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Subcountability Under Realizability

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1 Abstract Among the open problems in the metamathematics of constructivity catalogued by Michael Beeson in [2] are those of the consistency of IZF with SCDS and with SCMS. SCDS is the assertion that every discrete set is subcountable; SCMS asserts that every metric space is subcountable.

Solutions to the SCDS and SCMS problems serve not only to advance the techniques of model theory for intuitionism, but also to give some confirmation to three distinct insights into the overall structure of constructive mathematics. The first insight is that of Bishop, who suggested that even the strongest constructive systems should admit interpretations that reveal constructive mathematics as having "numerical meaning". The second insight is Greenleaf's intuition that the mainspring of the constructive theory of cardinality is not "raw size" of sets, as it is on the classical account, but rather logical or mathematical structure. The third is perhaps more a presupposition of traditional intuitionism than an insight. It was presupposed by Brouwer that the domain of significant mathematics includes only sets which are, from the classical standpoint, relatively small and highly structured.

By working the model-theoretic approach to realizability which derives from Kleene's work, we show that the full intuitionistic set theory IZF is consistent with SCDS and SCMS. In fact, both SCDS and SCMS hold in a realizability structure, $V(Kl)$, which is a close relative of forms of realizability presented by Beeson in [1].

We also formulate a new principle, SCAS, which is a strong generalization of both SCDS and SCMS. SCAS is the assertion that every set with strict apart-

*The style of reasoning exploited in this paper was developed jointly by the author and Dr. Giuseppe Rosolini at Oxford in 1980. At that time, the reasoning was applied to answering a question posed by Dana Scott. The results of this work have been reported elsewhere in [6]. The author wishes to thank Giuseppe Rosolini, Dana Scott, and Simon Thompson for their conversation, assistance, and encouragement.

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ness is subcountable. We prove that $V(Kl)$ satisfies SCAS. This fact reveals that the features of $V(Kl)$ underlying the truth of SCMS and of SCDS are far more constructive than one might initially have expected.

2 Background and motivation In 1968, Bishop encouraged mathematicians to see constructive mathematics as fundamentally numerical, or as having what he called "numerical meaning". According to Bishop, the link between the constructive and the numerical had been forged by Kronecker:

> In my book I proposed, in the spirit of Kronecker rather than Brouwer, that the integers are the only irreducible mathematical constructs. This is not an arbitrary restriction, but follows from the basic constructivist goal—that mathematics concern itself with the precise description of finitely performable abstract operations. It is an empirical fact that all such operations reduce to operations with the integers. [3]

Unfortunately, Bishop's proposed analysis in [3] of the Kroneckerian "numerical meaning" notion was overly restrictive. In attempting to make the idea precise, Bishop overspecified; he tied the notion directly to Gödel's *Dialectica* interpretation. Technical difficulties aside, this approach offends the spirit of traditional intuitionism which emphasizes insight and a direct apprehension of meaning unmediated by formalism.

This is not to say that there is no natural way to meld intuitionism with "Kroneckerianism". One can make the idea of numerical meaning concrete by specifying that a branch of mathematics has numerical meaning if every result of that branch is uniformly interpretable as a statement about the elements of some numerical structure. Arguably, the numerical structures are, at least, the functional types over ω together with subobjects and quotients of the types. With no further ado, one can see that a great deal of mathematics has numerical meaning in this sense if pure analysis does. We will show that, in the realizability structure $V(Kl)$, the basic structures of concern to analysts, the abstract metric spaces, are numerical structures. In particular, we will show that $V(Kl)$ satisfies SCMS: every metric space is a subobject of a quotient of ω, i.e., every metric space is subcountable. It follows that, in $V(Kl)$, analysis can go on over a small category. This means that there is a set containing all the structures one needs to get along in analysis.

In [2] Beeson asked whether the strongest formal system appropriate to Bishop's brand of constructivism, the formal system IZF, is consistent with SCMS. At the same time, he put a seemingly easier question. Beeson asked if it is consistent with IZF to take all the trivial metric spaces, the discrete spaces, to be subcountable. Here, a set S is discrete iff equality on S is classical:

$$\forall x, y \in S(x = y \vee \sim x = y).$$

Greenleaf has argued (in [5]), on the basis of a number of constructive cardinality results, that the cardinality of a set in constructive mathematics is determined not so much by its size but more by its structure. Classically, we think of cardinality as being measured extrinsically—by a set's functional relations with some standard measure, such as the ordinals. Constructively, says Greenleaf, we ought to think of cardinality as determined not by this extrinsic concept of size, but primarily by properties intrinsic to the set, properties which relate directly to its internal "proof-theoretic" structure. It would be some confirmation of this idea to show that, over a very natural model of IZF, those properties of sets which one easily construes as intrinsic (like discreteness and metricity) strongly affect the size of the set.

In the standard realizability model, the fundamental internal structure of a set is (ordinary) recursion-theoretic. Roughly, the more we know about the "decidability" of a set in $V(Kl)$ the more structure it has. The consistency results for SCDS and SCMS draw connections between this recursion-theoretic understanding of structure and Greenleaf's ideas. Familiar mathematical conditions on a set, like discreteness, are connected to recursive structure and thence via realizability to cardinality.

In dismissing the bulk of Cantor's theory of the cardinals (in particular, anything beyond the second number class), Brouwer was presupposing that all significant mathematics can go on in collections which are, classically at least, cardinally small or mathematically well-controlled. That $V(Kl) \Vdash$ SCMS shows that $V(Kl)$ is a universe of constructive mathematics which satisfies Brouwer's presupposition. Here, all of analysis can be "covered by subsets of ω"; every metric space is the image of some ω-subset. This means that when the situation is viewed from a classical standpoint all constructive metric spaces are of relatively low cardinality and that all relations on constructive metric spaces are representable as relations on some subpartition of ω.

3 Extensional, set-theoretic realizability Solutions to Beeson's problems and confirmation of our three insights come from extending Kleene's ideas on realizability to full set theory, and treating the resulting interpretation as an extensional class model of the theory.

Intuitionistic Zermelo-Fraenkel set theory, IZF, was proposed originally as a regimentation of the concept of constructive set exploited by Bishop in his "New Constructivism" (cf. [3]). It would be heuristically, but not foundationally, accurate to view IZF as a modification of ZF in accord with a minimal constraint. Briefly, IZF comes from conventional ZFC by dropping full AC, reformulating the Fraenkel-Skolem replacement axiom as a scheme of collection and putting transfinite induction on ϵ in place of foundation. The constraint which mandates these changes is the reasonable demand that no set theory qualify as constructive if the theory entails TND: $\phi \vee \sim\phi$. As is well known (cf. [2] and [6]), the conventional ZF axioms, even in Heyting's predicate logic, entail TND, tertium non datur.

To get down to particulars, IZF is formulated in the familiar single-sorted first-order language with ϵ as its sole nonlogical primitive. In this setting the axioms of IZF are all the instances of (1) through (8):

Axioms of IZF

(1) $\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$ (extensionality)

(2) $\exists z(x \in z \wedge y \in z)$ (pair)

(3) $\exists y\, \forall z\, \forall u \in x(z \in u \rightarrow z \in y)$ (union)

(4) $\exists y\, \forall z\, (z \in y \leftrightarrow z \in x \wedge \phi)$ (separation)

(5) $\exists y\, \forall z\, (\forall u \in z(u \in x) \rightarrow z \in y)$ (power)

(6) $\exists x\, (\varnothing \in x \wedge \forall\, y \in x\, \exists z \in x\, y \in z)$ (infinity)

(7) $\forall y \in x\, \exists z\, \phi \rightarrow \exists u\, \forall y \in x\, \exists z \in u\, \phi$ (collection)
(8) $\forall x\, (\forall y \in x\, \phi(y) \rightarrow \phi) \rightarrow \forall x\, \phi$. (transfinite induction)

(4) and (7) are schemata to which the usual restrictions apply.

In classical predicate logic, the IZF axioms are equivalent to those of ZF. In Heyting's intuitionistic logic, the equivalence fails; IZF derives neither a classical foundation nor the general law of excluded third, TND.

Our treatment of realizability has affinities to the standard treatment of Boolean-valued models for classical set theory. We work over a ground model of classical ZF. We usually allow AC in our ground model, but it is not relevant here. First, a "realizability universe", $V(Kl)$, is defined recursively

$$V(Kl)_0 = \varnothing$$
$$V(Kl)_{\alpha+1} = P(\omega \times V(K)_\alpha)$$
$$V(Kl) = \bigcup_{\beta<\lambda} V(K)_\beta.$$[1]

We define realizability ($\Vdash$) over $V(Kl)$ for the language of ZF extended with elements of $V(Kl)$, used autonomously. Here $f, g \in \omega$, and $a, b, c, d \in V(Kl)$.

Definition 1

$e \Vdash a \in b$	iff	$\exists c(\langle e_0, c\rangle \in b$ and $e_1 \Vdash a = c)$	(1)
$e \Vdash a = b$	iff	$\forall c, f(\langle f, c\rangle \in a$ implies that $\{e_0\}(f) \Vdash c \in b$ and $\langle f, c\rangle \in b$ implies that $\{e_1\}(f) \Vdash c \in a)$	(2)
$e \Vdash \phi \wedge \psi$	iff	$e_0 \Vdash \phi$ and $e_1 \Vdash \psi$	(3)
$e \Vdash \phi \vee \psi$	iff	either $e_0 = 0$ and $e_1 \Vdash \phi$ or $e_0 \neq 0$ and $e_1 \Vdash \psi$	(4)
$e \Vdash \phi \rightarrow \psi$	iff	$\forall f(f \Vdash \phi$ implies that $\{e\}(f) \Vdash \psi)$	(5)
$e \Vdash \sim\phi$	iff	$\forall f \sim f \Vdash \phi$	(6)
$e \Vdash \forall x\, \phi$	iff	$\forall a\; e \Vdash \phi(a)$	(7)
$e \Vdash \exists x\, \phi$	iff	$\exists a\; e \Vdash \phi(a)$	(8)

We say that $V(Kl)$ satisfies ϕ (in symbols, $V(Kl) \Vdash \phi$) whenever $\exists n\; n \Vdash \phi$.

Generally $\langle , \rangle$ represents set-theoretic pairing. $()_0$ and $()_1$ are for the first- and second-projections with respect to number-theoretic pairing. Occasionally, we will use $\langle , \rangle$ for number-theoretic pairs also, but we hope that context will sort things out.

Of course, there is a basic soundness theorem whose proof shows $V(Kl)$ to be a model of the intuitionistic set theory IZF. The result is well known and the details are straightforward.

Theorem 1 (Soundness) $V(Kl) \Vdash IZF + ECT + PA + RDC + UP + AC(\omega) + \sim KS + \sim CTP$.

ECT is Extended Church's Thesis for total functions, PA is Blass's Presentation Axiom, RDC is relativized dependent choice, AC(ω) is the axiom of choice over ω, UP is Troelstra's Uniformity Principle, KS is Kripke's Schema, and CTP is Church's Thesis for Partial functions. For explanation and further illumination of these principles, cf. [1] or [6].

4 SCDS Once we have shown that $V(Kl) \Vdash$ SCMS, it is really superfluous to give a separate proof that $V(Kl) \Vdash$ SCDS. This is because IZF $\vdash$ SCMS $\rightarrow$ SCDS and $V(Kl)$ is sound for IZF. We begin, however, with a proof that

$V(Kl) \Vdash$ SCDS because this more elementary result is so instructive. It reveals interesting internal features of our model and serves as a fine example of our approach.

Remarks:

(i) The operation $\|\langle\,,\,\rangle\|$ on $V(Kl)$ represents internal set-theoretic pairing, in that we have

$$\forall a, b \in V(Kl),\ V(Kl) \Vdash \|\langle a, b\rangle\| = \langle a, b\rangle.$$

(ii) $\underline{\omega} = \{\langle n, \underline{n}\rangle : n \in \omega\}$, where $\underline{n} = \{\langle m, \underline{m}\rangle : m \in n\}$. $\underline{\omega}$ represents ω in $V(Kl)$, in that $V(Kl) \Vdash \underline{\omega} = \omega$.

(iii) As far as membership is concerned, ω is realizability absolute. This means that for $m, n \in \omega$, $V(Kl) \Vdash \underline{m} = \underline{n}$ if and only if $m = n$, and $V(Kl) \Vdash \underline{m} \in \underline{n}$ if and only if $m < n$.

(iv) Since $V(Kl)$ is a model of intuitionistic logic plus identity, $\exists m \in \omega$ $m \Vdash \forall x(x = x)$. Fix i to be such an m.

As a preliminary we prove what is in itself an important result. Lemma 1 shows that the external or realizability notion of set, that notion of set which we see in the structure $V(Kl)$ from the outside, is represented internally also. We can put this more "philosophically" by saying that $V(Kl)$ has the means to reflect on its own underlying notion of constructive set and to express the result of reflection as another set in $V(Kl)$. But, even if you do not care to wax poetic about it, $\Vdash$ is a very useful gadget for the following proofs. V is the ground model against which we have defined $V(Kl)$.

Lemma 1 *For any* $s \in V(Kl)$, $\Vdash_s = \{\langle n, \|\langle \underline{n}, a\rangle\|\rangle : \dot{n} \Vdash a \in s\} \in V(Kl)$.

Proof: The lemma follows immediately from facts (a) and (b):

(a) For any $s \in V(Kl)$, $\{a\colon V(Kl) \Vdash a \in s\} \in V$
(b) For any $s \in V(Kl)$, $\{b\colon V(Kl) \Vdash s = b\} \in V$.

(a) and (b) are proved simultaneously by ordinal induction using items (1) and (2) of Definition 1.

On closer inspection, the realizability set $\Vdash_s$ proves to be a function whenever s is realizably discrete. In particular, $\Vdash_s$ turns out to be a function that "subcounts" s; in $V(Kl)$, $\Vdash_s$ is a function which maps a subset of ω onto s.

Lemma 2 $V(Kl) \Vdash$ *(s is discrete* $\Rightarrow \Vdash_s$ *is a function which "subcounts" s).*

Proof: As the realizability clause for $\rightarrow$ demands, we show how to compute, from the "numerical evidence" that the antecedent of the theorem-formula obtains, evidence for the truth of its consequent. Before we go into that, we should note something quite general about discrete sets in $V(Kl)$. We prove that, if s is realizably discrete, $n \Vdash a \in s$ and $n \Vdash b \in s$, then $V(Kl) \Vdash a = b$.

Assume that $e \Vdash$ "s is discrete" and that $n \Vdash a \in s$ and $n \Vdash b \in s$. Then,

$$e \Vdash \forall x, y \in s(x = y \vee \sim x = y),$$

so either

$$\{e\}(\langle n, n\rangle)_0 = 0 \text{ and } \{e\}(\langle n, n\rangle)_1 \Vdash a = b$$

or

$$\{e\}(\langle n, n\rangle)_0 \neq 0 \text{ and } \{e\}(\langle n, n\rangle)_1 \Vdash {\sim} a = b.$$

However, since $\langle n, n\rangle \Vdash a \in s \wedge a \in s$ and $i \Vdash a = a$, $\{e\}(\langle n, n\rangle)_0 = 0$. Therefore, $\{e\}(\langle n, n\rangle)_1 \Vdash a = b$. Hence, $V(Kl) \Vdash a = b$.

Now we can show that, realizably, if s is discrete, then $\Vdash_s$ is functional. Again, let $e \Vdash$ "s is discrete" and assume that

$$g \Vdash \|\langle a, b\rangle\| \in \Vdash_s$$
$$h \Vdash \|\langle a, c\rangle\| \in \Vdash_s.$$

Then, from the realizability clauses governing atomic statements (Definitions 1(1) and 1(2)), there are $\|\langle \underline{n}, d\rangle\|$, $\|\langle \underline{m}, d^*\rangle\| \in V(Kl)$ such that

$$\langle g_0, \|\langle \underline{n}, d\rangle\|\rangle \in \Vdash_s$$

and

$$g_1 \Vdash \|\langle a, b\rangle\| = \|\langle \underline{n}, d\rangle\|,$$

while

$$\langle h_0, \|\langle \underline{m}, d^*\rangle\| \in \Vdash_s$$

and

$$h_1 \Vdash \|\langle a, c\rangle\| = \|\langle \underline{m}, d^*\rangle\|.$$

Using the realizability of the obvious properties of pairs and of the transitivity of identity, we have

$$g_0 = n = m = h_0.$$

Without loss of generality, we can say that

$$g_0 \Vdash d \in s$$
$$g_0 \Vdash d^* \in s$$
$$g_1 \Vdash b = d,$$

and

$$h_1 \Vdash c = d^*.$$

From above, we have that $\{e\}(\langle g_0, g_0\rangle)_1 \Vdash d = d^*$. Another application of transitivity now yields functionality of $\Vdash_s$, that $V(Kl) \Vdash b = c$.

It is easy to see that $\Vdash_s$ is realizably onto and this will complete the proof. $\Vdash_s$ is realizably onto if and only if

$$V(Kl) \Vdash \forall x(x \in s \rightarrow \exists y(\langle y, x\rangle \in \Vdash_s)).$$

The latter is obviously true: Let $n \Vdash a \in s$, then,

$$\langle n, i\rangle \Vdash \|\langle \underline{n}, a\rangle\| \in \Vdash_s$$

and we are done.

The realizability of SCDS now follows immediately from the lemma and we have

Theorem 2 $\quad V(Kl) \Vdash SCDS$.

5 SCMS Our proof of the following:

Theorem 3 $V(Kl) \Vdash SCMS$

is an adaption, to a more complex case, of the SCDS theme. Again, we show that, realizably, if $\langle M, \rho \rangle$ is a metric space, then $\Vdash_M$ "subcounts" M. For this proof, we need to mention some familiar properties of metric spaces and a somewhat less familiar property of the reals in $V(Kl)$.

Remarks:

(i) As is well known, if $\langle M, \rho \rangle$ is a metric space, then ρ is a metric such that

(a) $\forall x, y \in M \exists r \in R(\langle x, y, r \rangle \in \rho)$
(b) $\forall x \in M \forall r \in R(\langle x, x, r \rangle \in \rho \Rightarrow r \approx \bar{0})$ and
(c) $\forall x, y \in M \forall s \in R(\langle x, y, s \rangle \in \rho \wedge s \approx \bar{0} \Rightarrow x = y)$.

R is the structure of Cauchy reals, $\bar{0}$ is the identically zero Cauchy sequence, and $\approx$ is the usual equality relation on R.

(ii) Let $\overline{CR} = \{\langle \langle e, f \rangle, \bar{e} \rangle$: e is the Turing index of a recursive Cauchy sequence, f is its modulus, and $\bar{e} = \{\langle n, \|\langle \underline{n}, \underline{\{e\}(n)} \rangle\| \rangle : n \in \omega\}\}$. $\overline{CR}$ is, internally, the set of Cauchy reals of $V(Kl)$:

$$V(Kl) \Vdash \overline{CR} = R.$$

A proof of Lemma 3 will suffice for the theorem.

Lemma 3 $V(Kl) \Vdash (\langle M, \rho \rangle$ *is a metric space* $\Rightarrow \Vdash_M$ *"subcounts"* $M)$.

Proof: Again, the only real difficulty is the proof that, under the appropriate assumptions, $\Vdash_M$ is realizably functional.

Assume that $n \Vdash a \in M$ and that $n \Vdash b \in M$. Also assume that $V(Kl) \Vdash$ "$\langle M, \rho \rangle$ is a metric space". Then, using Remark (i)(a), there is an $e \in \omega$ such that

$$e \Vdash \forall x, y \in M \; \exists r \in \overline{CR} \langle x, y, r \rangle \in \rho.$$

It follows from Remark (ii) and Definition 1 that

$$\exists r \{e\}(\langle n, n \rangle) \Vdash r \in \overline{CR} \wedge \|\langle a, a, r \rangle\| \in \rho$$

and

$$\{e\}(\langle n, n \rangle)_{01} \Vdash r = \overline{\{e\}(\langle n, n \rangle)_{000}}.$$

Also,

$$\exists s \{e\}(\langle n, n \rangle) \Vdash s \in \overline{CR} \wedge \| \langle a, b, s \rangle \| \in \rho$$

and

$$\{e\}(\langle n, n \rangle)_{01} \Vdash s = \overline{\{e\}(\langle n, n \rangle)_{000}}.$$

Therefore, $V(Kl) \Vdash r = s$. With the realizability of (i)(b), we have that

$$V(Kl) \Vdash r \approx \bar{0},$$

and immediately,

$$V(Kl) \Vdash s \approx \bar{0}.$$

Finally, a realizability witness for (i)(c) yields $V(Kl) \Vdash a = b$ and it is clear that a witness for this fact can be calculated effectively from n and a witness for $V(Kl) \Vdash$ "$\langle M, \rho\rangle$ is a metric space."

Corollary 1: $V(Kl) \Vdash$ *Every metric space has a subcountable basis.*

Proof: Immediate from the SCMS and soundness theorems.

Corollary 2: $V(Kl) \Vdash$ *The category of metric spaces is equivalent to a small category.*

Proof: Again, trivial.

6 Apartness and subcountability The style of reasoning exhibited above admits yet another generalization. Beyond giving a more general result, this will reveal the extent to which SCDS and SCMS depend for their realizability on nonconstructive principles. For instance, the truth of SCMS in $V(Kl)$ *appears* to rely on a somewhat incidental (and nonconstructive) fact about the Cauchy reals under realizability—that they correspond with the recursive reals of $\overline{CR}$. That the reliance is wholly apparent is shown by directing our attention to the more general notion of sets with *strict apartness relations.* Historically, apartness relations were introduced to satisfy the constructivists' need for a "positive concept" of inequality on the reals. We will prove that, in $V(Kl)$, every set admitting strict apartness is subcountable. This result entails $V(Kl) \Vdash \text{SCMS} \wedge \text{SCDS}$ and shows that all these rely upon no fact which is more strikingly nonconstructive than the admissibility of number realizability itself.

Definition 2 *$R \subseteq x \times x$ is a strict apartness (relation) on x if and only if, for all $y, z \in x$,*

(i) $y = z \leftrightarrow \sim R(y, z)$

(ii) $R(y, z) \leftrightarrow R(z, y)$, and

(iii) $R(y, z) \rightarrow \forall r \in x(R(y, r) \vee R(z, r))$.

Intuitionistically, the reals have a natural apartness on them given by the separation of Cauchy sequences. Every metric space then inherits a strict apartness from the reals via its metric.

We say that a set with a strict apartness on it is an *apartness space.* The claim that every apartness space is subcountable is abbreviated 'SCAS'.

Theorem 4 $V(Kl) \Vdash$ *SCAS.*

Proof: As in Lemma 2, we need only check that, in $V(Kl)$, $\Vdash_s$ is functional whenever s has R as a strict apartness. To that end, we assume that

$$e \Vdash \text{"}R \text{ is a strict apartness on } s\text{"}.$$

We also assume that $n \Vdash a \in s$ and $n \Vdash b \in s$. Let $m \Vdash R(a, b)$. Given the definition of strict apartness, we know that there is a partial recursive θ such that

$$\theta(m, n, e) \Vdash R(a, a) \vee R(b, a).$$

Because of the realizability of Definition 2(i), $\theta(m, n, e)_0 \neq 0$ and $\theta(m, n, e)_1 \Vdash R(b, a)$.

But, by the same token,

$$\theta(m, n, e) \Vdash R(a, b) \vee R(b, b).$$

Since $\theta(m, n, e)_0 \neq 0$, $V(Kl) \Vdash R(b, b)$. But this is impossible; by Definition 2(i), no element of s can be realizably apart from itself. Therefore, our original assumption, that $V(Kl) \Vdash R(a, b)$ is false. It now follows from part (6) of Definition 1 that $0 \Vdash \sim R(a, b)$. Another use of Definition 2 shows that there is a partial recursive Ψ such that $\Psi(e) \Vdash a = b$.

We can now replay the proof of Lemma 2 to show that $\Vdash_s$ subcounts s in $V(Kl)$.

It is a simple matter to check that the realizability of SCAS faithfully incorporates all the work we have accomplished so far:

Corollary 3 $V(Kl) \Vdash SCAS \wedge SCMS \wedge SCDS$.

Proof: We remarked *en passant* that, even constructively, every discrete set admits the discrete metric. Hence, IZF $\vdash$ SCMS $\rightarrow$ SCDS. Furthermore, IZF $\vdash$ SCAS $\rightarrow$ SCMS; as we remarked above, every metric space $\langle M, \rho\rangle$ inherits a strict apartness from the reals via ρ.

Unfortunately, it is not the case that apartness characterizes subcountability in $V(Kl)$; demonstrably, the converse of SCAS fails over $V(Kl)$:

Theorem 5 *In $V(Kl)$, there is a subcountable set which is not an apartness space.*

Proof: Equality on *a* set s is *stable* just in case it is constructively true that

$$\forall x, y \in S(\sim\sim x = y \rightarrow x = y).$$

It is obvious from Definition 2(i) that, on every apartness space, equality is stable. Hence, to prove our theorem it suffices to show that there is, realizably, a subcountable set on which equality is nonstable. The most direct approach would be to define, in $V(Kl)$, a nonstable equivalence relation on ω itself.

Let $T(x, y, z)$ represent the conventional set-theoretic formalization of the unary Kleene T-predicate. We set, for $e, f \in \omega$,

$$e \sim f \text{ iff } (\exists m\ T(e, e, m) \vee \sim\exists m\ T(e, e, m)) \leftrightarrow (\exists m\ T(f, f, m) \vee \sim\exists m\ T(f, f, m)).$$

We will abbreviate '$\exists m\ T(e, e, m) \vee \sim\exists m\ T(e, e, m)$' as $\Phi(e)$.

IZF $\vdash$ "$\{\langle e, f\rangle : e \sim f\}$ is an equivalence relation on ω", so, by soundness, $\{\langle e, f\rangle : e \sim f\}$ is an equivalence relation in $V(Kl)$. Assume that

$$V(Kl) \Vdash \forall e, f \in \omega(\sim\sim(e \sim f) \rightarrow (e \sim f)). \tag{1}$$

It follows that

$$V(Kl) \Vdash \forall e \in \omega(\sim\sim\Phi(e) \rightarrow \Phi(e)).$$

It is a matter of pure constructive logic that

$$\forall e \sim\sim \Phi(e).$$

Hence, we have

$$V(Kl) \Vdash \forall e \in \omega\ \Phi(e). \tag{2}$$

In [6], we have shown that (2) reduces to the solvability of the halting problem in Section 5. Therefore, assumption (1) is false and the theorem is proved.

7 ***Final comments*** As we mentioned before, our first proof of $V(Kl) \Vdash$ SCMS gives the distinct but faulty impression that the truth of the result depends essentially on the *recursiveness* of recursive realizability. The dependence is marked by the presence of principles like $R = \overline{CR}$ (or, alternatively, ECT). Our proof of $V(Kl) \Vdash$ SCMS via a proof of $V(Kl) \Vdash$ SCAS banishes that impression completely. Clearly, the truth of SCAS depends upon little more than the abstract notion of number realizability itself. In particular, the notion is that there is, on the natural numbers, a suitable representation of the applicative structure of constructive proofs. One now sees that SCAS (hence, SCMS and SCDS) will hold under any variant of realizability which exploits the abstract idea. Among such variants are hyperarithmetic realizability, and realizability based on the indices of functions recursive in some fixed degree.

Again and again, realizability shows off the sorts of "extra information" carried naturally by the constructive logical constants. The solution of the SCDS conjecture is a good (and simple) example of this information-bearing aspect of the constants. The discreteness of a set is seemingly a logical property, a constraint set down on the "logic" of equality over a set. Subcountability is clearly a mathematical property which is, on the face of it, totally independent of discreteness. Our proof of the consistency of SCDS shows that a (coded) proof, in IZF, of the *logical* discreteness of a set can be "unpacked" to give up the *mathematical* information that the set is realizably subcountable.

NOTE

1. The implicit trifurcation of the class of ordinals into zero, successors, and limits reveals the fact that we are defining $V(Kl)$ within a classical metatheory. It is not constructively true that every ordinal is either zero, a successor, or a limit and a fortiori, this classical result is not a consequence of IZF. The recursive definition of $V(Kl)$ and, indeed, all our metatheoretic work, could be carried out constructively and within the confines of IZF. But, out of sympathy for the classical reader, we have refrained from this constructivization. $V(Kl)$ can even be defined without reference to the ordinals by setting $V(Kl)_x = P(w \times \bigcup \{V(Kl)_y \colon y \in x\})$.

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