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The Hanf Numbers of Stationary Logic II: Comparison with Other Logics

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Abstract We show the ordering of the Hanf number of $\mathcal{L}_{\omega,\omega}(wo)$ (well ordering) $\mathcal{L}^c_{\omega,\omega}$ (quantification on countable sets), $\mathcal{L}_{\omega,\omega}(aa)$ (stationary logic), and second-order logic has no more restraints provable in ZFC than previously known (those independence proofs assume *CON*(ZFC only). We also get results on corresponding logics for $\mathcal{L}_{\lambda,\mu}$.

0 Introduction The stationary logic, denoted by $\mathcal{L}(aa)$ was introduced by Shelah [8]. Barwise, Kaufman, and Makkai [1] make a comprehensive research on it, proving for it the parallel of the good properties of $\mathcal{L}(Q)$. There has been much interest in this logic, being both manageable and strong (see Kaufman [5] and Shelah [10]).

Later some properties indicating its affinity to second-order logic were discovered. It is easy to see that countable cofinality logic is a sublogic of $\mathcal{L}(aa)$. By [10], for pairs φ, ψ of formulas in $\mathcal{L}_{\omega,\omega}(Q^{cf}_{\aleph_0})$, satisfying $\vdash \varphi \rightarrow \psi$ there is an interpolant in $\mathcal{L}(aa)$. By Kaufman and Shelah [6], for models of power $> \aleph_1$, we can express in $\mathcal{L}_{\omega,\omega}(aa)$ quantification on countable sets. Our main conclusion is (on the logics see Definition 1.1 or the abstract, on h, the Hanf numbers, see Definition 1.2):

Theorem 0.1 *The only restriction on the Hanf numbers of* $\mathcal{L}_{\omega,\omega}(wo)$, $\mathcal{L}^c_{\omega,\omega}$, $\mathcal{L}_{\omega,\omega}(aa)$, $\mathcal{L}^{II}_{\omega,\omega}$ *are*:
(a) $h(\mathcal{L}_{\omega,\omega}(wo)) \le h(\mathcal{L}^c_{\omega,\omega}) \le h(\mathcal{L}_{\omega,\omega}(aa)) \le h(\mathcal{L}^{II}_{\omega,\omega})$
(b) $h(\mathcal{L}^c_{\omega,\omega}) < h(\mathcal{L}^{II}_{\omega,\omega})$.

Proof: See 2.1 (neccessity), 2.2, 2.4, 2.5, and 3.3 (all five possibilities are consistent).

The independence results are proved assuming *CON*(ZFC) only and the re-

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sults are generalized to $\mathcal{L}_{\lambda^+,\omega}$. We do not always remember to write down the inequalities of the form $\mathcal{L}_{\lambda,\omega}(Q_1) < \mathcal{L}_{\mu,\omega}(Q_2)$. For some of the results when we generalize them to $\mathcal{L}_{\lambda^+,\omega}$ or $\mathcal{L}_{\lambda,\kappa}$ we need a stronger hypothesis. The proofs of the results on $h(\mathcal{L}_1) \leq h(\mathcal{L}_2)$ give really stronger information: we can interpret $\mathcal{L}_1$ in $\mathcal{L}_2$, usually here by using extra predicates, i.e., every formula in $\mathcal{L}_1$ is equivalent to a formula in $\Delta(\mathcal{L}_2)$; remember $\Delta(\mathcal{L}_2)$ is defined by: $\theta \in \Delta(\mathcal{L}_2)(\tau)$ is represented by (θ_1, θ_2), $\theta_e \in \mathcal{L}_2(\tau_e)$, $\tau_1 \cap \tau_2 = \tau$, $M \models \theta$ iff M can be expanded to a model of θ_1 iff M cannot be expanded to a model of θ_2 (so the requirement on (θ_1, θ_2) is strong). Note that this has two interpretations: one in which we allow τ_1, τ_2 to have new sorts hence new elements, the other in which we do not allow it. We use an intermediate course, we allow this but the number of new elements is the power set of the old. But for $\mathcal{L}^c_{\omega,\omega} \leq \mathcal{L}_{\omega,\omega}(aa)$, for models of power $\lambda = \lambda^{\aleph_0}$, we do not need new elements.

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Notation Let cardinals be denoted by $\lambda, \kappa, \mu, \chi$. Ordinals are denoted by $\alpha, \beta, \gamma, \xi, \zeta, i, j$. δ is a limit ordinal. Let $H(\lambda)$ be the family of sets whose transitive closure has cardinality $< \lambda$ (so for λ regular it is a model of ZFC$^-$, i.e., ZFC except the power set axiom: and for a strong limit a model of ZC). Let Lévy$(\lambda, \kappa) = \{f : f$ a function from some $\alpha < \lambda$ into $\kappa\}$.

Lévy $(\lambda, <\kappa) = \{f : f$ a partial function from $\lambda \times \kappa$ to κ, $|Dom f| < \lambda$, $f(\alpha, \beta) < 1 + \beta\}$.

Notation on logics $\mathcal{L}$ will be a logic, τ a vocabulary (i.e., set of predicates and function symbols, always with a fixed arity, usually finite). We assume that $\mathcal{L}(\tau)$ is a set of formulas, each with $< Oc_1(\mathcal{L})$ free variables and $< Oc(\mathcal{L})$ predicates and function symbols; $\mathcal{L}(\tau)$ is closed under first-order operations, substitutions, and relativizations and $\mathcal{L}(\tau)$ is a set (with τ and the family of variables sets). Two formulas are isomorphic if some mapping from the set of predicates, function symbols, and free variables of one onto those of another is one-to-one and map one formula to the other. We are assuming that up to isomorphism there is a set of $\mathcal{L}$-formulas, this number is denoted by $|\mathcal{L}|$. Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ mean $\mathcal{L}_1(\tau) \subseteq \mathcal{L}_2(\tau)$ for every vocabulary τ.

1 Preliminaries

Definition 1.1

(1) $\mathcal{L}_{\lambda,\kappa}$ is the logic in which $\bigwedge_{i\in I}(|I| < \lambda)$ and $(\exists x_0, \ldots, x_i \ldots)_{i\in J}(|J| < \kappa)$ are allowed, with $Oc_1(\mathcal{L}_{\lambda,\kappa}) = \kappa$ (so $\mathcal{L}_{\omega,\omega}$ is first-order logic)

(2) For a logic $\mathcal{L}$, $\mathcal{L}(wo)$ extends $\mathcal{L}$ by allowing the quantifier $(wo\, x, y)\varphi(x, y)$ saying $\langle \{x : \exists\varphi(x, y)\}, \varphi(x, y)\rangle$ is well ordering.

(3) For a logic $\mathcal{L}$, $\mathcal{L}^c = \Sigma(\exists^c)$ extends $\mathcal{L}$ by allowing monadic predicates as free variables and the quantifier $(\exists^c X)\varphi(X)$ saying there is a countable set X such that $\varphi(X)$.

(4) For a logic $\mathcal{L}$, $\mathcal{L}(aa)$ extends $\mathcal{L}$ by allowing monadic predicates as free variables and the quantifiers $(aaX)\varphi(X)$ saying that the collection of countable X satisfying φ contains a closed unbounded family of countable subsets of the model.

(5) For a logic $\mathcal{L}$, $\mathcal{L}^{II} = \mathcal{L}(\exists^{II})$ extends $\mathcal{L}$ by allowing binary predicates as free variables and the quantifiers $\exists R\varphi(R)$ saying there is a two-place relation R on the model satisfying φ.

(6) For $Q \in \{\exists^c, aa, \exists^{II}\}$, $\mathcal{L}'(Q)$ is defined similarly allowing a string

$$(Qx_1 \ldots Qx_i \ldots)_{i<\alpha}, |\alpha| < 0c_1(\mathcal{L}).$$

(7) Let $\mathcal{L}^c = \mathcal{L}(\exists^c)$, $\mathcal{L}^{wo} = \mathcal{L}(wo)$, $\mathcal{L}^{II} = \mathcal{L}(\exists^{II})$, $\mathcal{L}^{aa} = \mathcal{L}(aa)$.

Definition 1.2

(1) For a sentence ψ, let $h(\psi) = \sup\{|M|^+ : M \models \psi\}$ (so it is a cardinal (or infinity) and it is the first λ such that ψ has no model $\geq \lambda$).

(2) For a theory T, $h(T) = h(\bigwedge_{\psi\in T}\psi)$.

(3) For a logic $\mathcal{L}$ let $h(\mathcal{L}) = \sup\{h(\psi) : h(\psi) < \infty, \psi \in \mathcal{L}(\tau)$ for some vocabulary $\tau\}$.

(4) For a logic $\mathcal{L}$ and cardinal λ let $h(\mathcal{L},\lambda) = \sup\{h(\psi)$: for some vocabulary τ of power $< \lambda, \psi \in \mathcal{L}(\tau), h(\psi) < \infty\}$.

(5) For a logic $\mathcal{L}$ and cardinal λ let $hth(\mathcal{L},\lambda) = \sup\{h(T)$: for some vocabulary τ of power $< \lambda$, $T \subseteq \mathcal{L}(\tau)$, $h(T) < \infty\}$.

$$hth(\mathcal{L}) = H(\mathcal{L},\infty).$$

Claim 1.3

(1) For every $\psi \in \mathcal{L}$ for some $\varphi \in \mathcal{L}$, $[h(\psi) < \infty \to h(\psi) < h(\varphi) < \infty]$.

(2) $h(\mathcal{L})$ is strong limit.

(3) If $\mathcal{L}$ is closed under $\bigwedge_{\alpha<\alpha_0}$ for $\alpha_0 < \lambda$ then $cf[h(\mathcal{L})] \geq \lambda$.

(4) If the number of sentences in $\mathcal{L}$ (up to isomorphism) is $\leq \lambda$ then $cf[h(\mathcal{L})] \leq \lambda$.

Lemma 1.4 *Assume $\mathcal{L}$ is a logic $\subseteq \mathcal{L}^{II}_{\omega,\omega}$ and there is a function f from* Card *to* Card *such that*:

(a) *f is definable in $\mathcal{L}^{II}_{\omega,\omega}$, i.e., the class of two-sorted models $\langle\kappa, f(\kappa)\rangle$ is definable by some sentence of $L^{II}_{\omega,\omega}$ or even just*

(a)$^-$ *For some $\lambda^* < h(\mathcal{L}^{II}_{\omega,\omega})$ and $\varphi^* \in \mathcal{L}^{II}_{\omega,\omega}$ for $\kappa, \mu, \geq \lambda^*$, $\kappa < h(\mathcal{L}^{II}_{\omega,\omega})$, we have $\langle\kappa,\mu\rangle \models \varphi^*$ iff $\mu = f(\kappa)$.*

(b) *If $\psi \in \mathcal{L}$ has a model of power $\geq \kappa$ then ψ has a model M, $\kappa \leq \|M\| \leq f(\kappa)$.*

(c) *$\mathcal{L}$ is definable in $\mathcal{L}^{II}_{\omega,\omega}$, i.e., the class $\{(\psi, \tau, M) : \psi \in \mathcal{L}(\tau)$, M a τ-model, $M \models \psi\}$ is definable by a sentence in $\mathcal{L}^{II}_{\omega,\omega}$.*

(d) *For $\mu < h(\mathcal{L})$, $f(\mu) < h(\mathcal{L})$.*

Then $h(\mathcal{L}) < h(\mathcal{L}^{II}_{\omega,\omega})$.

Proof: Let $\psi_0 \in \mathcal{L}^{II}_{\omega,\omega}$ be such that $\lambda^* < h(\psi_0) < \infty$, where λ^*, φ^* are as in (a)$^-$. We can assume $h(\psi_0) < h(\mathcal{L})$ (otherwise the conclusion is trivial). Let $\psi \in \mathcal{L}^{II}_{\omega,\omega}$ say that for some λ, μ_0:

(i) The model M is isomorphic to some $(H(\lambda),\in)$, λ strong limit.

(ii) For every $\kappa < \lambda$, $M \models (\exists\mu \geq \kappa)$ [ψ_0 has a model of cardinality μ] $\vee$ $(\exists\mu \geq \kappa)[\langle\kappa,\mu\rangle \models \varphi^*]$.

(iii) $\mu_0 < \lambda$, ψ_0 has a model of power whose cardinality is in the interval (μ_0,λ).

(iv) For every $\kappa < \lambda$, $\kappa \geq \mu_0$, there is $\theta \in \mathcal{L}$ which has a model of cardinality in the interval (κ,λ), but for some $\kappa' \in (\kappa,\lambda)$ has no model of cardinality in the interval (κ',λ).

Now $(H(h(\mathcal{L})),\in)$ is a model of ψ and it has no models of larger cardinality.

We can prove similarly:

Lemma 1.5 *Suppose $\mathcal{L}_1,\mathcal{L}_2$ are logics and there is f*: Card $\to$ Card *such that*

(a) *For some $\lambda^* < h(\mathcal{L}_2)$ and $\varphi^* \in \mathcal{L}_2$ for $\kappa,\mu \geq \lambda^*$ we have: $\langle\kappa,\mu\rangle \models \varphi^*$ iff $\mu = f(\kappa)$.*

(b) *If $\psi \in \mathcal{L}_1$ has a model of cardinality $\geq \kappa$ then ψ has a model M, $\kappa \leq \|M\| \leq f(\kappa)$.*

(c) *$\mathcal{L}_1$ is definable in $\mathcal{L}_2$ just in the following weaker sense: for $K_1 = \{(\psi,\tau) : \psi \in \mathcal{L}_1(\tau)\}$, $K_2 = \{(M,\psi,\tau) : M \models \psi,\ \psi \in \mathcal{L}_1(\tau)\}$ there are $\psi_e \in \mathcal{L}_2$.*

$(\forall x)\,[x \in K_e \Leftrightarrow$ for some λ, some expansion of $(H(\lambda),\in,x)$ satisfies $\psi_e]$ and for every x $\{\lambda$: some expansion of $(H(\lambda),\in,x)$ satisfies $\psi_e\}$ is a bounded family of cardinals.

(d) *For $\mu < h(\mathcal{L}_1), f(\mu) < h(\mathcal{L}_1)$.*

(e) *$\|\mathcal{L}\| < h(\mathcal{L}_2)$, $\mathcal{L}^{II}_{\omega,\omega} \subseteq \mathcal{L}_2$.*

Then $h(\mathcal{L}_1) < h(\mathcal{L}_2)$.

Remark 1.6 Of course if the hypothesis 1.5 holds for $\mathcal{L}_1$ (and $\mathcal{L}_2$) then the conclusion holds for $\mathcal{L}_1', \mathcal{L}_2'$ whenever $\mathcal{L}_1' \subseteq \mathcal{L}_1$ and $\mathcal{L}_2 \subseteq \mathcal{L}_2'$.

Lemma 1.7

(1) *If $M \models \psi$, $\psi \in \mathcal{L}^{wo}$ then this is preserved by any forcing; this holds even for $\psi \in \mathcal{L}^{wo}_{\infty,\omega}$.*

(2) *If $M \models \psi$, $\psi \in \mathcal{L}^{aa}_{\omega,\omega}$ then this is preserved by any $\aleph_1$-complete forcing; this holds even for $\psi \in \mathcal{L}^{aa}_{\infty,\omega}$.*

(3) *If $M \models \psi$, $\psi \in \mathcal{L}^{c}_{\omega,\omega}$ this is preserved by forcing not adding new countable subsets of $|M|$ (this holds even for $\psi \in \mathcal{L}_{\infty,\omega_1}$).*

(4) *If $M \models \psi$, $\psi \in \mathcal{L}_{\infty,\lambda}$, λ regular, then this is preserved by forcing by P, where P does not add sequences of ordinals of length $< \lambda$. If P is $\aleph_1$-complete this holds for $\psi \in \mathcal{L}^{aa}_{\infty,\lambda}$.*

(5) *Suppose V_1, V_2 are models of set theory (with the same ordinals), $V_1 \subseteq V_2$, and letting $\lambda = h(\mathcal{L})^{V_1}$ where $\mathcal{L}$ is $\mathcal{L}^{wo}_{\omega,\omega}$ or $\mathcal{L}^{c}_{\mu,\omega}$ or $\mathcal{L}^{c}_{\mu,\kappa}$ (just a suitable downward Lowenheim–Skolem theorem is needed).*

If $\{A \subseteq \lambda : A$ bounded, $A \in V_1\} = \{A \subseteq \lambda : A$ bounded, $A \in V_2\}$ then $h((\mathcal{L})^{V_1} = h(\mathcal{L})^{V_2}$.

Proof: Left to the reader.

2 Independence for $\mathcal{L}^{c}_{\omega,\omega}$, $\mathcal{L}^{II}_{\omega,\omega}$

In this section we shall deal with the independence of the cases where $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^{c}_{\omega,\omega})$.

Lemma 2.1

(1) *For any logic $\mathcal{L}$, $h(\mathcal{L}(wo)) \leq h(\mathcal{L}^{c}) \leq h(\mathcal{L}(aa)) \leq h(\mathcal{L}^{II})$.*

(2) *For any logic $\mathcal{L}$, we have $h(\mathcal{L}^{c}_{\omega,\omega}) < h(\mathcal{L}^{II}_{\omega,\omega})$.*

(3) *For any logic $\mathcal{L}$, we have $h(\mathcal{L}^{c}_{\lambda^+,\omega}) < h(\mathcal{L}^{II}_{\lambda^+,\omega})$, moreover: if $\lambda < h(\mathcal{L}^{II}_{\mu,\omega})$ then $h(\mathcal{L}^{c}_{\lambda^+,\omega}) < h(\mathcal{L}^{II}_{\mu,\omega})$.*

Proof:

(1) By Kaufman and Shelah ([6], Theorem 4.1); only $\mathcal{L} = \mathcal{L}_{\omega,\omega}$ is discussed there, but it makes no difference, the nontrivial part is $h(\mathcal{L}^c) \leq h(\mathcal{L}^{aa})$;

(2) See [6];

(3) Use 1.5 for the function $f: f(\kappa) = (\kappa^{\aleph_0})^+$.

Lemma 2.2
(1) *If $V = L$ then $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^{c}_{\omega,\omega}) < h(\mathcal{L}^{aa}_{\omega,\omega}) = h(\mathcal{L}^{II}_{\omega,\omega})$.*
(2) *If $V = L$, then for any logic $\mathcal{L}$, $h(\mathcal{L}^{wo}) = h(\mathcal{L}^c) \leq h(\mathcal{L}^{aa}) = h(\mathcal{L}^{II})$.*

Proof: For (1), see [6]; and (2) is the same proof.

Fact 2.3 *For a regular cardinal λ and $\psi \in \mathcal{L}^{aa}_{\lambda,\lambda}$ the following are equivalent:*
(i) *For every μ large enough $\Vdash_{\text{Lévy}(\lambda,\mu)}$ "ψ has a model of power λ".*
(ii) *For some λ-complete forcing Q we have: $\Vdash_Q$ "ψ has a model of power $\geq \lambda$".*

Proof: (i) $\Rightarrow$ (ii): As Lévy(λ,μ) is a λ-complete forcing notion, (i) is a particular case of (ii).

(ii) $\Rightarrow$ (i): Let Q be a λ-complete forcing notion such that $\Vdash_Q$ "ψ has a model of cardinality $\geq \lambda$". Let μ be such that $\mu > |Q|, \Vdash_Q$ "ψ has a model of cardinality $\geq \lambda$ but $\leq \mu$" and $\mu = \mu^\lambda$. In $(V^Q)^{\text{Lévy}(\lambda,\mu)}$ ψ has a model of cardinality λ by 1.7(4).

But $(V^Q)^{\text{Lévy}(\lambda,\mu)}$ is $V^{\text{Lévy}(\lambda,\mu)}$ (see e.g. Kunen [7]).

Notation 2.3A Let $\mu_0[\psi,\lambda]$ be the first cardinal μ satisfying 2.3(i), if one exists, and λ otherwise.

Lemma 2.4
(1) *In some forcing extension of L, $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^{c}_{\omega,\omega}) < h(\mathcal{L}^{aa}_{\omega,\omega}) < h(\mathcal{L}^{II}_{\omega,\omega})$.*
(2) *Moreover, for $\lambda < h(\mathcal{L}^{II}_{\mu,\kappa})$, we have $h(\mathcal{L}^{aa}_{\lambda,\lambda}) < h(\mathcal{L}^{II}_{\mu,\kappa})$.*

Remark 2.4A If we want to have: $\lambda < h(\mathcal{L}^{aa}_{\mu,\lambda}) \Rightarrow h(\mathcal{L}^{c}_{\lambda,\omega}) < h(\mathcal{L}^{aa}_{\mu,\omega})$, we should define $\lambda_{i+1} = h(\mathcal{L}^{c}_{\mu_i^+,\omega})^+$.

Proof: Start with $V = L$. Let $\psi^* \in \mathcal{L}^{aa}_{\omega,\omega}$ a sentence such that $h(\mathcal{L}^{c}_{\omega,\omega}) < h(\psi^*) < \infty$ be chosen later. Let $\lambda_0 > h(\psi^*)$ be regular, $\lambda_0 < h(\mathcal{L}^{II}_{\omega,\omega})$. We define an iterated forcing $\langle P_i, Q_j : i \leq \infty, j < \infty\rangle$ and cardinals λ_i such that:

(a) iteration is with set support (so P_∞ is a class forcing).
(b) λ_i is regular cardinal.
(c) $\lambda_i \geq \sum_{j<i} \lambda_j$, and λ_i is the first regular cardinal $\geq \sum_{j<i}(\lambda_j + \mu_j)^+$ (when $i > 0$).
(d) $Q_i (\in V^{P_i})$ is λ_i-complete.
(e) Let $\{\psi^i_\alpha : \alpha < \lambda_i\}$ be the set of all $\mathcal{L}^{aa}_{\lambda_i,\lambda_i}$ sentences (up to isomorphism) in V^{P_i}.

We define in V^{P_i}, Q_i to be Lévy(λ_i,μ_i) where μ_i is the successor of $\sup\{\mu_0[\psi, \lambda_i]^{P_i} : \psi \in \mathcal{L}_{\lambda_i,\lambda_i}\}$ and so $\lambda_{i+1} = \mu_i^+$.

Our model is V^{P_∞}. Clearly the λ_i are not collapsed (as well as limits of λ_i and $\chi < \lambda_0$) and other successor cardinals $\geq \lambda_0$ are collapsed. So in V^{P_∞}, for regular $\chi \geq \lambda_0$, if $\psi \in \mathcal{L}^{aa}_{\chi,\chi}$ has a model of cardinality $\geq \chi$ then it has a model of cardinality χ. As clearly $h(\mathcal{L}^{II}_{\omega,\omega}) > \lambda_0$, we get by 1.4 $h(\mathcal{L}^{aa}_{\omega,\omega}) < h(\mathcal{L}^{II}_{\omega,\omega})$ (as well as (2)).

By the Lowenheim–Skolem theorem, using 1.7(5) for $\psi \in \mathcal{L}^{wo}_{\omega,\omega}$ or $\psi \in \mathcal{L}^{c}_{\omega,\omega}$, $h(\psi)$ does not change (being ∞ or $< \lambda_0$) hence (in V^{P_∞}) $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^{wo}_{\omega,\omega})^V$; $h(\mathcal{L}^{c}_{\omega,\omega}) = h(\mathcal{L}^{c}_{\omega,\omega})^V$. Hence (in V^{P_∞}) $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^{c}_{\omega,\omega})$ as this holds in L.

We still have to choose $\psi^* \in \mathcal{L}^{aa}_{\omega,\omega}$ and prove that in V^{P_∞} we have $h(\mathcal{L}^{c}_{\omega,\omega}) < h(\mathcal{L}^{aa}_{\omega,\omega})$. There is $\psi^* \in \mathcal{L}^{aa}_{\omega,\omega}$, $L \models$ "$h(\mathcal{L}^{c}_{\omega,\omega}) < h(\psi^*) < \infty$" (by 2.2).

Clearly for any such ψ^*, $V^{P_\infty} \models$ "$h(\mathcal{L}^{c}_{\omega,\omega}) < h(\psi^*)$" (as no new subset of $h(\psi^*)$ is added), but we need also $V^{P_\infty} \models$ "$h(\psi^*) < \infty$"; but checking the sentences produced in [6] proof of Theorem 4.3 (for proving $L \models h(\mathcal{L}^{aa}) = h(\mathcal{L}^{II})$), they are like that. So $V^{P_\infty} \models$ "$h(\mathcal{L}^{c}_{\omega,\omega}) < h(\mathcal{L}^{aa}_{\omega,\omega})$".

Lemma 2.5
(1) *In some forcing extension of L we have* $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^{c}_{\omega,\omega}) = h(\mathcal{L}^{aa}_{\omega,\omega}) < h(\mathcal{L}^{II}_{\omega,\omega})$.
(2) *In fact for any logic* $\mathcal{L}$ *we have* $h(\mathcal{L}^{wo}) = h(\mathcal{L}^{c}) = h(\mathcal{L}^{aa})$.
(3) *For* $\lambda < h(\mathcal{L}^{II}_{\mu,\kappa})$ *then,* $h(\mathcal{L}^{aa}_{\lambda,\lambda}) = h(\mathcal{L}^{II}_{\mu,\kappa})$.

Proof: We start with $V = L$. We define a (full set support) iteration, $\bar{Q} = \{P_i, \mathbf{Q}_i : i \text{ an ordinal}\}$ ($\mathbf{Q}_i$ – a P_i name) and cardinals λ_i such that

(a) λ_i is regular $\geq \aleph_1 + |P_i|$, for i limit $\lambda_i = (\sum_{j<i} \lambda_i)^+$.
(b) $\mathbf{Q}_i$ is λ_i-complete.
(c) If i is even, $G_i \subseteq P_i$ the generic set (remember $\mathbf{Q}_i \in V^{P_i}$) then let the set of elements of P_i be listed as $\{p^i_\alpha : \alpha < \lambda_i\}$, and Q_i will be the product of the Lévy collapses of $\aleph_{\lambda_i\omega+4\alpha+2+m}$ to $\aleph_{\lambda_i\omega+4\alpha+1+m}$ for $\alpha < \lambda_i$ such that: $[p^i_\alpha \in G_i \Rightarrow m = 0]$ and $[p^i_\alpha \notin G_i \Rightarrow m = 1]$. Let $\lambda_{i+1} = \aleph_{\lambda_i\omega+\lambda_i+1}$.
(d) If i is odd, let $\{\psi^i_\alpha : \alpha < \lambda_i\}$ list all sentences of $\mathcal{L}^{aa}_{\lambda_i,\lambda_i}$ in a rich enough vocabulary (of cardinality λ_i). For each α, if there is a λ_i-complete forcing notion Q (which is a set) and (in V^{P_i}) $\Vdash_Q$ "there is a model of ψ^i_α of cardinality $\geq \lambda_i$" then let μ^i_α be such that $\Vdash_{\text{Lévy}(\lambda_i,\mu^i_\alpha)}$ "ψ^i_α has a model of cardinality λ_i". Otherwise $\mu^i_\alpha = \lambda_i$.

Note that μ^i_α exists by 2.3.
Let $Q_i = \text{Lévy}(\lambda_i, < \lambda_{i+1})$ where $\lambda_{i+1} = (\lambda_i + \sum_{\alpha<\lambda_i} \mu^i_\alpha)^{++}$.
Let $G_\infty \subseteq P_\infty$ be generic over V and $V[G_\infty]$ be our model. Note in $V[G_\infty]$,

(*) $[i \text{ odd} \Rightarrow \lambda_{i+1} = \lambda_i^+]$.
$[i \text{ even} \Rightarrow \lambda_{i+1} = \lambda_i^{+(\lambda_i\omega+1)}]$.
$[i \text{ limit} \rightarrow \lambda_i = (\sum_{j<i} \lambda_j)^+]$.

For $\mu = \lambda_{2j+1}$, if $\psi \in \mathcal{L}^{aa}_{\mu,\mu}$ has a model of cardinality $\geq \mu$ then it has a model of cardinality λ (by 2.3 + 1.7(4)). By (*) we deduce that $V^{P_\infty} \models$ "if $\psi \in \mathcal{L}^{aa}_{\lambda,\chi}$ has a model of cardinality $> \lambda$ then it has a model M, $\lambda < \|M\| < \aleph_{\lambda^+}$".

So 1.5 is applicable to show $h(\mathcal{L}^{aa}_{\omega,\omega}) < h(\mathcal{L}^{II}_{\omega,\omega})$ (and by 1.6 and 1.7) also 2.5(3) holds.

Why $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^{aa}_{\omega,\omega})$? Let ψ^* describe $(L_\lambda, \in, G_\infty \cap \bigcup_{i<\delta} P_i)$.

If $M \models \psi^*$, then for some α and G, $M \cong (L_\alpha, \in, G)$, so without loss of generality equality holds. Now if $\lambda < |\alpha|$, $M \models$ "λ is a [regular] cardinal of L" *iff* λ is a [regular] cardinal of L. Also we know that for every ordinal ζ, if in L, $\lambda_{2i} \leq \aleph_\zeta < \lambda_{2i+1}$, ζ divisible by four, then forcing by P_∞ collapses at most one of the cardinals $\aleph_{\zeta+1}, \aleph_{\zeta+2}, \aleph_{\zeta+3}, \aleph_{\zeta+4}$ of L; if $\lambda_i\omega \leq \zeta < \lambda_i\omega + \lambda_i$ then exactly one.

We assume ψ^* say so, and so when $\aleph^L_{\zeta+4} \leq |\alpha|$ the answer in M to the question "which of $\aleph_{\zeta+1}, \aleph_{\zeta+2}, \aleph_{\zeta+3}, \aleph_{\zeta+4}$ is collapsed" is the right one. So when $\lambda_{2i+1} < |\alpha|$, we can in M reconstruct $G_\infty \cap P_{2i}$ (see choice of Q_{2i}).

But $V^{P_\infty} \models$ "$\lambda_{2i+1} \leq \aleph_{\lambda_{2i}(\omega+1)+1}$ and $\lambda_{2i+2} = (\lambda_{2i+1})^+$ and for limit δ we have $\lambda_\delta = (\sum_{i<\delta} \lambda_i)^+$".

The rest is as in [6], proof of 4.3.

3 $h(\mathcal{L}^{wo}_{\lambda,\omega})$ is O.K. but for $h(\mathcal{L}_{\aleph_3,\omega})$ large cardinals are needed and sufficient In Section 2 we deal with the three cases for which $h(\mathcal{L}^{wo}_{\omega,\omega}) = h(\mathcal{L}^{c}_{\omega,\omega})$. Here we deal with the three cases where $h(\mathcal{L}^{wo}_{\omega,\omega}) < h(\mathcal{L}^{c}_{\omega,\omega})$. The new part is Lemma 3.2, and then in 3.3 we get the desired conclusion. For dealing with $\mathcal{L}_{\lambda^+,\omega}$ we do not assume CON(ZFC) alone, we assume the existence of a class of large cardinals (weaker than measurability). By 3.4 at least if $\lambda \geq \aleph_3 + (2^{\aleph_0})^+$, something of this sort is necessary.

Fact 3.1 *The following are equivalent for $\psi \in \mathcal{L}^{wo}_{\omega,\omega}$ or even $\psi \in \mathcal{L}^{wo}_{\infty,\omega}$:*
(i) *For every μ large enough $\Vdash_{\text{Lévy}(\aleph_0,<\mu)}$ "$h(\psi) = \infty$".*
(ii) *For some (set) forcing notion P we have $\Vdash_P$ "$h(\psi) = \infty$".*

Proof: Similar to the proof of 2.3.

Notation 3.1A Let the first μ satisfying (i) be $\mu_1(\psi)$ (and $\aleph_0$ if there is no such μ).

Lemma 3.2 $(V = L)$.
(1) *For some (set) forcing notion P*

$$\Vdash_P \text{“}h(\mathcal{L}^{wo}_{\omega,\omega}) < h(\mathcal{L}^{c}_{\omega,\omega})\text{”}$$

and this is preserved by $h(\mathcal{L}^{wo}_{\omega,\omega})^+$-complete forcing".
(2) *In (1) we can use* Lévy$(\aleph_0, <\mu)$ *for some $\mu > cf\mu = \aleph_0$.*
(3) *We can use instead* Cohen$(\mu) = \{f : f$ *a finite function from μ to* $(0,1)\}$. *So cardinals are not collapsed.*

Proof: (1) Let $\mu^* = \sup\{\mu_1(\psi) : \psi \in \mathcal{L}^{wo}_{\omega,\omega}\}$.

We now define a finite support iteration $\langle P_i, Q_n : i \leq \omega, n < \omega\rangle$ and μ_n as follows:

$\mu_0 = \mu^*$
$Q_0 = \text{Lévy}(\aleph_0, \mu_0)$
for $n \geq 0$, μ_{n+1} is $h(\mathcal{L}^{wo}_{\omega,\omega})^{V^{P_n}}$
$Q_n = \text{Lévy}(\aleph_0, \mu_n)$.

Let $\mu = (\sum \mu_n)$. Note that P_ω satisfies the $\mu^+ - c.c.$
Now V^{P_ω} is our model. Note

(*) $V^{P_\omega} \models G.C.H. + \aleph_1 = \mu^+$, and $V = L[\mathbf{R},<]$ for any well ordering of $\mathbf{R}$.

Note that in $\mathfrak{B} = (\omega \cup \mathcal{P}(\omega)^{V^{P_\omega}}; o, +, \times, \in)$ we can define by first-order formulas (representing ordinals by well ordering of ω):

(a) $\bigcup_n \mu_n$ (maximal countable ordinal which is a cardinal in L_{μ^+})
(b) L_{μ^+} hence $\langle \mu_n : n < \omega \rangle$ (by induction remembering the Lowenheim-Skolem theorem) hence the iteration (really we can omit this as P_ω is just Lévy $(\aleph_0, < \mu)$)
(c) the set $\mathbf{R}^- =_{def} \{r \in \mathbf{R}$: for some n, and $G \subseteq P_n$ generic over $V, r \in V[G]\}$. And for $r \in \mathbf{R}^-$
(d) $H_r = \{\psi \in \mathfrak{L}^{wo}_{\omega,\omega} : L[r] \models h(\psi) < \infty\}$ as it is equal to $\{\psi \in \mathfrak{L}^{wo}_{\omega,\omega} : L[r] \models h(\psi) < \bigcup_n \mu_n\}$.

[Note that P_n's are homogeneous, hence $h(\psi)$ does not depend on $G \subseteq P_n$].

So by 3.1 and the choice of μ_0, we can define in that model $\mathfrak{B}$

$$H^* = \{\psi \in \mathfrak{L}^{wo}_{\omega,\omega} : h(\psi)^{V^{P_\omega}} < \infty\}.$$

[How? It is $\bigcap\{H_r : r \in \mathbf{R}^-\}$, remembering 3.1.]

Let $\lambda = h(\mathfrak{L}^{wo}_{\omega,\omega})$ (in V^{P_ω}).

Now we define a sentence $\varphi \in \mathfrak{L}^{c}_{\omega,\omega}$: it just describes $(H(\lambda), \in)$: it says

(i) Enough axioms of ZFC hold.
(ii) Every countable bounded set of ordinals is represented.
(iii) On every infinite cardinal α there is a model M_α with universe α satisfying some $\psi \in H^*$ (which we have shown is definable in any model M of φ).

So we have proved the first assertion from 3.2. Now λ-complete forcing, preserve trivially "$h(\psi) \geq \mu$" as it preserves satisfaction for $\mathfrak{L}^{wo}_{\omega,\omega}$. It preserves "$h(\psi) < \infty$". As this is equivalent to "$h(\psi) < \lambda$", the forcing adds no new model power $< \lambda$, and the Lowenheim–Skolem theorem finishes the argument.

(2) We have proved it in the proof of (1).

(3) A similar proof, replacing $\mu_1(\psi)$ by $\mu_1' =$ first μ such that $\Vdash_{Cohen(\mu)}$ "$h(\psi) = \infty$" if there is one, $\aleph_0$ otherwise.

Conclusion 3.3 *For some forcing extensions of L:*
(1) $h(\mathfrak{L}^{wo}_{\omega,\omega}) < h(\mathfrak{L}^{c}_{\omega,\omega}) < h(\mathfrak{L}^{aa}_{\omega,\omega}) = h(\mathfrak{L}^{II}_{\omega,\omega})$.
(2) $h(\mathfrak{L}^{wo}_{\omega,\omega}) < h(\mathfrak{L}^{c}_{\omega,\omega}) = h(\mathfrak{L}^{aa}_{\omega,\omega}) < h(\mathfrak{L}^{II}_{\omega,\omega})$.
(3) $h(\mathfrak{L}^{wo}_{\omega,\omega}) < h(\mathfrak{L}^{c}_{\omega,\omega}) < h(\mathfrak{L}^{aa}_{\omega,\omega}) < h(\mathfrak{L}^{II}_{\omega,\omega})$.

Proof: Combine 3.2 with Section 2.

Claim 3.4 $(\neg 0\#)$: *For $\lambda \geq \aleph_3 + (2^{\aleph_0})^+$ we have $h(\mathfrak{L}^{wo}_{\lambda,\omega}) = h(\mathfrak{L}^{c}_{\lambda,\omega})$.*

Remark The logics are essentially equivalent.

Proof: If $\psi \in \mathfrak{L}^{wo}_{\lambda,\omega}$ says M is, for some α, $(L_\alpha[A], \in)$ (up to isomorphism), $\alpha > 2^{\aleph_0}$, $A \subseteq 2^{\aleph_0}$, every subset of ω is in $L_{(2^{\aleph_0})}[A]$, and $\alpha \geq \omega_2$, and $\{\delta < \aleph_2 : cf\delta = \aleph_0$ in $L_{\omega_2}[A]\} = \{\delta < \aleph_2 : cf\delta = \aleph_0\}$ then by Jensen's covering lemma $[\beta < |\alpha| \Rightarrow$ every countable subset of β is represented in the model$]$.

Claim 3.5 *Suppose that:*
(*) *For every χ for some μ, $\mu \rightarrow (\omega_1)^{<\omega}_\chi$ or even just*
(**) *For every χ for some μ, $\mu \rightarrow_{BG} (c)^{<\omega}_\chi$, which means: for every $f: [\mu]^{<\omega} \rightarrow \chi$ for some $\langle \gamma_n : n < \omega \rangle$ for every $\alpha < \omega_1$, for some $Y \subseteq \mu$, Y has order type α and $\bigwedge_n (\forall w \in [Y]^n)[\gamma_n = f(w)]$.*

Then for every λ, $h(\mathfrak{L}^{wo}_{\lambda^+,\omega}) < h(\mathfrak{L}^{c}_{\lambda^+,\omega})$.

Remark 3.5A
(1) The property $(**)$ was discovered by Baumgartner and Galvin [3]; proving

$$\mu \to_{BG} (c)^{<\omega}_{\chi} \text{ iff } \mu \geq h(\mathcal{L}^{wo}_{\chi^+,\omega}).$$

(2) See ([6], 4.2) (for $\lambda = \omega$).

Proof: There is a sentence $\psi \in \mathcal{L}^{c}_{\omega,\omega}$ such that for $\chi \leq \mu$: there is a model M of ψ, $\|M\| = \mu$, $|P^M| = \lambda$, iff $(\forall \alpha < \mu)\alpha \not\to_{BG} (c)^{<\omega}_{\chi}$.

On $K = K^V$ (the core model of V) see Dodd [4].

Claim 3.6 *Suppose $V = K$, and $(**)$ (from 3.5), then*
(1) *For every λ we have $h(\mathcal{L}^{wo}_{\lambda^+,\omega}) < h(\mathcal{L}^{c}_{\lambda^+,\omega}) < h(\mathcal{L}^{aa}_{\lambda^+,\omega}) = h(\mathcal{L}^{II}_{\lambda^+,\omega})$.*
(2) *For every $\mathcal{L}$, $h(\mathcal{L}^{aa}) = h(\mathcal{L}^{II})$.*

Proof: (1) First inequality by the observation above, the second inequality follows from last equality + Th 2.1(3), last equality see (2).

(Note: If $cf\delta > \aleph_0$ in $\mathcal{L}^{aa}_{\omega,\omega}$ we can say for $A \subseteq \delta$ whether $\{\alpha < \delta : cf\alpha = \aleph_0, \alpha \in A\}$ is a stationary subset of δ).

(2) As in [6].

Observation 3.7 There is $\psi \in \mathcal{L}^{c}_{\omega,\omega}$ such that $M \models \psi$ *iff* M is isomorphic to K_α for some α.

It is known (see [3] and [4]):

Fact 3.8 *If in V there is, e.g., a measurable cardinal in* Card, *then $K \models (**)$.*

Claim 3.9 *Suppose $V = K$ and $(**)$ holds. For some forcing extension $V[G_\infty]$ of V, $V[G_\infty] \models$ "$(**)$ and for every λ, $h(\mathcal{L}^{wo}_{\lambda^+,\omega}) < h(\mathcal{L}^{c}_{\lambda^+,\omega}) < h(\mathcal{L}^{aa}_{\lambda^+,\omega}) < h(\mathcal{L}^{II}_{\lambda^+,\omega})$".*

Proof: Similar to 2.4(1) except that we want to preserve $(**)$. We define by induction on α an iterated forcing $\langle P_i, Q_j \leq \alpha, j < \alpha\rangle$ with set support and cardinals λ_i increasing such that:

(i) $\lambda_0 = \aleph_2$.
(ii) $\lambda_\delta = (\sum_{i<\delta} \lambda_i + |P_\delta|)^+$.
(iii) if λ_i, P_i are defined, let μ_i be $\lambda_i^+ + \bigcup\{\mu_0[\psi, \lambda_i] : \psi \in \mathcal{L}^{aa}_{\lambda_i,\lambda_i}\}$.

$Q_i = \text{Lévy}(\lambda_i^+, \mu_i^+)$ (in V^{P_i}) and λ_{i+1} is minimal such that $\lambda_{i+1} \to_{BG} (c)^{<\omega}_{\mu_i^{++}}$ and $\lambda_{i+1} \leq h(\mathcal{L}^{c}_{\mu_i^+,\omega})$.

We leave the rest to the reader.

Claim 3.10 *Suppose $V = K$ and $(**)$ holds. For some forcing extension $V[G_\infty]$ of V, $V[G_\infty] \models (**)$ (hence the conclusion of 3.7) and for every λ,*

$$h(\mathcal{L}^{wo}_{\lambda^+,\omega}) < h(\mathcal{L}^{c}_{\lambda^+,\omega}) = h(\mathcal{L}^{aa}_{\lambda^+,\omega}) < h(\mathcal{L}^{II}_{\lambda^+,\omega}).$$

Proof: Combine the proofs of 3.9 and 2.5.

4 Lowering consistency strength

We present here some alternative proofs with lower consistency strength than in Section 3. Specifically 4.1, 4.3, and 3.2(3) justify the restriction $\lambda \geq \aleph_3 + (2^{\aleph_0})^+$ in 3.4.

Lemma 4.1 *Let $V = L$. Then there is a forcing notion $P \in L$, not adding reals, such that for $G \subseteq P$ generic over V, in $V[G]$:*

(a) $h(\mathfrak{L}^{wo}_{\omega_1,\omega}) < h(\mathfrak{L}^{c}_{\omega_1,\omega})$.

(b) *No $\aleph_1$-complete forcing notion (or even forcing notion satisfying the $\mathbf{I}$-condition, $\mathbf{I}$ a set of $\aleph_2^{V[G]}$-complete ideals from L) changes the truth value of "$h(\psi) < \infty$" for $\psi \in L^{wo}_{\omega_1,\omega}$.*

(c) *There is a sentence $\psi \in \mathfrak{L}^{c}_{\omega,\omega}$ whose class of models of power $\geq \aleph_2$ is just $\{L_\alpha[G] : \alpha \geq \aleph_2\}$ (and note $P \in L_{\aleph_2}[V[G]]$).*

(d) $h(\mathfrak{L}^{c}_{\lambda^+,\omega}) < h(\mathfrak{L}^{aa}_{\lambda^+,\omega}) = h(\mathfrak{L}^{II}_{\lambda^+,\omega})$.

Remark 4.1A In the proof below, coding generic sets by the decision which L-cardinals are collapsed is replaced here by "which L-regular cardinal have in V cofinality $\aleph_0$ and which cofinality $\aleph_1$".

Proof: Let $\mathbf{I}(\mu, \kappa)$ be, e.g. the class of filters D which are λ-complete over some λ (this in V), where $\mu \leq \lambda < \kappa$, $|\bigcup D| < \kappa$.

We define by induction on n, α_n, β_n, $\lambda_{i,j}$, $\mu_{i,j}$, $\langle P_i, \mathbf{Q}_j : i \leq \alpha_n, j < \alpha_n\rangle$, and f_n such that letting $\mathbf{I}_{i,j} = \mathbf{I}(\lambda_{i,j}, \mu_{i,j})$:

(A) $\alpha_0 = 0$, $\alpha_{n+1} > \alpha_n$.

(B) $\langle P_i, \mathbf{Q}_j, \mu_j : i \leq \alpha_n,\ j < \alpha_n\rangle$ is an *RCS* iteration suitable for $x_{\alpha_n} = \langle \mathbf{I}_{i,j}, \lambda_{i,j}, \mu^{i}_{i,j}, i < j \leq \alpha_n$, i not strongly inaccessible$\rangle$. See Shelah [9], Ch. XI or Shelah [11], Ch. XI, particularly Definition 6.1.

(C) f_n is a one-to-one function from P_i onto some ordinal β_n, extending $\bigcup_{e<n} f_e$.

G_α will denote a generic subset of P_α.

For $n = 0$ there is nothing to do.

For $n + 1$, note that forcing by P_{α_n} does not add new reals. So $(\mathfrak{L}^{wo}_{\omega_1,\omega})^V = (\mathfrak{L}^{wo}_{\omega_1,\omega})^{V[G_{\alpha_n}]}$ and let $\{\psi_i : i < \omega_1\}$ be a list of the sentences (up to isomorphism).

We now (i.e., for defining α_{n+1} etc.) define by induction on $\zeta < \omega_1$, $Q_{\alpha_n+\zeta}$, $x_{\alpha_n+\zeta+1}$ as follows:

(a) $\langle P_i, \mathbf{Q}_j : i \leq \alpha_n + \zeta\rangle$ is $x_{\alpha_n+\zeta+1}$-suitable *RCS* iteration.

(b) If there is $\mathbf{Q}_{\alpha_n+\zeta}$, a $P_{\alpha_n+\zeta}$-name of a forcing notion satisfying the $\mathbf{I}((|P_{\alpha_n+\zeta}| + \sup\{\lambda_{i,j} : i < j \leq \alpha_n + \zeta\})^+, \kappa)$-condition for some κ such that $\Vdash_{P_{\alpha_n+\zeta+1} * Q}$ "ψ_ζ has arbitrarily large models" then $\mathbf{Q}_{\alpha_n+\zeta}$ is like that, otherwise it is, e.g., Lévy $(\aleph_1, 2^{\aleph_1})$.

Next let $\mu_\zeta = h(\mathfrak{L}^{wo}_{\omega_1,\omega})^{V[G_{\alpha_n+\omega_1}]}$, $Q_{\alpha_n+\omega_1} =$ Lévy $(\aleph_1, \mu_\zeta^+)$. Now (where $\langle , \rangle$ is Godel's pairing function on ordinals), let in $V[G_{\alpha_n+\omega_1}] : A_n = \{\langle f_n(p), f_n(q)\rangle : p, q \in P_{\alpha_n} \vDash p \leq q$ and $p \neq q\} \cup \{\langle f_n(p), f_n(p)\rangle : q \in G_{\alpha_n+\omega}\}$ and let $\gamma_n = \sup\{\langle f_n(p), f_n(q)\rangle : p, q \in P_{\alpha_n+\omega_1}\}$. Now we define $Q_{\alpha_n+\omega_1+i}$ by induction on $i \leq \gamma_n$:

$Q_{\alpha_n+\omega_1}$ is Lévy $(\aleph_1, \aleph_2)^{V[G_{\alpha_n+\omega_1}]}$,

$Q_{\alpha_n+\omega_1+1+2i+1}$ is Lévy $(\aleph_1, \aleph_2)\,[V[G_{\alpha_n+\omega_1+1+2i+1}]]$,

$Q_{\alpha_n+\omega_1+1+2i}$ is Namba forcing (of $V[G_{\alpha_n+\omega_1+1+i}]$) if $i \in A_n$ and Lévy $(\aleph_1, \aleph_\zeta)^{V(n,i)}$ where $V(n,i) = V[G_{\alpha_n+\omega_1+1+2i}]$ if $i \notin A_n$.

Now let $\alpha_{n+1} = \alpha_n + \omega_1 + 2\gamma_n$, $\lambda_{n+1} = |P_{\alpha_n+\omega_1+2\gamma_n}|$, and define f_{n+1}.

We leave the rest to the reader. (We can make our free choices as the first legal candidates in L.)

Conclusion 4.2

(1) *We can do the forcing from* 2.4, 2.5 *to the universe we got in* 4.1 *getting corresponding results* (*for* $\mathcal{L}_{\omega_1,\omega}(Q)$*'s, with CH and G.C.H.*): *so we need CON*(ZFC) *only.*

(2) *The same holds for* 4.3 *for the* $\mathcal{L}_{\omega_2,\omega}(Q)$*'s* (*so we use CON*(ZFC + "*the class of ordinals in Mahlo*") *only*) [*similar proof*].

Lemma 4.3 *Suppose* $V = L$, (*for simplicity*) *and* ∞ *is a Mahlo cardinal* (*i.e., every closed unbounded class of cardinals has a regular member*). *Then there is an accessible cardinal* λ *and a forcing notion* $P \subseteq H(\lambda)$, *such that*:

(a) P *satisfies the* λ*-c.c., does not add reals and collapse every* $\mu \in (\aleph_1, \lambda)$: *and* $\Vdash_P$ "*G.C.H.* + λ *is* $\aleph_2$" *and* $|P| = \lambda$.

(b) $h(\mathcal{L}^{wo}_{\omega_2,\omega}) < h(\mathcal{L}^{c}_{\omega,\omega})$.

(c) *There is a sentence* $\psi \in \mathcal{L}^{c}_{\omega,\omega}$ *whose class of models of power* $\geq \aleph_2$ *is just suitable expansions of* $\{L_\alpha[G] : \alpha \geq \aleph_2\}$.

Proof: Like 4.1, but instead of induction on $n < \omega$ we do induction on $\gamma < \infty$, and in the induction only we first do the coding $(Q_{\alpha_n+\omega_1+i}, i < \gamma)$ (so that for c), we say that for some club of C of ω_2, for $\delta \in C$, we are coding the set of sentence in $\mathcal{L}_{|\delta|^+}[G \cap P_\delta]$.

Do we really need the large cardinal hypothesis in 4.3 (and so in 4.2(2))?

Claim 4.4 *Suppose* $0^\# \notin V$ *and* $\aleph_2^V$ *is a successor cardinal in L and* $2^{\aleph_0} = \aleph_1$ *then for some sentence* $\psi \in \mathcal{L}^{wo}_{\omega_2,\omega}$, *its models are exactly suitable expansions of* $(L_\alpha, \mathcal{P}_{<\aleph_1}(\alpha))$, *where* α *is the last L-cardinal* $< \aleph_2^V$.

Hence, $h(\mathcal{L}^{wo}_{\omega_2,\omega}) = h(\mathcal{L}^{c}_{\omega_2,\omega})$.

Proof: Should be clear.

Concluding Remarks 4.5 Still we do not settle the exact consistency strength. In fact, e.g., if $\aleph_2^V$ is the first L-inaccessible, we can still prove the last sentence of 4.4. For $h(\mathcal{L}^{wo}_{\omega_2,\omega}) < h(\mathcal{L}^{c}_{\omega_1,\omega})$ with $2^{\aleph_0} = \aleph_2$ we can generalize Lemma 4.3 to this case (using Shelah [11], Ch. XV). Also, there is a gap in consistency strength in Section 3 for $\lambda \geq \aleph_3 + (2^{\aleph_0})^+$. It is not hard to show that if $\lambda \geq \aleph_2 + 2^{\aleph_0}$, $cf\lambda > \aleph_0$ and for some $A \subseteq \lambda$, A does not exist, then $h(\mathcal{L}^{wo}_{\lambda^+,\omega}) = h(\mathcal{L}^{c}_{\lambda^+,\omega})$.

Added in proof: In Lemma 4.3 a sufficient consistency strength of "G.C.H. + $h(\mathcal{L}^{wo}_{\omega_2,\omega}) < h(\mathcal{L}^{c}_{\omega_2,\omega})$" is:

> there is an inaccessible cardinal λ such that for every $\varphi, x \in L_\lambda$ if, for a class of ordinals α, $(L_\alpha, \in) \models \varphi[x, \lambda]$ then for some $\lambda' < \lambda$, for a class of ordinals α, $(L_\alpha, \in) \models \varphi[x, \lambda']$.

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