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# Sums of Finitely Many Ordinals of Various Kinds

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***Abstract*** The ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$ are said to be *pairwise-noncommutative* if for all $i, j = 1, 2, \ldots, n$, if $i \neq j$, then $\alpha_i + \alpha_j \neq \alpha_j + \alpha_i$. For positive integers $n$ and $k$, let $\Sigma_n$ be the symmetric group on $n$ letters and let $E_n$ (respectively $L_n$, $S_n$, $T_n$, or $P_n$) be the set of all $k$ for which there exist $n$ (not necessarily distinct) nonzero ordinals (respectively, limit ordinals, successor ordinals, infinite successor ordinals, or pairwise-noncommutative ordinals) such that $\sum_{i=1}^{n} \alpha_{\phi(i)}$ takes on exactly $k$ values as $\phi$ ranges over $\Sigma_n$. Then for all $n \geq 1$, $E_n = L_n = S_n = T_n$; $min\, P_n = n$, and $max\, P_n = max\, E_n$. Furthermore, $P_1 = E_1$, $P_2 = E_2$, $P_3 = E_3 - \{1, 2\}$, and $P_4 = E_4 - \{1, 2, 3, 11\}$.

***1 Introduction*** Addition of ordinal numbers depends upon the order of the summands. For each positive integer $n$, the maximum number, $m_n$, of distinct values that can be assumed by a sum of $n$ nonzero ordinal numbers in all $n!$ permutations of the summands has been calculated by Erdös [1] and Wakulicz [3] and [4]. The first few values of $m_n$ are as follows: $m_1 = 1$, $m_2 = 2$, $m_3 = 5$, $m_4 = 13$, $m_5 = 33$, $m_6 = 81$, $m_7 = 193$, $m_8 = 449$; moreover, it is known that $\lim_{n\to\infty} \dfrac{m_n}{n!} = 0$.

Let $n$ and $k$ be positive integers. Let $\Sigma_n$ be the symmetric group on $n$ letters. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any $n$ (not necessarily distinct) nonzero ordinals. We will say that $\alpha_1, \alpha_2, \ldots, \alpha_n$ *yield k* sums if $\left\{\sum_{i=1}^{n} \alpha_{\phi(i)} : \phi \in \Sigma_n\right\}$ is a $k$-element set. Let $E_n$ be the set of all integers $k$ for which there exist $n$ (not necessarily distinct) nonzero ordinals that yield $k$ sums. It is known that $E_n = \{1, 2, 3, \ldots, m_n\}$ for $n = 1, 2, 3, 4, 6, 7$, and 8 ([2], [5], and [6]), that $E_5 = \{1, 2, 3, \ldots, 29\} \cup \{31, 32, 33\}$ ([3]), and that $E_n$ is properly included in $\{1, 2, 3, \ldots, m_n\}$ for all $n \geq 9$ ([7]).

For every ordinal number $\alpha > 0$, let

(1) $\alpha = \omega^{\lambda_1}a_1 + \omega^{\lambda_2}a_2 + \ldots + \omega^{\lambda_r}a_r$

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be the *(Cantor) normal form* of $\alpha$; here $r$, $a_1, a_2, \ldots, a_r$ are positive integers and $\lambda_1 > \lambda_2 > \ldots > \lambda_r \geq 0$ are ordinals. $\lambda_1$ is called the *degree of* $\alpha$ (written, "*deg* $\alpha$") and $\alpha_1$, *the leading coefficient of* $\alpha$. By the *remainder of* $\alpha$, we mean $\omega^{\lambda_2}a_2 + \ldots + \omega^{\lambda_r}a_r$ (or zero, if $r = 1$). By the *remainder form of* $\alpha$, we mean $\omega^{\lambda_1}a_1 + \rho_1$, where $\lambda_1$ is the degree of $\alpha$, $a_1$ is the leading coefficient of $\alpha$, and $\rho_1$ is the remainder of $\alpha$.

The ordinal numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are said to be *nonoverlapping* if for each $i, j\ (\neq i) = 1, 2, \ldots, n$, whenever $\lambda_i = deg\,\alpha_i > deg\,\alpha_j = \lambda_j$, then (1), the normal form of $\alpha_i$, consists of terms all of which are of degree $>\lambda_j$. Addition of nonoverlapping ordinals is considerably simpler than in the general case, and is considered in [6]. Here and in [8] we consider the addition of various other types of ordinals.

***2 Limit ordinals; successor ordinals*** Let $L_n$ be the set of all integers $k$ for which there exist $n$ (not necessarily distinct) limit ordinals that yield $k$ sums; let $S_n$ be the set of all integers $k$ for which there exist $n$ (not necessarily distinct) successor ordinals that yield $k$ sums, and let $T_n$ be the set of all integers $k$ for which there exist $k$ (not necessarily distinct) infinite successor ordinals that yield $k$ sums.

**Theorem 1** *For all* $n = 1, 2, 3, \ldots, E_n = L_n = S_n = T_n$.

*Proof:* Clearly, $L_n \subseteq E_n$ and $T_n \subseteq S_n \subseteq E_n$.

For any nonzero ordinal $\alpha$ whose normal form is given by (1), let

$$\alpha' = \omega^{\lambda_1+1}a_1 + \omega^{\lambda_2+1}a_2 + \ldots + \omega^{\lambda_r+1}a_r$$

$$\alpha'' = \begin{cases} \alpha + 1, & \text{if } \alpha \text{ is infinite} \\ \alpha\quad, & \text{if } \alpha \text{ is finite} \end{cases}$$

and

$$\alpha''' = \omega^{\lambda_1+1}a_1 + \omega^{\lambda_2+1}a_2 + \ldots + \omega^{\lambda_r+1}a_r + 1.$$

Let $k \in E_n$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ yield $k$ sums. Suppose that for some $\phi \in \Sigma_n$,

$$\sum_{i=1}^{n} \alpha_{\phi(i)} = \omega^{\delta_1}b_1 + \omega^{\delta_2}b_2 + \ldots + \omega^{\delta_s}b_s.$$

Then

$$\sum_{i=1}^{n} (\alpha_{\phi(i)}') = \omega^{\delta_1+1}b_1 + \omega^{\delta_2+1}b_2 + \ldots + \omega^{\delta_s+1}b_s$$

so that $\alpha_1', \alpha_2', \ldots, \alpha_n'$ yield $k$ sums, and consequently, $E_n \subseteq L_n$. Clearly $1 \in S_n$ for all $n$. To see that $E_n \subseteq S_n$ for all $n$, we can assume that at least one of the ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$ is infinite. Then

$$\sum_{i=1}^{n} (\alpha_{\phi(i)}'') = \left(\sum_{i=1}^{n} \alpha_{\phi(i)}\right) + 1$$

so that $\alpha_1'', \alpha_2'', \ldots, \alpha_n''$ yields $k$ sums, and $E_n \subseteq S_n$. Finally,

$$\sum_{i=1}^{n} (\alpha_{\phi(i)}''') = \omega^{\delta_1+1}b_1 + \omega^{\delta_2+1}b_2 + \ldots + \omega^{\delta_s+1}b_s + 1$$

so that $\alpha_1'''$, $\alpha_2'''$, $\ldots$, $\alpha_n'''$ yield $k$ sums, and $E_n \subseteq T_n$. Thus for all $n$, $E_n = L_n = S_n = T_n$, as was to be proved.

***3 Pairwise-noncommutative ordinals*** Let $\alpha = \omega^{\lambda_1}a_1 + \rho$ and $\beta = \omega^{\mu_1}b_1 + \sigma$ be the remainder forms of the nonzero ordinals $\alpha$ and $\beta$, respectively. Then it is well-known that $\alpha + \beta = \beta + \alpha$ if and only if $\lambda_1 = \mu_1$ and $\rho = \sigma$. In other words, two nonzero ordinals commute if and only if they agree in their degrees and in their remainders.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any $n$ nonzero ordinals. Then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are said to be *pairwise-noncommutative* if for all $i, j = 1, 2, \ldots, n$, if $i \neq j$, then $\alpha_i + \alpha_j \neq \alpha_j + \alpha_i$. In many of the examples in [2], [3], [5], and [6], ordinals repeat, or more than one ordinal is finite, or several ordinals are integral multiples of $\omega$. These examples thus make use of $n$ ordinals, at least two of which commute. Addition of pairwise-noncommutative ordinals is considerably more restrictive.

For each $n$, let $P_n$ be the set of all integers $k$ for which there exist $n$ pairwise-noncommutative ordinals that yield $k$ sums.

**Lemma 1** *Suppose that for ordinals $\alpha$ and $\beta$, $\alpha + \beta \neq \beta + \alpha$.*
*If $deg\,\beta < deg\,\alpha$, then*

$$\alpha = \beta + \alpha < \alpha + \beta.$$

*If $deg\,\beta = deg\,\alpha$ and $rem\,\beta < rem\,\alpha$, then*

$$\alpha + \beta < \beta + \alpha.$$

**Theorem 2** *For all $n \geq 1$, $min\,P_n = n$ and $max\,P_n = m_n$.*

*Proof:* We first show that for all $n \geq 1$, every set of $n$ pairwise-noncommutative ordinals yields at least $n$ distinct sums. For $n = 1$, this is obvious.

Let $n > 1$ and suppose that for $1 \leq k < n$, every set of $k$ pairwise-noncommutative ordinals yields at least $k$ distinct sums. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are pairwise-noncommutative ordinals and $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. If $deg\,\alpha_{n-1} < deg\,\alpha_n$, let $A_1, A_2, \ldots, A_{n-1}$ be sums for $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that $A_1 < A_2 < \ldots < A_{n-1}$. Then $deg\,A_1 < deg\,\alpha_n$ so that, by Lemma 1,

$$A_1 + \alpha_n = \alpha_n < \alpha_n + A_1 < \alpha_n + A_2 < \ldots < \alpha_n + A_{n-1}$$

and consequently,

$$A_1 + \alpha_n,\ \alpha_n + A_1,\ \alpha_n + A_2, \ldots, \alpha_n + A_{n-1}$$

are $n$ distinct sums for $\alpha_1, \alpha_2, \ldots, \alpha_n$.

If $deg\,\alpha_{n-1} = deg\,\alpha_n$, let $m$ be the smallest index for which $deg\,\alpha_m = deg\,\alpha_n$. If $m = 1$ and if $A_1, A_2, \ldots, A_{n-1}$ are distinct sums for $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that $A_1 < A_2 < \ldots < A_{n-1}$, then because $\alpha_1, \alpha_2, \ldots, \alpha_n, A_1, A_2, \ldots,$ and $A_{n-1}$ are all of the same degree and because

$$rem\,A_1 = rem\,\alpha_1 < rem\,\alpha_2 = rem\,A_2 < \ldots < \\ rem\,\alpha_{n-1} = rem\,A_{n-1} < rem\,\alpha_n$$

it follows that

$$\alpha_n + A_1 < \alpha_n + A_2 < \ldots < \alpha_n + A_{n-1} < A_{n-1} + \alpha_n$$

so that $\alpha_n + A_i$, $i = 1, 2, \ldots, n-1$, together with $A_{n-1} + \alpha_n$ are $n$ distinct sums for $\alpha_1, \alpha_2, \ldots, \alpha_n$.

If $m > 1$, then let

$$\begin{aligned}
B_m &= \sum_{i=1}^{m-1} \alpha_i + \sum_{i=m+1}^{n-1} \alpha_i + \alpha_m = \sum_{i=m+1}^{n-1} \alpha_i + \alpha_m \\
B_{m+1} &= \sum_{i=1}^{m} \alpha_i + \sum_{i=m+2}^{n-1} \alpha_i + \alpha_{m+1} = \alpha_m + \sum_{i=m+2}^{n-1} \alpha_i + \alpha_{m+1} \\
&\vdots \\
B_{n-1} &= \sum_{i=1}^{n-1} \alpha_i = \sum_{i=m}^{n-1} \alpha_i.
\end{aligned}$$

Then $B_m < B_{m+1} < \ldots < B_{n-1}$, so that $B_m, B_{m+1}, \ldots, B_{n-1}$ are $n - m$ distinct sums for $\alpha_m, \alpha_{m+1}, \ldots, \alpha_{n-1}$ as well as for $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. Moreover,

(2) $\alpha_n + B_m < \alpha_n + B_{m+1} < \ldots < \alpha_n + B_{n-1}$

so that $\alpha_n + B_i$, $i = m, m+1, \ldots, n-1$, are $n - m$ distinct sums for $\alpha_1, \alpha_2, \ldots, \alpha_n$. Furthermore, by the inductive hypothesis, there are (at least) $m$ distinct sums, $C_1, C_2, \ldots, C_{m-1}, C_n$ for $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \alpha_n$. We can assume that $C_1 < C_2 < \ldots < C_{m-1} < C_n$, and consequently,

$$B_{n-1} + C_1 < B_{n-1} + C_2 < \ldots < B_{n-1} + C_{m-1} < B_{n-1} + C_n.$$

Each of the ordinals $B_{n-1} + C_i$, $i = 1, 2, \ldots, m-1, n$, is a sum for $\alpha_1, \alpha_2, \ldots, \alpha_n$. Finally, using the lemma, we see that

(3) $\alpha_n + B_{n-1} < B_{n-1} + \alpha_n \leq B_{n-1} + C_1 < B_{n-1} + C_2 < \ldots <$
$B_{n-1} + C_{m-1} < B_{n-1} + C_n$

so that by (2) and (3),

$$\begin{aligned}
\alpha_n + B_m < \alpha_n + B_{m+1} < \ldots < \alpha_n + B_{n-1} < B_{n-1} + C_1 \\
< B_{n-1} + C_2 < \ldots < B_{n-1} + C_{m-1} < B_{n-1} + C_n.
\end{aligned}$$

This proves that $\alpha_n + B_i$, $i = m, m+1, \ldots, n-1$, together with $B_{n-1} + C_j$, $j = 1, 2, \ldots, m-1, n$, are $n$ distinct sums for $\alpha_1, \alpha_2, \ldots, \alpha_n$.

For all $n \geq 1$, $\omega + 1, \omega + 2, \ldots, \omega + n$ are $n$ pairwise-noncommutative ordinals with sums $\omega n + 1, \omega n + 2, \ldots, \omega n + n$. Thus, $\min P_n = n$. Finally, Wakulicz [3] has shown, in effect, that the maximal sum, $m_n$, for $E_n$ can always be obtained by using $n$ pairwise-noncommutative ordinals of the form

$$\begin{gathered}
\omega^{2r};\ \omega^{2r-1} + \omega^{2r-2},\ \omega^{2r-1}\cdot 2 + \omega^{2r-2}\cdot 2,\ \omega^{2r-1}\cdot 4 + \omega^{2r-2}\cdot 3, \\
\ldots, \omega^{2r-1}\cdot 2^{x_r - 1} + \omega^{2r-2}\cdot x_r; \\
\omega^{2r-3} + \omega^{2r-4},\ \omega^{2r-3}\cdot 2 + \omega^{2r-4}\cdot 2,\ \omega^{2r-3}\cdot 4 + \omega^{2r-4}\cdot 3, \\
\ldots, \omega^{2r-3}\cdot 2^{x_{r-1} - 1} + \omega^{2r-4}\cdot x_{r-1}; \\
\vdots \\
\omega + 1,\ \omega 2 + 2,\ \omega 4 + 3, \ldots, \omega 2^{x_1 - 1} + x_1,
\end{gathered}$$

where $x_1 + x_2 + \ldots + x_r = n - 1$.

**Corollary** *$P_1 = \{1\} = E_1$ and $P_2 = \{2\} = E_2 - \{1\}$.*

**Theorem 3** *Let $n \geq 2$. Then the following integers are in $P_n$:*
(a) $n, n + 1, \ldots, 2n - 2$
(b) *For $n \geq 3$ and for $1 \leq \ell \leq n - 2$, all integers of the form* $(n - 2)^2 + \ell(n - 2) + 2$
(c) *For $n \geq 4$,* $n(n - 1)$
(d) *For $n \geq 5$,* $n^2 - 2$
(e) $2^{n-1}$
(f) $n^2 - 3n + 3$
(g) $n^2 - 3n + 4$.

*Proof:* Unless otherwise indicated, assume $n \geq 2$.

(a) For $1 \leq \ell \leq n - 1$, the $n$ pairwise-noncommutative ordinals $\omega + 1, \omega + 2, \ldots, \omega + (n - 1)$, and $\ell$ have sums

$$\omega(n - 1) + 1, \omega(n - 1) + 2, \ldots, \omega(n - 1) + n - 1 + \ell.$$

Thus $\{n, n + 1, \ldots, 2n - 2\} \subseteq P_n$.

(b) Let $n \geq 3$ and let $1 \leq \ell \leq n - 2$. Then $\omega^2, \omega + 1, \omega + 2, \ldots, \omega + (n - 2)$, and $\ell$ have sums $\omega^2$, $\omega^2 + \ell$; $\omega^2 \cdot i + j$, where $1 \leq i \leq n - 2$ and $1 \leq j \leq n - 2 + \ell$. Consequently, for each such $\ell$ there are $(n - 2)^2 + \ell(n - 2) + 2$ distinct sums.

(c) For $n \geq 4$, the ordinals $\omega^2$; $\omega 2 + 1, \omega + 2, \omega + 3, \omega + (n - 1)$ yield the $n(n - 1)$ distinct sums $\omega^2$; $\omega^2 + \omega + j$, for $2 \leq j \leq n - 1$; $\omega^2 + \omega i + j$, for $2 \leq i \leq n$ and $1 \leq j \leq n - 1$.

(d) For $n \geq 5$, the ordinals $\omega^2, \omega 3 + 1, \omega + 2, \omega + 3, \ldots, \omega + (n - 1)$ have as sums $\omega^2$; $\omega^2 + \omega i + j$, for $i = 1, 2$ and $2 \leq j \leq n - 1$; $\omega^2 + \omega i + j$, for $3 \leq i \leq n + 1$ and $1 \leq j \leq n - 1$. Thus there are $(n - 1)^2 + 2(n - 2) + 1$, or $n^2 - 2$ distinct sums.

(e) The ordinals $\omega^{n-1}, \omega^{n-2}, \ldots, \omega^2, \omega, 1$ yield $2^{n-1}$ distinct sums.

(f) The ordinals $\omega^2, \omega^2 + \omega, \omega, \omega + 1, \omega + 2, \ldots, \omega + (n - 3)$ have as sums $\omega^2 \cdot 2$; $\omega^2 \cdot 2 + \omega i + j$, for $1 \leq i \leq n - 1$ and $0 \leq j \leq n - 3$. There are $n^2 - 3n + 3$ distinct sums.

(g) The ordinals $\omega^2, \omega + 2, \omega + 4, \ldots, \omega + 2(n - 2), 2$ have as sums $\omega^2$, $\omega^2 + 2$; $\omega^2 + \omega i + 2j$, for $1 \leq i \leq n - 2$ and $1 \leq j \leq n - 1$. There are $n^2 - 3n + 4$ distinct sums.

**Theorem 4** $P_3 = \{3, 4, 5\} = E_3 - \{1, 2\}$.

*Proof:* $P_3 \subseteq E_3 = \{3, 4, 5\}$. Moreover $\{3, 5\} \subseteq P_3$ by Theorem 2 and $4 \in P_3$ by part (a) of Theorem 3.

**Lemma 2** *In order for* 4 *ordinals to yield* 11 *or more different sums, one of these must have highest degree and the other three must have the same degree.*

*Proof:* Given any 4 ordinals, let $\delta$ be the highest degree of any of these. Then it is easily seen that if all 4 ordinals are of degree $\delta$, there are at most 4 different sums; if 3 of the ordinals are of degree $\delta$, there are at most 6 different sums; if 2 of the ordinals are of degree $\delta$ there are at most 10 different sums.

Now suppose that exactly one of the ordinals, $\alpha_\delta$, is of degree $\delta$. Let $\gamma$ be the highest degree among the other 3 ordinals. If exactly one ordinal, $\alpha_\gamma$, is of degree $\gamma$ and if $\beta_1$ and $\beta_2$ are the remaining ordinals, there are at most 10 possible sums: $\alpha_\delta$, $\alpha_\delta + \beta_1$, $\alpha_\delta + \beta_2$, $\alpha_\delta + \beta_1 + \beta_2$, $\alpha_\delta + \beta_2 + \beta_1$, $\alpha_\delta + \alpha_\gamma$, $\alpha_\delta + \alpha_\gamma + \beta_1$, $\alpha_\delta + \alpha_\gamma + \beta_2$, $\alpha_\delta + \alpha_\gamma + \beta_1 + \beta_2$, and $\alpha_\delta + \alpha_\gamma + \beta_2 + \beta_1$. If exactly two ordinals, $\alpha_1$ and $\alpha_2$ are of degree $\gamma$ and if $\beta$ is the remaining ordinal, there are at most 10 possible sums: $\alpha_\delta$, $\alpha_\delta + \beta$, $\alpha_\delta + \alpha_1$, $\alpha_\delta + \alpha_2$, $\alpha_\delta + \alpha_1 + \alpha_2$, $\alpha_\delta + \alpha_2 + \alpha_1$, $\alpha_\delta + \alpha_1 + \beta, \alpha_\delta + \alpha_2 + \beta$, $\alpha_\delta + \alpha_1 + \alpha_2 + \beta$, and $\alpha_\delta + \alpha_2 + \alpha_1 + \beta$. The lemma is thereby established.

**Lemma 3** 4 *pairwise noncommutative ordinals cannot yield* 11 *sums.*

*Proof:* By Lemma 2, it suffices to consider ordinals with remainder form $\omega^\gamma \ell_i + \rho_i$ for $i = 1, 2$, and 3 together with $\alpha$, where $deg\,(\alpha) > \gamma$. Clearly, because the ordinals are pairwise-noncommutative, $\rho_1$, $\rho_2$, and $\rho_3$ are distinct. We can assume $\ell_1 \leq \ell_2 \leq \ell_3$. The possible sums for these ordinals are then

$$\begin{aligned}
\beta_1 &= \alpha \\
\beta_2 &= \alpha + \omega^\gamma \ell_1 + \rho_1 \\
\beta_3 &= \alpha + \omega^\gamma \ell_2 + \rho_2 \\
\beta_4 &= \alpha + \omega^\gamma \ell_3 + \rho_3 \\
\beta_5 &= \alpha + \omega^\gamma(\ell_1 + \ell_2) + \rho_1 \\
\beta_6 &= \alpha + \omega^\gamma(\ell_1 + \ell_2) + \rho_2 \\
\beta_7 &= \alpha + \omega^\gamma(\ell_1 + \ell_3) + \rho_1 \\
\beta_8 &= \alpha + \omega^\gamma(\ell_1 + \ell_3) + \rho_3 \\
\beta_9 &= \alpha + \omega^\gamma(\ell_2 + \ell_3) + \rho_2 \\
\beta_{10} &= \alpha + \omega^\gamma(\ell_2 + \ell_3) + \rho_3 \\
\beta_{11} &= \alpha + \omega^\gamma(\ell_1 + \ell_2 + \ell_3) + \rho_1 \\
\beta_{12} &= \alpha + \omega^\gamma(\ell_1 + \ell_2 + \ell_3) + \rho_2 \\
\beta_{13} &= \alpha + \omega^\gamma(\ell_1 + \ell_2 + \ell_3) + \rho_3
\end{aligned}$$

Some of these 13 sums may be the same.

If $\ell_1 = \ell_2 = \ell_3$, then $\beta_5 = \beta_7$, $\beta_6 = \beta_9$, and $\beta_8 = \beta_{10}$, so that there are 10 distinct sums.

If $\ell_1 = \ell_2 < \ell_3$, then $\beta_8 = \beta_{10}$, and there are 12 distinct sums.

If $\ell_1 < \ell_2 = \ell_3$, then $\beta_5 = \beta_7$, and there are 12 distinct sums.

If $\ell_1 < \ell_2 < \ell_3$, then there are 13 distinct sums.

**Theorem 5** $P_4 = \{4, 5, 6, 7, 8, 9, 10, 12, 13\}$
$= E_4 - \{1, 2, 3, 11\}$.

*Proof:* By [5] together with Theorem 2 and Lemma 3 of this paper, $P_4 \subseteq \{4, 5, 6, 7, 8, 9, 10, 12, 13\}$. Moreover, $\{4, 13\} \subseteq P_4$ by Theorem 2, $\{5, 6\} \subseteq P_4$ by Theorem 3 (a), $\{8, 10\} \subseteq P_4$ by Theorem 3 (b), $12 \in P_4$ by Theorem 3 (c), and $7 \in P_4$ by Theorem 3 (f). Finally, the ordinals $\omega^2$, $\omega$, $\omega 2 + 1$, and 1 have 9 distinct sums: $\omega^2$, $\omega^2 + 1$, $\omega^2 + \omega$, $\omega^2 + \omega + 1$, $\omega^2 + \omega 2 + 1$, $\omega^2 + \omega 2 + 2$, $\omega^2 + \omega 3$, $\omega^2 + \omega 3 + 1$, and $\omega^2 + \omega 3 + 2$.

The cases of 5 and 6 pairwise-noncommutative ordinals will be considered in [8].

## REFERENCES

[1] Erdös, P., "Some remarks on set theory," *Proceedings of the American Mathematical Society*, vol. 1 (1950), pp. 127–141.

[2] Sierpiński, W., "Sur les series infinies de nombres ordinaux," *Fundamenta Mathematicae*, vol. 36 (1949), pp. 248–253.

[3] Wakulicz, A., "Sur la somme d'un nombre fini de nombres ordinaux," *Fundamenta Mathematicae*, vol. 36 (1949), pp. 254–266.

[4] Wakulicz, A., "Correction au trauvail "Sur les sommes d'un nombres fini de nombres ordinaux" de A. Wakulicz," *Fundamenta Mathematicae*, vol. 38 (1951), p. 239.

[5] Wakulicz, A., "Sur les sommes de quatre nombres ordinaux," *Polska Akademia Umiejetności, Sprawozdania z Czynności i Posiedzén*, vol. 42 (1952), pp. 23–28.

[6] Zuckerman, M., "Sums of at most 8 ordinals," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 19 (1973), pp. 435–446.

[7] Zuckerman, M., "Sums of at least 9 ordinals," *Notre Dame Journal of Formal Logic*, vol. 14 (1973), pp. 263–268.

[8] Zuckerman, M., "Sums of 5 or 6 pairwise-noncommutative ordinals," to appear in *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*.

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