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Classifying Pairs of Equivalence Relations

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Abstract Let E_0, E_1 be equivalence relations such that the number of E_0-classes or E_1-classes in any class of the join of E_0 and E_1 is bounded. We study classification theory (according to Shelah) for these pairs of equivalence relations.

Roughly speaking, Shelah's classification problem is to find under which conditions, given a (countable) complete first-order theory T, the isomorphism types of models of T can be characterized by invariants such as cardinal numbers or something similar. It is simple to see that if, for all sufficiently large cardinals λ, T has too many (namely 2^λ) nonisomorphic models of power λ, then this assignment of invariants cannot be done. Shelah shows that, in some sense, the converse is also true: if T has fewer than 2^λ nonisomorphic models in some uncountable λ, then a classification of isomorphism types of models of T is possible (see Baldwin [1] for a more complete and precise exposition of the whole matter).

Here we are interested in the classification problem for complete theories of two equivalence relations E_0, E_1. The general analysis of this case is complicated owing to the Rogers theorem that the theory of two equivalence relations is undecidable (Rogers [7]). Hence we limit ourselves to the theories T satisfying the condition (+): if E denotes the equivalence relation generated by E_0 and E_1, then there exists $h \in \omega$ such that any E-class contains at most either h E_0-classes or h E_1-classes.

The first part of the paper (Sections 1–3) is devoted to characterizing in this context the basic tools of Shelah's classification theory: regular types, SR types, orthogonal types, and so on. In particular, we will see that any theory T satisfying (+) is superstable and monadically stable. Section 4 shows that T is presentable and shallow and satisfies the existence property, and hence is classifiable in the Shelah sense.

We already studied theories of pairs of equivalence relations in Toffalori [9], where we dealt with the problem of determining the theories that are categori-

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cal in some infinite power. We assume that the reader is familiar with [9], mainly with Sections 1 and 2. References for basic stability theories include especially [1], but also Makkai [6] and Harrington and Makkai [4]. The existence property is explained, for instance, in Hart [5], while monadically stable (or tree decomposable) theories are treated in Baldwin and Shelah [3]. As usual, we will assume that all models of a theory T as above are elementarily embedded in a very saturated model U.

Finally, let me thank the referee for suggesting several improvements.

*1 M and M** Let (M, E_0, E_1) be a structure with two equivalence relations E_0, E_1. Let E denote the equivalence relation generated by E_0 and E_1. We assume

(+) there is $h \in \omega$ such that, for all $a \in U$, the E-class of a contains at most h classes of either E_0 or E_1.

Notice that (+) implies that E is $\emptyset$-definable. Moreover, if (+) holds, then we can decompose M as $\bigcup_{i<h}(M_i^0 \cup M_i^1)$ where, for all $i < h$, $M_i^0 = \{a \in M: E(M,a)$ contains exactly $i+1$ classes of $E_0\}$ and $M_i^1 = \{a \in M - \bigcup_{j<h} M_j^0: E(M,a)$ contains exactly $i+1$ classes of $E_1\}$. Notice that the models of the theory of M are just the structures N of the form

$$N = \bigcup_{i<h} (N_i^0 \cup N_i^1)$$

where, for all $i < h$ and $e = 0$ or 1, N_i^e is elementarily equivalent to M_i^e; hence, they decompose in a uniform way as finite disjoint unions of structures whose E-classes contain the same (finite) number of classes of either E_0 or E_1. Then it is straightforward to see that we can assume without loss of generality that any E-class of M contains exactly h classes of E_0.

Let us build now a new structure (M^*, E_0, E_1, P) (where P is a 1-ary relation symbol).

(i) M^* includes M, and $M^* - M$ consists of a new element $x(X)$ for any class X of $E_0 \cap E_1$ in M, and of a new element $x(X_0, X_1)$ for any pair of classes X_0, X_1 of E_0, E_1 respectively in M such that $X_0 \cap X_1 = \emptyset$ but, for all $a_0 \in X_0$ and $a_1 \in X_1$, $M \models E(a_0, a_1)$.

(ii) $P(M^*) = M^* - M$.

(iii) For any $a \in M^*$ and $e = 0, 1$, put $a_e = a$ if $a \in M$, $a_e =$ an element of X if $a = x(X)$ for some X, and $a_e =$ an element of X_e if $a = x(X_0, X_1)$ for some X_0, X_1. Then, for all $a, b \in M^*$, define $M^* \models E_e(a, b)$ if and only if $M \models E_e(a_e, b_e)$ (notice that this definition does not depend on the choice of a_e and b_e).

It is easy to see that the following properties hold in M^*.

1. For any $e = 0, 1$, $E_e(M^{*2})$ is an equivalence relation and $E_e(M^{*2}) \cap M^2 = E_e(M^2)$.
2. $E(M^{*2}) \cap M^2 = E(M^2)$.
3. E_0 and E_1 permute in M^* (hence $E = R_2^0 = R_2^1 = R_2$, according to the notation of [9]).

4. For all $a \in M^*$, there is $a' \in M$ such that $M^* \models E(a,a')$, and, for all $a \in M$, $E(M^*,a) = (E(M,a))^*$.
5. For any class X of $E_0 \cap E_1$ in M, $(E_0 \cap E_1)(x(X), M^*) = X \cup \{x(X)\}$; for any pair of classes X_0, X_1 of E_0, E_1 respectively in M such that $X_0 \cap X_1 = \emptyset$ but, for all $a_0 \in X_0$ and $a_1 \in X_1$, $M \models E(a_0, a_1)$, $(E_0 \cap E_1)(x(X_0, X_1), M) = \{x(X_0, X_1)\}$.
6. M^* satisfies (+); furthermore, if every E-class in M contains exactly h E_0-classes, then the same is true in M^*.

We also point out that

(a) for all $a \in M^*$, there is a unique $x \in (E_0 \cap E_1)(M^*, a)$ satisfying $P(v)$;
(b) for all $a_0, a_1 \in M$ such that $E(a_0, a_1)$ holds in M (or, equivalently, in M^*) and $E_0(M, a_0) \cap E_1(M, a_1) = \emptyset$, there is a unique $x \in M^*$ such that $\models E_0(x, a_0) \wedge E_1(x, a_1)$;
(c) for all $a \in M^*$ with $|(E_0 \cap E_1)(M^*, x)| = 1$, there exist $a_0, a_1 \in M$ such that $M \models E(a_0, a_1)$ and $M^* \models E_0(a, a_0) \wedge E_1(a, a_1)$.

Now let $M = (M, E_0, E_1)$, $M' = (M', E_0, E_1)$ be structures satisfying (+). By proceeding as in [9], one can easily show that the following propositions hold.

P1 *$M \simeq M'$ iff $M^* \simeq M'^*$.*
P2 *Let $(\overline{M}, E_0, E_1, P)$ satisfy (+); put $M = (\neg P(\overline{M}), E_0 \cap M^2, E_1 \cap M^2)$, and assume that (a), (b), and (c) hold; then $\overline{M} \simeq M^*$.*
P3 *$M \equiv M'$ iff $M^* \equiv M'^*$.*

Therefore, if $T = Th(M)$ and $T^* = Th(M^*)$, then the models of T^* are exactly the structures M'^* with $M' \models T$. Furthermore,

- $M'|\simeq \to M'^*|\simeq$ defines a bijection between the sets of isomorphism types of models of T and T^*;
- let $\overline{M} = (\overline{M}, E_0, E_1, P)$ satisfy (+) and (a), (b), and (c), $\bar{T} = Th(\overline{M})$; if $M = \neg P(\overline{M})$ and $T = Th(M)$, then $\bar{T} = T^*$.

Therefore there is no loss of generality for our purposes in restricting ourselves to consider the classification problem for theories of structures (M, E_0, E_1, P) where E_0 and E_1 are permuting equivalence relations satisfying (+) (and even admitting exactly h E_0-classes in any E-class), and P is a 1-ary relation choosing an element in every $E_0 \cap E_1$-class of M. In fact, if M satisfies the further conditions (a), (b), and (c), then $M \simeq (\neg P(M))^*$.

Hence in the following sections T will always denote the theory of such a structure, and L its first-order language.

2 1-types and h-types We already saw in [9] that

Theorem 1 *T is superstable.*

The proof relies on a simple counting types argument and, more generally, on the analysis of the nonalgebraic 1-types over a model M of T we shall sketch below. We also recall that we gave in [9] an example of a non-ω-stable theory T of the type we are studying. Then let $M \models T$, $p \in S_1(M)$, p be nonalgebraic.

(A) There is $a \in M$ such that $E_1(v, a) \in p$; with no loss of generality, we can suppose $E_0(v, a) \in p$. Then, these conditions fully determine p; in particular $\neg P(v) \in p$.

(B) For all $m \in M$, $\neg E_1(v, m) \in p$; there exist $a_0, \ldots, a_{h-1} \in M$ pairwise equivalent in E_1 and inequivalent in E_0, and $\alpha_0, \ldots, \alpha_{h-1} \in \omega^* \cup \{|U|\}$ ($\omega^* = \omega - \{0\}$ from now on) such that

$$E_0(v, a_0) \in p$$

$$\text{“}|E_0(U, a_j) \cap E_1(U, v)| = \alpha_j\text{” is in } p \text{ for all } j < h.$$

(Notice that this condition can be expressed by a single formula when $\alpha_j \in \omega$, and by a denumberable set of formulas otherwise.) Then p is fully determined by these formulas (and hence by the sequences $a_0, \ldots, a_{h-1}$ and $\alpha_0, \ldots, \alpha_{h-1}$) together with $\neg E_1(v, m)$ for all $m \in M$ and either $P(v)$ or $\neg P(v)$.

(C) For all $m \in M$, $\neg E_0(v, m) \in p$. Then p is fully determined by this condition and by $p \restriction \varnothing$.

Hence we have to study $S_1(\varnothing)$. It is more convenient for our purposes to examine the h-types over $\varnothing$ of sequences $\bar{x} = (x_0, \ldots, x_{h-1}) \in U^h$ satisfying

$$\bigwedge_{i<j<h} (E_1(v_i, v_j) \wedge \neg E_0(v_i, v_j)).$$

Let $r \in S_h(\varnothing)$ contain this formula. Assign r the following invariants (1)–(3):

(1) The sequence $\tau(r) \in 2^h$ such that, for all $j < h$,

$$\tau(r)(j) = \begin{cases} 1 & \text{if } P(v_j) \in r, \\ 0 & \text{otherwise.} \end{cases}$$

(2) The sequence $\alpha(r) \in (\omega^* \cup \{|U|\})^h$ such that, for all $j < h$,

$$\alpha(r)(j) = |(E_0 \cap E_1)(U, v_j)|$$

(r can express $\alpha(r)(j)$ for all $j < h$; in fact, if there is $n \in \omega^*$ such that $\exists ! n w(E_0(v_j, w) \wedge E_1(v_j, w)) \in r$, then $\alpha(r)(j) = n$, while, if $\exists > n w(E_0(v_j, w) \wedge E_1(v_j, w)) \in r$ for all $n \in \omega$, then $\alpha(r)(j) = |U|$).

Let us introduce now the invariant (3) in a more informal way. Consider a realization $\bar{x}$ of r; let $(\alpha_0, \ldots, \alpha_{h-1})$ be any sequence of cardinal numbers, all greater than 0, and either finite or equalling $|U|$. We wish to estimate how many E_1-classes X in the E-class of $\bar{x}$ in U satisfy

$$|X \cap E_0(U, x_j)| = \alpha_j \text{ for every } j < h.$$

Clearly, if all the a_j's are finite, then r can express the power of the set of these classes. Assume now that there is some $j < h$ with α_j infinite. Then we can distinguish two cases: If there is $k \in \omega$ such that no E_1-class X in the E-class of $\bar{x}$ satisfies

$$|X \cap E_0(U, x_j)| = \alpha_j \text{ if } \alpha_j \text{ is finite,}$$

$$|X \cap E_0(U, x_j)| \text{ is finite and } {>}k \text{ otherwise,}$$

then r can express how many E_1-classes X in the E-class of $\bar{x}$ satisfy

$$|X \cap E_0(U, x_j)| = \alpha_j \text{ for every } j < h$$

(just because "$= |U|$" is equivalent to "$>k$"). Otherwise, r can only witness that a k playing the above role does not exist (and a compactness argument gives $|U|$-many E_1-classes X as required).

(3) The function $f(r)$ of $(\omega^* \cup \{|U|\})^h$ into $\omega \cup \{-1, |U|\}$ defined in the following way. Let $(\alpha_0, \dots, \alpha_{h-1})$ be any sequence in $(\omega^* \cup \{|U|\})^h$;
 (3.1) if α_j is finite for all $j < h$, then $f(r)(\alpha_0, \dots, \alpha_{h-1})$ is the power of the set of E_1-classes X in the E-class of any realization $\bar{x}$ of r such that
$$|X \cap E_0(U, x_j)| = \alpha_j \text{ for every } j < h;$$
 (3.2) if α_j is infinite for some $j < h$, but there is $k \in \omega$ such that, for all $\beta_0, \dots, \beta_{h-1} \in \omega^*$ with $\beta_j = \alpha_j$ if α_j is finite, $\beta_j > k$ otherwise, $f(r)(\beta_0, \dots, \beta_{h-1}) = 0$, then $f(r)(\alpha_0, \dots, \alpha_{h-1})$ is defined as in (3.1);
 (3.3) otherwise, $f(r)(\alpha_0, \dots, \alpha_{h-1}) = -1$.

Lemma 1 *Let $r, r' \in S_h(\emptyset)$ contain $\bigwedge_{i<j<h}(E_1(v_i, v_j) \wedge \neg E_0(v_i, v_j))$. Then $r = r'$ if and only if $(\tau(r), \alpha(r), f(r)) = (\tau(r'), \alpha(r'), f(r'))$.*

Proof: ($\Rightarrow$) is trivial.

($\Leftarrow$) Suppose $(\tau(r), \alpha(r), f(r)) = (\tau(r'), \alpha(r'), f(r'))$. Let $\bar{x} \models r$, $\bar{x}' \models r'$, we claim that there exists an isomorphism from $E(U, x_0)$ onto $E(U, x_0')$ sending $\bar{x}$ into $\bar{x}'$. As this isomorphism can be easily extended to an automorphism of U, it follows that $\bar{x} \equiv \bar{x}'$ and hence $r = r'$.

Let $\alpha_0, \dots, \alpha_{h-1} \in \omega^* \cup \{|U|\}$. If $(\alpha_0, \dots, \alpha_{h-1})$ satisfies the conditions of (3.1) or (3.2), then both $E(U, x_0)$ and $E(U, x_0')$ contain exactly $f(r)(\alpha_0, \dots, \alpha_{h-1}) = f(r')(\alpha_0, \dots, \alpha_{h-1})$ E_1-classes X satisfying, for all $j < h$,
$$|X \cap E_0(U, x_j)| = \alpha_j, \quad |X \cap E_0(U, x_j')| = \alpha_j$$
respectively.

Otherwise (3.3) holds, and $f(r)(\alpha_0, \dots, \alpha_{h-1}) = f(r')(\alpha_0, \dots, \alpha_{h-1}) = -1$. Also, for any $k \in \omega$, there exist $\lambda_0, \dots, \lambda_{h-1} \in \omega^*$ such that $\lambda_j = \alpha_j$ when α_j is finite, $\lambda_j > k$ otherwise, and
$$f(r)(\lambda_0, \dots, \lambda_{h-1}) = f(r')(\lambda_0, \dots, \lambda_{h-1}) \neq 0$$
(and both r and r' can recognize that this is the case). Consider the following formulas over $\bar{x}$
$$\bigwedge_{j<h} E_0(v_j, x_j) \wedge \bigwedge_{i<j<h} E_1(v_i, v_j),$$
$$\bigwedge_{\alpha_j \text{ finite}} \exists ! \alpha_j z (E_0(v_j, z) \wedge E_1(v_j, z)),$$
$$\bigwedge_{\alpha_j \text{ infinite}} \exists > nz (E_0(v_j, z) \wedge E_1(v_j, z)) \text{ for all } n \in \omega,$$
$$\bigwedge_{j<h} P(v_j) \text{ (for instance).}$$
An easy compactness argument shows that the set of these formulas is consistent; furthermore it can be enlarged in a unique way to a type over $\bar{x}$; this type admits $|U|$-many pairwise E_1-inequivalent realizations in U. Hence there are

$|U|$ E_1-classes X in $E(U, x_0)$ such that, for all $j < h$, $|X \cap E_0(U, x_j)|$ equals α_j if α_j is finite, and $|U|$ otherwise.

The same holds for $\bar{x}'$. Since $\tau(r) = \tau(r')$ and $\alpha(r) = \alpha(r')$, it is easy at this point to build the required isomorphism between $E(U, x_0)$ and $E(U, x'_0)$.

Now let M be a model of T, and consider nonalgebraic types $r \in S_h(M)$ containing $\bigwedge_{i<j<h}(E_1(v_i, v_j) \wedge \neg E_0(v_i, v_j))$. One could classify them in a way similar to the one we followed before for 1-types. We omit here the details, and we only emphasize the particular case when, for all $m \in M$, $E_0(v_0, m) \notin r$. Notice that these formulas and $r \restriction \emptyset$ isolate r.

Let us discuss now the connection between 1-types and h-types of the above kind. First consider types over $\emptyset$ and notice that every 1-type p over $\emptyset$ can be enlarged to a type $r \in S_h(\emptyset)$ with $\bigwedge_{i<j<h}(E_1(v_i, v_j) \wedge \neg E_0(v_i, v_j)) \in r$ just taking a sequence x in U such that $x_0 \models p$ and $\models \bigwedge_{i<j<h}(E_1(x_i, x_j) \wedge \neg E_0(x_i, x_j))$, and putting $r = tp(\bar{x}/\emptyset)$. Of course this extension r is not uniquely determined. However, if $r, r' \in S_h(\emptyset)$ contain $\bigwedge_{i<j<h}(E_1(v_i, v_j) \wedge \neg E_0(v_i, v_j))$ and $\bigwedge_{0<j<h} P(v_j)$, and $P(v_0) \in r$ if and only if $P(v_0) \in r'$, and p, p' denote $tp(x_0/\emptyset)$, $tp(x'_0/\emptyset)$ respectively for $\bar{x} \models r$ and $\bar{x}' \models r'$, then

$$p = p'$$

if and only if there is $s \in S_h$ such that $s(0) = 0$ and $r'(v_0, \ldots, v_{h-1}) = r(v_{s(0)}, \ldots, v_{s(h-1)})$.

($\Leftarrow$) is trivial.

($\Rightarrow$) Let $x_0 \models p$, enlarge x_0 to a sequence $\bar{x}$ realizing r, and to a sequence $\bar{x}'$ realizing r'. Define $s \in S_h$ by putting, for all $i, j < h$,

$$s(i) = j \text{ iff } \models E_0(x'_i, x_j) \wedge E_1(x'_i, x_j).$$

Then s is as claimed.

Now let us look at nonalgebraic types over a model M of T. Here we only notice that, if $p \in S_1(M)$ contains $\neg E_0(v_0, m)$ for all $m \in M$, then, just proceeding as above, one can enlarge p to a type $r \in S_h(M)$. This extension is not unique, but, if r, r' are such extensions, then $r \not\perp r'$; furthermore, both r and r' are not orthogonal to p (this fact will be shown in the next section—Lemma 5).

Corollary *The following propositions are equivalent*:

(i) *T is ω-stable*;

(ii) *$S_1(\emptyset)$ is countable*;

(iii) $\{r \in S_h(\emptyset) : \bigwedge_{i<j<h}(E_1(v_i, v_j) \wedge \neg E_0(v_i, v_j)) \in r\}$ *is countable.*

Proof: (i) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (ii). Every type $p \in S_1(\emptyset)$ can be extended in at most finitely many ways to a type r as in (iii).

(ii) $\Rightarrow$ (i) follows from the analysis we gave of the 1-types over an arbitrary model of T.

Corollary *T satisfies Vaught's Conjecture* (*namely T has either $2^{\aleph_0}$ or at most $\aleph_0$ nonisomorphic countable models*).

Proof: If T is ω-stable, then it suffices to refer to the Shelah theorem stating that all ω-stable theories satisfy Vaught's conjecture (Shelah, Harrington, Makkai [8]).

If T is not ω-stable, then $|S_1(\emptyset)| = 2^{\aleph_0}$, hence T needs $2^{\aleph_0}$ nonisomorphic countable models to realize all the 1-types over $\emptyset$.

Notice that, consequently, Vaught's Conjecture holds even for an arbitrary theory of two equivalence relations satisfying (+). In order to simplify the exposition in the forthcoming sections, let me introduce the following conventions:

- From now on, h-type will always mean h-type containing the formula $\bigwedge_{i<j<h}(E_1(v_i, v_j) \wedge \neg E_0(v_i, v_j))$.
- For every h-type r (over $\emptyset$, or over a model M of T), we will denote by $p(r)$ the 1-type of x_0 where $\bar{x}$ is any realization of r.
- For every h-type r, we will put $f(r) = f(r \upharpoonright \emptyset)$.

3 *The Shelah first level*

Lemma 2 *Let $M \models T$, $A \supseteq M$, $p' \in S_1(A)$, $p = p' \upharpoonright M$. Then p' forks over M if and only if one of the following conditions holds*:
(i) *p satisfies* (A) *and there is $x \in A$ such that $v = x \in p'$*;
(ii) *p satisfies* (B) *and there is $x \in A$ such that $E_1(v,x)$ is in p'*;
(iii) *p satisfies* (C) *and there is $x \in A$ such that $E(v,x)$ is in p'.*

The proof is just the same as Lemma 5 in [9]. Notice also that the forking extensions of h-types over M can be characterized in a similar way. In particular, if $M \models T$, $A \supseteq M$, $r' \in S_h(A)$, $r = r' \upharpoonright M$, and r contains $\neg E_0(v_0, m)$ for all $m \in M$, then r' forks over M if and only if there is $x \in A$ such that $E(v_0, x) \in r'$.

Lemma 3 *Let M be a model of T, p be a nonalgebraic* 1*-type over M. Then $RU(p) \leq 3$, p is regular and p is trivial.*

Proof: Let p satisfy (A). For every $M' > M$, $p|M'$ is the only extension of p over M' that does not represent $v = w$. This implies that every forking extension p' of p over M' is algebraic, hence $RU(p) = 1$ and p is regular. Furthermore, $\not\perp_M$ equals $=$ in $p(U)$, so that, for every $I \subseteq p(U)$, independence is equivalent to pairwise independence in I. It follows that p is trivial.

Assume now that (B) holds. For every $M' > M$, $p|M'$ is the only extension of p over M' that does not represent $E_1(v, w)$. This implies that any forking extension has $RU \leq 1$; hence $RU(p) \leq 2$. Also, if x realizes any forking extension p' of p over M' and $y \models p|M'$, then $tp(y/M' \cup x)$ does not represent $E_1(v, w)$ (otherwise $p|M'$ represents this formula, too), hence $y \perp_{M'} x$ and $p' \perp^a p|M'$. Thus p is regular. Finally, M equals E_1 in $p(U)$, and again, for all $I \subseteq p(U)$, independence and pairwise independence are equivalent in I. Therefore, p is trivial.

Suppose now that p satisfies (C). If $M' > M$, then $p|M'$ is the only extension of p over M' that does not contain $E(v,m)$ for any $m \in M'$. If p' is any forking extension of p over M', then for some $m \in M'$ $E(v,m)$ and hence $E_0(v,m)$ is in p. Hence $RU(p') \leq 2$ and $RU(p) \leq 3$. Also, if $x \models p'$ and $y \models p|M'$, then $y \perp_{M'} x$, hence $p' \perp^a p|M'$. Then p is regular. Finally, $\not\perp_M$ equals E in $p(U)$, so for sets of realizations of p, independence is equivalent to pairwise independence, and again p is trivial.

We have implicitly shown that, for every p as above, the pregeometry $(p(U), cl)$ associated to p (where, for all $S \subseteq p(U)$, $cl(S) = \{y \in p(U): y \not\downarrow_M S\}$) is degenerate.

Notice also that an argument similar to the previous one proves that every nonalgebraic h-type r over M is regular and trivial.

The regularity of types over a model of T and even over any subset of U could be proved also as a consequence of the following result.

Lemma 4 *T is monadically stable (tree decomposable in the sense of Baldwin and Shelah* [3]).

Proof: We will show that T is 3-tree decomposable. Let M be any model of T, $\lambda = |M|$. Define a tree $I \subseteq \lambda^{\leq 3}$ and a set $\{(M(\eta), N(\eta)) : \eta \in I\}$ of pairs of models of T in the following way:

(0) $M_{\langle\rangle} = M$, $N_{\langle\rangle} = N$ is any countable elementary submodel of M.

(1) Let $\{X(\eta) : \eta < \mu\}$ be a list of E-classes in M (for a suitable $\mu \leq \lambda$). Then μ is just the set of elements of I of length 1. Moreover, for every $\eta < \mu$, put $M(\eta) = N \cup X(\eta)$. We claim that $M(\eta)$ is an elementary submodel of M (and hence a model of T). It suffices to show that $M(\eta)$ satisfies the Tarski–Vaught criterion. Suppose $\bar{a} \in M(\eta)$ and $M \models \exists v \varphi(v, \bar{a})$. Let $b \in M$ satisfy $\varphi(v, \bar{a})$. If $b \in M(\eta)$, then we are done. If there is $x \in N$ such that $\models E_1(b, x)$ (without loss of generality $\models E_0(b, x)$), then the $E_0 \cap E_1$-class of x in N is infinite, and we can build an automorphism of M fixing $\bar{a}$ and sending b in N; so, again, we are done. Suppose now that, for all $x \in N$, $\models \neg E_1(b, x)$ but, for some $x \in N$, $\models E_0(b, x)$; let $x_0 = x, x_1, \ldots, x_{h-1}$ be pairwise E_1-equivalent and E_0-inequivalent elements of N, $b_0 = b, \ldots, b_{h-1}$ satisfy $E_0(v_j, x_j)$ for all $j < h$ and $E_1(v_i, v_j)$ for all $i < j < h$. Then $tp(\bar{b}/N)$ is defined by the previous formulas and $|(E_0 \cap E_1)(U, b_j)|$ for $j < h$. Hence there is $k \in \omega$ such that

$$\{E_0(v_j, x_j) : j < h\} \cup \{E_1(v_i, v_j) : i < j < h\}$$
$$\cup \{\exists > k w(E_0(v_i, w) \wedge E_1(v_i, w))\} \vdash \varphi(v, \bar{a}),$$

hence there exists $b' \in N$ satisfying $\varphi(v, \bar{a})$. Finally, assume that $\models \neg E(b, x)$ for all $x \in M(\eta)$. Then $tp(x/M(\eta))$ is fully determined by $tp(b/\emptyset) \cup \{\neg E(v, x): x \in M(\eta)\}$; hence there are $\vartheta(v) \in tp(b/\emptyset)$ and a finite $A \subseteq M(\eta)$ such that

$$\{\vartheta(v)\} \cup \{\neg E(v, x) : x \in A\} \vdash \varphi(v, \bar{a}).$$

Hence there does exist some $b' \in N$ such that $\models \varphi(b', \bar{a})$. This concludes the proof of the claim. Hence we can put, for all $\eta < \mu$, $N(\eta) =$ some countable elementary submodel of $M(\eta)$ such that $N < N(\eta)$ and $N(\eta) \cap X(\eta) \neq \emptyset$.

(2) For every $\eta < \mu$ let $\{Y(\nu) : \nu < \nu(\eta)\}$ be a list of E_1-classes in $X(\eta)$ (for a suitable $\nu(\eta) \leq \mu$). Then the elements of I of length 2 are just the $\eta \frown \nu$ for $\eta < \mu$, $\nu < \nu(\eta)$. Put $\tau = \eta \frown \nu$, and define $M(\tau) = N(\eta) \cup Y(\nu)$ (as before one can see that this is an elementary submodel of $M(\eta)$), $N(\tau) =$ a countable elementary substructure of $M(\tau)$ satisfying $N(\eta) < N(\tau)$ and $N(\tau) \cap Y(\nu) \neq \emptyset$.

(3) Finally, for every $\tau = \eta \frown \nu$ as above, let $\{c(\rho) : \rho < \rho(\mu, \nu)\}$ be a list of elements of $Y(\nu) - N(\tau)$; let $\delta = \tau \frown \rho$, and put $M(\delta) = N(\delta) = N(\tau) \cup \{c(\rho)\}$ (this is easily seen to be a submodel of $M(\tau)$); the δ's are the elements of I of length 3.

Clearly $M = \bigcup_{\eta \in I} N(\eta)$, and, for $\eta, \rho \in I$ with $\eta \subseteq \rho$, $N(\eta) < N(\rho) < M(\rho) < M(\eta)$. Moreover it is straightforward to define, for every $\eta \in I$ with length ≤ 2, a set C of conditions (in the sense of [3]), and a map $\sigma = \sigma(\eta)$ from $\{E_0, E_1, P\}$ to C such that $M(\eta)$ is the free union with respect to σ over $N(\eta)$ of the $M(\eta \frown \rho)$'s for $\eta \frown \rho \in I$, and M is the free union with respect to σ over $N(\eta)$ of the $M(\eta \frown \rho)$'s ($\eta \frown \rho$ as above) together with the $M(\tau) \cup N(\eta)$'s where $\tau \in I$, τ is not an initial segment of η but every proper initial segment of τ is a proper initial segment of η.

It follows that M is decomposed by $\{(M(\eta), N(\eta)) : \eta \in I\}$. Hence T is 3-tree decomposable.

We want to characterize now the orthogonality relation among types. Recall that for trivial stationary types non-orthogonality is equivalent to not almost orthogonality (see [2], for instance).

Lemma 5 *Let $M \models T$, $p \in S_1(M)$ be nonalgebraic, $r \in S_h(M)$ satisfy $p(r) = p$. Then $p \not\perp r$.*

Proof: As p and r are trivial, it suffices to show $p \overset{a}{\not\perp} r$. And this is trivial, too.

In particular, if $r, r' \in S_h(M)$, and $p(r) = p(r') = p$, then $r \not\perp r'$ (as r, r', p are regular, and $\not\perp$ is an equivalence relation among regular types).

Lemma 6 *Let $M \models T$, $r, r' \in S_h(M)$ contain $\neg E(v, m)$ for all $m \in M$. Then the following propositions are equivalent*:

(i) $r \not\perp r'$;

(ii) $r \overset{a}{\not\perp} r'$;

(iii) *there is $s \in S_h$ such that $f(r) = f(s(r'))$ (where $s(r')$ denotes $tp(x_{s(0)}, \ldots, x_{s(h-1)}/M)$ where $(x_0, \ldots, x_{h-1})$ is any realization of r').*

Proof: Clearly it suffices to show the equivalence between (ii) and (iii).

(ii) ⇒ (iii). Let $\bar{x} \models r$, $\bar{x}' \models r'$ satisfy $\bar{x} \underset{M}{\not\downarrow} \bar{x}'$. Hence, for any $i < h$, there is a unique $j < h$ such that $\models E_0(x_i, x'_j)$. Define $s(i) = j$. Then $s \in S_h$ and $f(r) = f(s(r'))$.

(iii) ⇒ (ii). Let $\bar{x} \models r$; $s(r')$ is fully determined by the formulas $\neg E(v_0, m)$ for $m \in M$, and by $(\tau(s(r')), \alpha(s(r')), f(s(r')))$. Furthermore, $f(r)(\alpha(s(r')) = f(s(r'))(\alpha(s(r'))) \neq 0$, hence $s(r')$ in realized in $E(U, x_0)$, and $r \overset{a}{\not\perp} r'$.

Lemma 7 *Let $M \models T$, $p, p' \in S_1(M)$ be nonalgebraic. Then the following propositions are equivalent*:

(i) $p \not\perp p'$;

(ii) $p \overset{a}{\not\perp} p'$;

(iii) *p and p' satisfy one of the conditions* (1)–(3) *below.*

(1) *There is an M such that $E_0(v, a) \wedge E_1(v, a) \in p \cap p'$ (so $p = p'$).*

(2) *There are $a_0, \ldots, a_{h-1} \in M$ pairwise equivalent in E_1 and inequivalent in E_0, and $\alpha_0, \ldots, \alpha_{h-1} \in \omega^* \cup \{|U|\}$ such that*

$$\neg E_1(v, m) \in p \cap p' \textit{ for all } m \in M,$$

$$E_0(v, a_i) \in p,\ E_0(v, a_j) \in p' \textit{ for some } i, j < h,$$

$$\text{“}|E_1(U, v) \cap E_0(U, a_j)| = \alpha_j\text{” is in } p \cap p' \textit{ for all } j < h.$$

(3) *For all* $m \in M$, $\neg E(v,m) \in p \cap p'$; *if* $r, r' \in S_h(M)$ *and* $p(r) = p$, $p(r') = p'$, *then* $r \overset{a}{\not\perp} r'$.

Notice that, owing to Lemmas 5 and 6, the choice of r, r' is inessential; $r \overset{a}{\not\perp} r'$ is characterized in Lemma 6.

Proof: The equivalence of (i) and (ii) follows from the triviality of p and p', while the equivalence of (ii) and (iii) is a straightforward consequence of Lemma 2.

Let us concentrate our attention now on the case T ω-stable. SR-types (where SR = strongly regular) play a basic role under this hypothesis. Hence let us see which 1-types over a model M of an ω-stable T are SR.

Lemma 8 *Let T be ω-stable, $M \models T$, p be a nonalgebraic* 1*-type over M satisfying* (A) *or* (B). *Then p is SR.*

Proof: First suppose that p satisfies (A). Let $x \models p$, $y \in M(x) - M$; then $y \underset{M}{\not\perp} x$ and $y = x$. Hence, p is SR via the formula $v = v$. Assume now (B). Again, if $x \models p$ and $y \in M(x) - M$, then $y \underset{M}{\not\perp} x$, and so $\models E_1(y,x)$. Hence $y \models p$, provided y satisfies the conjunction of $E_0(v, a_0)$ and $P(v)$ if $\models P(x)$, $\neg P(v)$ otherwise. Then p is SR.

It remains to examine the case (C).

Example Let T be the theory of two equivalence relations E_0 and E_1 such that

(i) $E_1 \subseteq E_0$;
(ii) E_0 has infinitely many classes;
(iii) for all $n \in \omega^*$, every E_0-class contains exactly one E_1-class of power n.

Then, for all $x, y \in U$, $x \equiv y$ iff $|E_1(U,x)| = |E_1(U,y)|$ and $\models P(x) \leftrightarrow P(y)$. In particular, $|S_1(\varnothing)| = \aleph_0$, and T is ω-stable. Let $M \models T$, p be the 1-type over M defined by $\neg E_0(v,m)$ for all $m \in M$ and $\exists^\infty w E_1(v,w)$ (besides $P(v)$, for instance). Then p satisfies (C), and p is regular. However p is not SR; in fact, consider the 1-type q over M defined by $\neg E_0(v,m)$ for all $m \in M$, $\exists! n w E_1(v,w)$ for some $n \in \omega^*$, and $P(v)$. First of all, q is SR; for, if $x \models q$ and $y \in M(x) - M$, then $y \underset{M}{\not\perp} x$, so $\models E_0(x,y)$ and consequently $y \models q$ provided $\models \exists! n w E_1(y,w) \wedge P(y)$. Furthermore $q \leq_{RK} p$, but $q \not\geq_{RK} p$, then p is not RK-minimal, and hence p cannot be SR. Notice that $p \not\perp q$.

Then, let T be ω-stable, $M \models T$, $p \in S_1(M)$ satisfy (C). We wish to find under which conditions p is SR.

Let $r \in S_h(M)$ be such that $p(r) = p$. Assume for simplicity $\bigwedge_{0<i<h} P(v_i) \in r$, and put $S(r) = \{j < h : \alpha(r)(j) \text{ is finite}\}$. Notice that, for every $x_0 \models p$, one can build a sequence $\bar{x} = (x_0, \dots, x_{h-1}) \in M(x_0)$ realizing r; moreover, for any $y_0 \in M(x_0) - M$, $y_0 \underset{M}{\not\perp} x_0$, hence $\models E(x_0, y_0)$.

Case 1. There is a $k \in \omega$ such that, for all $\beta_0, \dots, \beta_{h-1} \in \omega^*$ satisfying $\beta_j = \alpha(r)(j)$ when $j \in S(r)$ and $\beta_j > k$ when $j \in h - S(r)$, $f(r)(\beta_0, \dots, \beta_{h-1}) = 0$.

Notice that this is the case when $S(r) = h$. Then p is SR. In fact, for any $s \in S_h$, define a formula $\varphi(s)(\bar{w})$ in the following way:

(a) If $f(s(r)) = f(r)$, then put $\varphi(s)(\bar{w}) : w_0 = w_0$.
(b) If $f(s(r)) \neq f(r)$ and there is $\bar{\lambda} = (\lambda_0, \ldots, \lambda_{h-1}) \in (\omega^*)^h$ such that $f(s(r))(\bar{\lambda}) \neq f(r)(\bar{\lambda})$, then fix such a sequence $\bar{\lambda}$; if $f(r)(\bar{\lambda})$ is finite, then put

$$\varphi(s)(\bar{w}) : f(tp(\bar{w}/\varnothing))(\bar{\lambda}) = f(r)(\bar{\lambda})$$

(this can be written by a suitable first-order formula); otherwise $f(r)(\bar{\lambda}) = |U|$, but then $f(s(r))(\bar{\lambda})$ is finite, and we set

$$\varphi(s)(\bar{w}) : f(tp(\bar{w}/\varnothing))(\bar{\lambda}) \neq f(s(r))(\bar{\lambda}).$$

(c) If $f(s(r)) \neq f(r)$ but (b) does not hold, then there is $\bar{\lambda} = (\lambda_0, \ldots, \lambda_{h-1}) \in (\omega^* \cup \{|U|\})^h$ such that $\lambda_j = |U|$ for some $j < h$ and $f(r)(\bar{\lambda}) \neq f(s(r))(\bar{\lambda})$; choose such a sequence $\bar{\lambda}$ and notice that, as $f(r)$ and $f(s(r))$ are identically equal on $(\omega^*)^h$, for all $\mu_0, \ldots, \mu_{h-1} \in \omega^* \cup \{|U|\}$,

$$f(r)(\bar{\mu}) = -1 \quad \text{iff } f(s(r))(\bar{\mu}) = -1.$$

In particular both $f(r)(\bar{\lambda})$ and $f(s(r))(\bar{\lambda})$ are different from -1, and hence either $f(r)(\bar{\lambda})$ or $f(s(r))(\bar{\lambda})$ is in ω; then we can define $\varphi(s)(\bar{w})$ in a way similar to (b).

Finally, let $\varphi_0(\bar{w})$ be the formula

$$\bigwedge_{i<j<h} (E_1(w_i, w_j) \wedge \neg E_0(w_i, w_j)) \wedge \bigwedge_{0<j<h} P(w_j) \wedge \bigwedge_{j \in S(r)}$$

$$\exists ! \alpha(r)(j) z (E_0(w_j, z) \wedge E_1(w_j, z)) \wedge \bigwedge_{j \in h - S(r)} \exists > kz$$

$$(E_0(w_j, z) \wedge E_1(w_j, z)) \wedge \bigwedge_{s \in S_h} \varphi(s)(\bar{w})$$

and let $\varphi(\bar{w})$ be the conjunction of $\varphi_0(\bar{w})$ with $P(w_0)$ or $\neg P(w_0)$ provided that $P(v) \in p$ or $\neg P(v) \in p$. Then the formula

$$\vartheta(v) : \exists \bar{w}(v = w_0 \wedge \varphi(\bar{w}))$$

makes p SR. In fact $\vartheta(v) \in p$; moreover let $x_0 \models p$, $\bar{x} = (x_0, \ldots, x_{h-1})$ be a realization of r in $M(x_0)$; then, for every $y_0 \in M(x_0) - M$ satisfying $\vartheta(v)$, there exists $\bar{y} = (y_0, \ldots, y_{h-1}) \in M(x_0)$ such that $\models \bigwedge_{i<j<h} (E_1(y_i, y_j) \wedge \neg E_0(y_i, y_j))$ and

$$\tau(tp(\bar{y}/\varnothing)) = \tau(r), \alpha(tp(\bar{y}/\varnothing)) = \alpha(r).$$

Furthermore there is $s \in S_h$ such that, for all $j < h$, $\models E_0(y_j, x_{s(j)})$. It follows $f(tp(\bar{y}/\varnothing)) = f(s(r))$. As $\models \varphi(s)(\bar{y})$, it must be $f(s(r)) = f(r)$. Then $\bar{y} \models r \restriction \varnothing$, so that $\bar{y} \models r$ and $y_0 \models p$.

Case 2. For all $k \in \omega$, there are $\alpha_0(k), \ldots, \alpha_{h-1}(k) \in \omega^*$ such that $\alpha_j(k) = \alpha(r)(j)$ for any $j \in S(r)$, $\alpha_j(k) > k$ for any $j \in h - S(r)$, and $f(r)(\alpha_0(k), \ldots, \alpha_{h-1}(k)) \neq 0$.

The following conditions fully determine a type $r(k) \in S_h(M)$:

(i) for all $m \in M$, $\neg E(v_0, m)$;
(ii) for all $j < h$, $\alpha(r(k))(j) = \alpha_j(k)$;
(iii) $f(r(k)) = f(r)$;
(iv) finally, $\tau(r(k))$ is defined in some arbitrary way, for instance by setting $\tau(r(k)) = \tau(r)$, as is possible if we assume $k \geq 2$.

Let $p(k) = p(r(k))$. Then $S(r(k) = h$, so that $p(k)$ is SR. However $p(k) \leq_{RK} p$ and $p(k) \not\sim_{RK} p$, and consequently p is not *RK*-minimal, and hence p cannot be SR. In fact it suffices to prove $r(k) \leq_{RK} r$, $r \not\leq_{RK} r(k)$ (since $p(k) \sim_{RK} r(k)$ and $p \sim_{RK} r$).

- $r(k) \leq_{RK} r$: for every $\bar{x} \models r$, there is $\bar{y} \in M(\bar{x})$ such that $\bar{y} \models r(k)$ and even $\models \bigwedge_{j<h} E_0(y_j, x_j)$.
- $r \not\leq_{RK} r(k)$: let $\bar{y} \models r(k)$, suppose towards a contradiction that $M(\bar{y})$ contains a sequence $\bar{x}$ realizing r. Then $\bar{x} \underset{M}{\not\downarrow} \bar{y}$, and hence $\models E(x_0, y_0)$; moreover there is $s \in S_h$ such that, for any $j < h$, $\models E_0(x_j, y_{s(j)})$; in particular $f(r) = f(s(r(k))) = f(s(r))$.

  Then $tp(\bar{x}/M \cup \bar{y})$ is fully determined by these formulas:

  1. $\bigwedge_{i<j<h} E_1(v_i, v_j) \wedge \bigwedge_{j<h} E_0(v_j, y_{s(j)})$,
  2. the formula expressing $\tau(r)$,
  3. the formulas expressing $\alpha(r)$, namely:

  $$\exists!\alpha(r)(j)z(E_0(v_j, z) \wedge E_1(v_j, z)) \text{ for all } j \in S(r),$$

  $$\exists > nz(E_0(v_j, z) \wedge E_1(v_j, z)) \text{ for all } j \in h - S(r) \text{ and } n \in \omega.$$

  Moreover $tp(\bar{x}/M \cup \bar{y})$ is isolated, and hence is determined by a finite set $\Delta(\bar{v})$ of the previous formulas. Let $t \in \omega$ ($t \geq 2$) be such that, for any formula

  $$\exists > nz(E_0(v_j, z) \wedge E_1(v_j, z)) \quad (j \in h - S(r), n \in \omega)$$

  occurring in $\Delta(\bar{v})$, $t > n$. We know that $M(\bar{y})$ contains a sequence $\bar{x}'$ realizing $r(t)$ (and even $\models \bigwedge_{j<h} E_0(x'_j, x_j)$). In particular $\models \Delta(\bar{x}')$; however $\bar{x}'$ does not realize r, a contradiction. Therefore $r \not\leq_{RK} r(k)$.

We can summarize the previous results by means of the following:

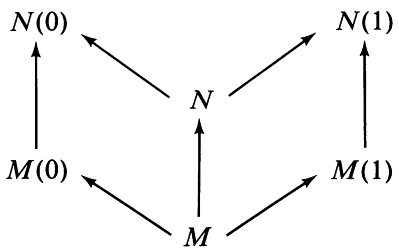
Proposition *Let T be ω-stable, $M \models T$, $p \in S_1(M)$ satisfy* (C), *$r \in S_h(M)$ be such that $p(r) = p$, and $S(r) = \{j < h : \alpha(r)(j) \in \omega\}$. Then p is* SR *if and only if there is $k \in \omega$ such that, for all $\beta_0, \ldots, \beta_{h-1} \in \omega^*$ with $\beta_j = \alpha(r)(j)$ when $j \in S(r)$ and $\beta_j > k$ when $j \in h - S(r)$, $f(r)(\bar{\beta}) = 0$.*

4 The Shelah second level This section is devoted to proving that every theory T satisfying the assumptions of Section 1 is classifiable according to Shelah. We already noticed that T is superstable, so we have to show that T is presentable (has NDOP), shallow, and satisfies the existence property (has NOTOP). On the other hand, T is also monadically stable, and every monadically stable theory satisfies both NDOP and NOTOP: this fact seems to be folklore, but, as far as I known, unpublished. So let met include here a proof for the sake of completeness.

Theorem 2 *Let T be any monadically stable theory. Then T is presentable and has the existence property.*

Proof: First let me show that, if $M, M(0), M(1)$ are models of T such that M is an elementary submodel of both $M(0)$ and $M(1)$ and $M(0) \underset{M}{\downarrow} M(1)$, then $M(0) \cup M(1)$ is a model of T. First assume that M, $M(0)$, $M(1)$ are a-models of T. Let M' be the a-model of T a-prime over $M(0) \cup M(1)$, we claim that $M' = M(0) \cup M(1)$. Suppose toward a contradiction that there exists $x \in M' - (M(0) \cup M(1))$; then, for every $e = 0$ or 1, $x \underset{M(e)}{\not\downarrow} M(1-e)$, so there is a sequence $\bar{y}(1-e)$ in $M(1-e) - M$ so that $x \underset{M(e)}{\not\downarrow} \bar{y}(1-e)$. As T is monadically stable, there is $y(1-e) \in M(1-e) - M$ so that $x \underset{M(e)}{\not\downarrow} y(1-e)$ (see [3], 4.2.6). Again using monadic stability, we obtain $x \underset{M}{\not\downarrow} y(1-e)$ for any $e = 0,1$ (as $\underset{M(e)}{\not\downarrow} \subseteq \underset{M}{\not\downarrow}$, see [3], 4.2.12) and finally, owing to the transitivity of $\underset{M}{\not\downarrow}$ for monadically stable theories, $y(0) \underset{M}{\not\downarrow} y(1)$, a contradiction.

Assume now that M, $M(0)$, and $M(1)$ are arbitrary models of T. Build an independent diagram



where N, $N(0)$, $N(1)$ are a-models of T, and the arrows denote elementary embeddings (see [6] and [4]). Then $N(0) \underset{N}{\not\downarrow} N(1)$, and hence $N(0) \cup N(1)$ is an a-model of T. Furthermore, for every formula $\varphi(\bar{v}, \bar{w})$ and for every $\bar{a} \in M(0) \cup M(1)$, if there is $\bar{b} \in N(0) \cup N(1)$ satisfying $\models\varphi(\bar{b}, \bar{a})$, then there is $\bar{a}' \in M(0) \cup M(1)$ such that $\models\varphi(\bar{a}', \bar{a})$; then $M(0) \cup M(1)$ is an elementary submodel of $N(0) \cup N(1)$, in particular $M(0) \cup M(1)$ is a model of T.

At this point the existence property is trivial; in fact, for all models M, $M(0)$, $M(1)$ of T satisfying $M < M(0)$, $M < M(1)$, and $M(0) \underset{M}{\downarrow} M(1)$, $M(0) \cup M(1)$ is a model of T, clearly prime and atomic over itself.

So we have only to show that T is presentable; hence consider a-models M, $M(0)$, $M(1)$ of T satisfying the previous assumptions; let p be any nonalgebraic 1-type over $M(0) \cup M(1)$ (= the a-model a-prime over itself), then we need to prove that either $p \not\perp M(0)$ or $p \not\perp M(1)$. If there is $e = 0,1$ such that p does not fork over $M(e)$, then we are done. Otherwise, let x realize p, then both $x \underset{M(0)}{\not\downarrow} M(1)$ and $x \underset{M(1)}{\not\downarrow} M(0)$, and we get a contradiction by proceeding just as above.

Theorem 3 *Let T be a theory satisfying the assumptions of Section* 1. *Then T is presentable, shallow with depth* ≤ 3, *and satisfies the existence property. In particular, T is classifiable in the Shelah sense.*

Proof: Owing to the previous remarks, we have only to show $Dp(t) \leq 3$. Let M be an a-model of T, and $p \in S_1(M)$ be nonalgebraic, hence regular.

First assume that p satisfies (A). Let $x \models p$, $M' = M[x]$ = the a-model of T a-prime over $M \cup \{x\}$, $y \in M' - M$. Then $y \not\downarrow_M x$ and hence $y = x$. It follows $M' = M \cup \{x\}$, and hence every nonalgebraic 1-type over M' does not fork over M. Hence $Dp(p) = 0$.

Suppose now that (B) holds. Let x, M', y be as above. This time $y \not\downarrow_M x$ implies $\models E_1(x, y)$, so that $M' = M \cup E_1(M', x)$. Then let q' be a nonalgebraic 1-type over M'. If there is $a' \in M' - M$ such that $E_0(v, a') \wedge E_1(v, a') \in q'$, then $q' \perp M$ and $Dp(q') = 0$. In the other cases, $q' \not\perp M$. Hence $Dp(p) \leq 1$.

Finally, assume (C). Now $M' = M \cup E(M', x)$. Let q' be a nonalgebraic 1-type over M'. If q' satisfies (A) for some $a' \in M' - M$, then $q' \perp M$ and $Dp(q') = 0$. If q' satisfies (B) for some sequence $a'_0, \ldots, a'_{h-1}$ of elements in $M' - M$, then $q' \perp M$ and $Dp(q') \leq 1$. In the other cases $q \not\perp M$. Consequently $Dp(p) \leq 2$.

In conclusion, T is shallow and $Dp(T) \leq 3$ (we already gave implicit examples of theories T of depth 1, 2, and 3 in [9]).

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