argue the contrapositive. Assume that $n - 1$ is nonprime. Then, let k be the least number such that $1 < k < n - 1$ and $n - 1$ is divisible by k. Suppose that $n - 1 = jk$. Consider now the set $\mu = \{1, k + 1, 2k + 1, \ldots, jk + 1\}$ where $jk + 1 = n$. Notice first that $2 \notin \mu$. Otherwise $k + 1 = 2$, and k is not the least number >1 that divides $n - 1$. Notice next that μ is closed with respect to $\neg$. For let $pk + 1$ $(0 \leqslant p \leqslant j)$ be an arbitrary member of μ. Then, $\neg(pk + 1) = n - (pk + 1) + 1 = n - pk - 1 + 1 = jk + 1 - pk = (j - p)k + 1 \in \mu$. Finally, notice that μ is closed with respect to $\rightarrow$. For let $pk + 1$ $(0 \leqslant p \leqslant j)$ and $qk + 1$ $(0 \leqslant q \leqslant j)$ be any two members of μ. Then, $(pk + 1) \rightarrow (qk + 1) = \max[1, qk + 1 - (pk + 1) + 1] = \max[1, qk + 1 - pk - 1 + 1] = \max[1, (q - p)k + 1] \in \mu$. Thus, $2 \notin \mu$, and μ is closed with respect to both $\neg$ and $\rightarrow$. It follows that no function having the value 2 when each of its arguments is from μ can be defined in terms of $\neg$ and $\rightarrow$. But some such functions are pure. So, not all pure functions are $Ł_n$-functions, and the set of $Ł_n$-functions is distinct from the set of pure n-valued functions.

Lemma 2 *If the functional incompleteness of $Ł_n$ is minimal, then $n - 1$ is prime.*

Proof: For the contrapositive, assume that $n - 1$ is not prime. By Lemma 1 the set of $Ł_n$-functions is distinct from the set of pure n-valued functions. It is easily verified that $\{1, n\}$ is closed under both $\neg$ and $\rightarrow$. Thus, all $Ł_n$-functions are pure. So there is some pure n-valued function that is not an $Ł_n$-function, i.e., there is some function f such that $\{1, n\}$ is closed under f, and f is not definable from $\neg$ and $\rightarrow$. Thus, $\{1, n\}$ is closed under each member of $\{\neg, \rightarrow, f\}$, and $\{\neg, \rightarrow, f\}$ is therefore functionally incomplete. Since f is non-Łukasiewiczian, the functional incompleteness of $\{\neg, \rightarrow\}$ is not minimal.

Subsequent proofs will make use of the familiar Łukasiewiczian & and $\vee$ where

$$(p \,\&\, q) = \max[p, q]$$

and

$$(p \vee q) = \min[p, q].$$

They will also make use of the H- and J-functions of Rosser and Turquette [5]. The H-functions are Łukasiewiczian and so defined that they have the following property for each i $(1 \leqslant i \leqslant n)$:

$$H_i(p) = \max[1, n - (p - 1)i].$$

Similarly, the J-functions are Łukasiewiczian and have the property:

$$\text{(J)} \qquad J_i(p) = \begin{cases} 1 \text{ if } p = i \\ n \text{ if } p \neq i. \end{cases}$$

Regarding the former of these functions we now establish:

Lemma 3 *If $n - 1$ is prime and $2 < p < n$, then there exists a Łukasiewiczian function H_k such that $1 < H_k(p) < p$.*

Proof: Assume that $n - 1$ is prime and that $2 < p < n$. Let k be the largest integer such that $(p - 1)k < n$. Then, $H_k(p) = n - (p - 1)k$. It follows that (a): $1 < H_k(p)$. For assume otherwise. Then $H_k(p) = 1$, i.e., $n - (p - 1)k = 1$. So $n - 1 = (p - 1)k$. Since $n - 1$ is prime and $2 < p$, $k = 1$. Thus, $n - 1 = p - 1$, and $n = p$. But this contradicts the assumption that $p < n$. (b) $H_k(p) < p$. For a contradiction, assume that $p \leqslant H_k(p)$. Then, $p \leqslant n - (p - 1)k$. Whence $p + (p - 1)k \leqslant n$. Thus, $(p - 1) + (p - 1)k < n$. So $(p - 1)(k + 1) < n$. But this contradicts the assumption that k is the largest integer for which $(p - 1)k < n$.

Next we observe that:

Lemma 4 *There is a Łukasiewiczian function $F_i(p)$ with the property*

$$F_i(p) = \begin{cases} i \text{ if } p = 2 \\ n \text{ if } p \neq 2. \end{cases}$$

Proof: Let $F_i(p) = \neg H_{i-1}(p)$ & $J_2(p)$. From (J) it is clear that $F_i(p) = n$ if $p \neq 2$. Assume that $p = 2$. Then, $J_i(p) = 1$. So $F_i(p) = \neg H_{i-1}(2) = \neg\max[1, n - (2 - 1)(i - 1)] = \neg\max[1, n - (i - 1)] = \neg(n - (i - 1)) = (n - (n - (i - 1))) + 1 = i$.

With the help of Lemma 4 and the H- and J-functions we can now establish that

Lemma 5 *If $n - 1$ is prime and $1 < i < n$, then there exists a Łukasiewiczian function G_i such that*

$$G_i(p) = \begin{cases} 2 \text{ if } p = i \\ n \text{ if } p \neq i \end{cases}$$

Proof: It is clear from Lemma 4 that we can let $G_2(p) = F_2(p)$. Assume now that $G_2, \ldots, G_{i-1}$ have already been defined. By hypothesis and Lemma 3, there exists a k such that $1 < H_k(i) < i$. Let $H_k(i) = j$. Since $j < i$, G_j has already been defined, and $G_j(H_k(p)) = 2$ if $p = i$. Since $J_i(p) = 1$ if $p = i$, $G_j(H_k(p))$ & $J_i(p) = 2$ if $p = i$. And, since $J_i(p) = n$ if $p \neq i$, $G_j(H_k(p))$ & $J_i(p) = n$ if $p \neq i$. Therefore, $G_i(p)$ may be defined as $G_j(H_k(p))$ & $J_i(p)$.

We are now in a position to prove that

Lemma 6 *If $n - 1$ is prime, then the set of $Ł_n$-functions is exactly the set of pure n-valued functions.*

Proof: It was earlier observed that all $Ł_n$-functions are pure. So to prove the lemma it will suffice to prove that all pure n-valued functions are $Ł_n$-functions under the hypothesis that $n - 1$ is prime. Let f be any n-valued function of (say) degree m. Consider an arbitrary row i from a table that characterizes f.

$p_1 \ldots$	p_m	$f(p_1, \ldots, p_m)$	
.	.	.	
.	.	.	
.	.	.	
α_1	α_m	β	(row i)
.	.	.	
.	.	.	
.	.	.	

Observe now that we can write a *representative* formula R_i that has the value β on row i and the value n on every other row. *Case 1*. $\beta = 1$. Let $R_i = J_{\alpha_1}(p_1)$ & ... & $J_{\alpha_m}(p_m)$. *Case 2*. β is one of the *nonclassical* values $2, \ldots, n - 1$. Since f is pure, at least one of $\alpha_1, \ldots, \alpha_m$ must be nonclassical. Consider now the formula $V(p_1)$ & ... & $V(p_m)$ where for each j from 1 to m: $V(p_j) = J_{\alpha_j}(p_j)$ if α_j is classical, and $V(p_j) = G_{\alpha_j}(p_j)$ if α_j is nonclassical. From (J) and Lemma 5 it is clear that $V(p_1)$ & ... & $V(p_m)$ has the value 2 on row i and the value n on every other row. Thus, by Lemma 4, $F_\beta(V(p_1)$ & ... & $V(p_m))$ has the value β on row i and the value n on every other row. So let $R_i = F_\beta(V(p_1)$ & ... & $V(p_m))$. *Case 3*. $\beta = n$. Let $R_i = \neg(p_1 \rightarrow p_1)$. It is now clear that $f(p_1, \ldots, p_m)$ can be defined as $R_1 \vee \ldots \vee R_k$ where $R_1, \ldots, R_k$ are the k ($= n^m$) representative formulas of the rows of the table characterizing f.

From Lemmas 1 and 6 we may conclude that

Theorem 1 *The set of $Ł_n$-functions is exactly the set of pure n-valued functions if and only if $n - 1$ is prime.*

Next we prove that

Lemma 7 *If $n - 1$ is prime, then the functional incompleteness of $Ł_n$ is minimal.*

Proof: Assume that $n - 1$ is prime and that f is non-Łukasiewiczian. From Theorem 1 it follows that f is impure. Thus there are elements $\alpha_1, \ldots, \alpha_m$ each of which is either 1 or n such that the value of $f(p_1, \ldots, p_m)$ is non-classical (say j) when the values of $p_1, \ldots, p_m$ are respectively $\alpha_1, \ldots, \alpha_m$. Consider the formula $f(p_1^*, \ldots, p_m^*)$ where p_i^* is $(p \rightarrow p)$ or $\neg(p \rightarrow p)$ according as α_i is 1 or n. It is clear that the value of $f(p_1^*, \ldots, p_m^*)$ is j for all assignments of elements to p. Then, by Lemma 5, the value of $G_j(f(p_1^*, \ldots, p_m^*))$ is uniformly 2. Thus, Słupecki's T-function can be defined in terms of $\{\neg, \rightarrow, f\}$. But $\{\neg, \rightarrow, T\}$ is known to be functionally complete. (See Rosser and Turquette [5], pp. 23-25.) So the lemma is proved.

From Lemmas 2 and 7 we may conclude that

Theorem 2 *The functional completeness of $Ł_n$ is minimal if and only if $n - 1$ is prime.*

Further results relating to functionally complete extensions of $\{\neg, \rightarrow\}$ may be found in [1], [2], and [4].

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