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# Provable Fixed Points in $I\Delta_0 + \Omega_1$

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**Abstract** It is shown that of the results of de Jongh–Montagna (Provable fixed points, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 1988) on provable fixed points in PA at least the positive part can be obtained for the system of bounded arithmetic of Wilkie and Paris (On the scheme of induction for bounded arithmetic formulas, *Annals of Pure and Applied Logic*, 1987). The methods used include use of a weakening of the scheme of Sigma-completeness due to A. Visser (this volume) which is valid for bounded arithmetic. The results imply that the results on shortenings of proofs due to Parikh (Existence and feasibility in arithmetic, *The Journal of Symbolic Logic*, 1971) apply to bounded arithmetic.

***1 Introduction***[1] This work should be considered as part of the general investigation into the arithmetical system $I\Delta_0 + \Omega_1$. We will present a refinement to $I\Delta_0 + \Omega_1$ of a result stated in de Jongh and Montagna [4] on witness comparison formulas having only provable fixed points in PA.

Briefly, let us introduce the arithmetical system and some of its properties: $I\Delta_0 + \Omega_1$ (cf. Paris and Wilkie [7]) is a set of axioms expressing the elementary arithmetic properties of the basic symbols 0, ′, +, *, ≤ (in the following we will refer to the obvious first-order language containing these symbols as $S$) together with the bounded induction schema $I\Delta_0$ (defined in $S$):

$$\forall x, z(\varphi(x,0) \wedge \forall y \leq z.(\varphi(x,y) \rightarrow \varphi(x,y')) \rightarrow \forall y \leq z\varphi(x,z)) \quad (\varphi \in \Delta_0)$$

plus the $S$-sentence $\Omega_1$ expressing $\forall x \exists y. \omega_1(x) = y$, where $\omega_1(x) := x^{|x|}$ and $|-|$ is the length function for the binary representation of $x$.

We note that by the following result of Verbrugge [10]:

If NP $\neq$ CO-NP then

$$\nvdash_{I\Delta_0+\Omega_1} \forall b, c(\exists a(\mathrm{Prf}(a,c) \wedge \forall z \leq a\, \neg\mathrm{Prf}(z,b)) \rightarrow \mathrm{Pr}(\ulcorner \exists a \mathrm{Prf}(a,c) \wedge \forall z \leq a \neg\mathrm{Prf}(z,b) \urcorner))$$

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it seems highly unlikely that the principle of $\Sigma_1$-*completeness*, i.e.,

$$\varphi \rightarrow \mathrm{Pr}(\ulcorner\varphi\urcorner) \quad \text{for } \varphi \in \Sigma_1,$$

is provable in $I\Delta_0 + \Omega_1$. However, it can be shown that $I\Delta_0 + \Omega_1$ proves *Švejdar's principle* (cf. Švejdar [9]): i.e.,

$$\vdash_{I\Delta_0+\Omega_1}\mathrm{Pr}(\ulcorner\varphi\urcorner) \rightarrow \mathrm{Pr}(\ulcorner\exists a(\mathrm{Prf}(a,\ulcorner\psi\urcorner) \wedge \forall z \leq a\ \neg\mathrm{Prf}(z,\ulcorner\varphi\urcorner)) \rightarrow \psi\urcorner) \qquad \text{(for all } \varphi, \psi)$$

(cf. Verbrugge [10]) and *Visser's principle* (cf. Visser [11])

$$\vdash_{I\Delta_0+\Omega_1}\mathrm{Pr}(\ulcorner C(S) \rightarrow s'\urcorner) \rightarrow \mathrm{Pr}(\ulcorner s'\urcorner)$$

where $C(S) = \bigwedge\{s \rightarrow \mathrm{Pr}(\ulcorner s\urcorner) : s \in S.\}$, $S$ is a finite set of $\Sigma_1$-sentences, and $s'$ is a $\Sigma_1$-sentence.

In [7], Buss [1], and [10] ample motivation for the general study of $I\Delta_0 + \Omega_1$ is given. Therefore, we will turn our attention here directly to the more specific aim of this paper.

In Parikh [6] it is shown that for each primitive recursive function $g$, there is a $\Sigma_1$-sentence $s$ such that $\vdash_{\mathrm{PA}} s$ and

$$g(\mu z.\mathrm{Prf}_{\mathrm{PA}}(z,\ulcorner\mathrm{Pr}_{\mathrm{PA}}(\ulcorner s\urcorner)\urcorner)) < \mu z.\mathrm{Prf}_{\mathrm{PA}}(z,\ulcorner s\urcorner). \tag{$*$}$$

In [4] Parikh's result is analyzed in the modal context $R$ (cf. Guaspari and Solovay [5]) when $g$ is the identity function; a simpler proof is presented, based on the fact that ($*$) has *only provable fixed points*. Furthermore, a characterization is given for pairs of modal formulas $B(p)$ and $C(p)$ such that for each arithmetical interpretation $*$, if $\vdash_{\mathrm{PA}} p^* \leftrightarrow (\Box B(p) \prec \Box C(p))^*$, then $\vdash_{\mathrm{PA}} p^*$: $\Box B(p) \prec \Box C(p)$ has only provable fixed points in PA. In de Jongh and Montagna [3] the result is extended to arbitrary $g$ which are provably recursive in PA.

Our aim is to refine the positive part of the proof of [4], the part in which it is shown that the formulas specified do indeed have only provable fixed points in PA, to a weaker modal system in which the $\Sigma$-*completeness axiom* (i.e., the corresponding modal version of the $\Sigma_1$-completeness principle) does not hold.

In Section 3, it is shown that the modal version (V) of Visser's principle playing the role of a weak version of $\Sigma$-completeness suffices to obtain the refined theorem we are looking for.

What is provable in the weak modal system including (V) is clearly provable in $I\Delta_0 + \Omega_1$ under every arithmetical interpretation; therefore, it follows that PA has no witness comparison formulas having only provable fixed points which the system $I\Delta_0 + \Omega_1$ does not already have.

Based on the result obtained in Section 3, in Section 4 we present the independence between (V) and the modal version (Šv) of Švejdar's principle (see Definitions 2.1(c) and (b)). In particular we give a counterexample to show that (Šv) does not imply (V), which gives an insight to understand why Švejdar's schema cannot play much of a role in the study of formulas having only provable fixed points.

In the appendix we give some proofs, mainly due to Visser [11], of modal principles derivable from Visser's principle.

*2 Modal systems and Kripke semantics* In this section we will briefly introduce the modal systems that we are going to work with, together with the associated Kripke semantics.

Formulas of our system are built up from propositional atoms using the Boolean connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\top$, $\bot$, a unary modality $\Box$, and binary witness comparisons $<$, $\leq$, where $<$ and $\leq$ are applicable only to those formulas having $\Box$ as principal connective. The following definition will introduce the list of modal systems.

**Definition 2.1**
(a) $B^-$(*Basic System*) is the modal system $L$ (Prl in Smoryński [8] (including its rules: modus ponens and necessitation) to which the following *order axioms* are added (see de Jongh [2]):
(01) $\Box A \rightarrow (\Box A \leq \Box B \vee \Box B < \Box A)$
(02) $\Box A \leq \Box B \rightarrow \Box A$
(03) $\Box A \leq \Box B \wedge \Box B \leq \Box C \rightarrow \Box A \leq \Box C$
(04) $\Box A < \Box B \leftrightarrow (\Box A \leq \Box B \wedge \neg(\Box B \leq \Box A))$
(b) $Z^-$ (cf. Švejdar [9]) is the system $B^-$ plus Švejdar's schema:

**(Šv)** $\Box A \rightarrow \Box(\Box B \leq \Box A \rightarrow B)$ for all formulas $A, B$

(c) $BV^-$ is the system $B^-$ plus Visser's schema:

**(V)** $\Box(C(S) \rightarrow s') \rightarrow \Box s'$

where $C(S) = \bigwedge\{s \rightarrow \Box s : s \in S\}$, $S$ is a finite set of $\Sigma$-formulas, and $s'$ is a $\Sigma$-formula. A $\Sigma$-formula is in this context a formula in the closure of the set of $\Box$-formulas, $<$-formulas, and $\leq$-formulas under $\wedge$ and $\vee$.
(d) $B, BV, Z$ are respectively the system $B^-$, $BV^-$, and $Z^-$ with the rule $\Box E$, i.e. $\Box A/A$ added.

Let $A(p)$ be some formula of $B$ of the form $\Box B(p) \leq \Box C(p)$. As in [4] we take $BC^-$, $BVC^-$, and $ZC^-$ to be the systems $B^-$, $BV^-$, and $Z^-$, respectively, plus the axiom $c \leftrightarrow A(c)$ (analogous notation is used for the systems $B$, $BV$, and $Z$). Since a different system is defined for different choice of $A$ it would be more appropriate to name the systems $BC(A)^-$, $BVC(A)^-$, and $ZC(A)^-$. But, as it will always be clear in the sequel which formula $A$ is intended, we will refrain from doing so, in order not to unnecessarily complicate the notation.

**Definition 2.2** A *model for* $B^-$ is a finite, tree-ordered Kripke-model for $L$ in which witness comparison formulas are treated as atomic formulas and in which every instance of (01)–(04) is forced at each node.

**Definition 2.3** *Models for* $BV^-$, $Z^-$ are Kripke-models for $B^-$ where, respectively, (V), (Šv) is forced at each node.

It is appropriate to remark that, just as is pointed out in [10] for the system $Z^-$, also for $BV^-$ the forcing for witness comparison formulas in $BV^-$-Kripke-models is *not persistent*, i.e. it does not necessary hold that if $j \Vdash \Box A \leq \Box B$ (resp. $j \Vdash \Box A < \Box B$) and $jRk$ then $k \Vdash \Box A \leq \Box B$ (resp. $k \Vdash \Box A < \Box B$).

No modal-completeness theorem or even a general extension lemma has been established for $BV$ (for $Z$, Švejdar did establish these in [9]).

***3 Witness comparison formulas having only provable fixed points in BV***
Theorem 3.3 of [4] reads:

*If $B(p)$ and $C(p)$ are L-formulas (i.e. do not contain witness comparisons), possibly containing propositional variables other than p, then $A(p) \equiv \Box B(p) \preceq \Box C(p)$ has only provable fixed points in R iff*
(i) $\vdash_L B(T)$
(ii) $\vdash_L \Box^+(\Box B(\bot) \to \Box C(\bot)) \to \Box^{k+1}\bot$, *for some k (where $\Box^+ D$ abbreviates $D \wedge \Box D$).*

Our aim is to obtain a characterization for a witness comparison formula to have only provable fixed points in *BV*. The result presented in this section constitutes a refinement of the theorem proved by de Jongh and Montagna; the proof that we present is syntactical and based on a different approach characterized by the proof of the following theorem:

**Theorem 3.1** *Let $B(p)$ and $C(p)$ be L-formulas. If*
(i) $\vdash_L B(T)$
(ii) $\vdash_L \Box^+(\Box B(\bot) \to \Box C(\bot)) \to \Box^{k+1}\bot$, *for some k, then $A(p) \equiv \Box B(p) \preceq \Box C(p)$ has only provable fixed points in BV.*

Some preparatory lemmas are needed. In the following we assume that (i) and (ii) of Theorem 3.1 hold, the systems $BC^-$, $BVC^-$, and $BVC$ refer to the $A(p)$ of this theorem. Some results already proved by Visser (cf. [4]) for his principle and used in the proof of the following lemmas are given in the appendix.

**Lemma 3.2** $\vdash_{BC^-} \Box^+ \neg c \to \Box^{k+1}\bot$.

*Proof:*
1. $\vdash_{B^-} \Box\neg c \to \Box(c \leftrightarrow \bot)$
$\to \Box^+((\Box B(c) \leftrightarrow \Box B(\bot)) \wedge (\Box C(c) \leftrightarrow \Box C(\bot)))$
$\to (\Box^+(\Box B(c) \to \Box C(c)) \to \Box^{k+1}\bot)$ (by (ii) and the Substitution Lemma (cf. Smoryński [8]))
2. $\vdash_{BC^-} \Box^+\neg c \to \Box^+(\Box B(c) \to \Box C(c))$ (by obvious properties of $\preceq$)
3. $\vdash_{BC^-} \Box^+\neg c \to \Box^{k+1}\bot$ (by 1 and 2).

**Lemma 3.3** $\vdash_L \Box c \to \Box B(c)$.

*Proof:*
1. $\vdash_L c \to B(T)$ (by (i))
2. $\vdash_L \Box c \to \Box B(T)$
3. $\vdash_L \Box c \to \Box(c \leftrightarrow T)$
4. $\vdash_L \Box c \to \Box B(c)$ (by 2 and 3).

**Lemma 3.4** $\vdash_L \Box^+ c \to \Box^+ B(c)$.

*Proof:*
1. $\vdash_L c \to B(T)$ (by (i))
2. $\vdash_L \Box^+ c \to \Box^+ B(T)$
3. $\vdash_L \Box^+ c \to \Box^+(c \leftrightarrow T)$
$\to (\Box^+ B(c) \leftrightarrow \Box^+ B(T))$
4. $\vdash_L \Box^+ c \to \Box^+ B(c)$ (by 2 and 3).

**Lemma 3.5** $\vdash_L \Box^{k+1}\bot \to (\Box C(\bot) \to B(\bot))$.

*Proof:* We claim that, if $\vdash_L \Box^+(\Box B \to \Box C) \to \Box^{k-1}\bot$, then $\vdash_L \Box^{k+1}\bot \to (\Box C \to B)$, where $B, C$ are arbitrary $L$-formulas. For suppose not, then a model $\mathbf{M}$ exists such that $\mathbf{M} \models \Box^+(\Box B \to \Box C) \to \Box^{k+1}\bot$ and $w \Vdash \Box^{k+1}\bot \wedge \Box C$, $\mathbf{w} \nVdash \mathbf{B}$, for some node $w$ in $\mathbf{M}$. Take the submodel of $\mathbf{M}$ generated by $w$ and add a tail of nodes below $w$ of such a length that the new model gets a root $x$ of level greater than or equal to $k + 1$ (end nodes are counted as having level 0). Clearly none of the nodes added below $w$ can force $\Box B$ but all of them force $\Box^+(\Box B \to \Box C)$. By hypothesis, $x \Vdash \Box^{k+1}\bot$ and this gives a contradiction, which proves our claim

By the claim and (ii) it follows that: $\vdash_L \Box^{k+1}\bot \to (\Box C(\bot) \to B(\bot))$.

**Lemma 3.6** $\vdash_{BC}\text{-}\Box^+\neg c \to \Box^+ B(c)$.

*Proof:*

1. $\vdash_L \Box^+\neg c \to (\Box C(c) \leftrightarrow \Box C(\bot)) \wedge (B(c) \leftrightarrow B(\bot))$
2. $\vdash_{BC}\text{-}\Box^+\neg c \to (\Box C(c) \to B(c))$ (by Lemmas 3.2 and 3.5)
3. $\vdash_{BC}\text{-}\neg c \to (\Box B(c) \to \Box C(c))$ (by obvious properties of $\leq$)
4. $\vdash_{BC}\text{-}\Box^+\neg c \to (\Box B(c) \to B(c))$ (by 2 and 3)
5. $\vdash_{BC}\text{-}\Box^+\neg c \to \Box(\Box B(c) \to B(c))$
   $\to \Box B(c)$ (by formalized Löb)
6. $\to B(c)$ (by 4).

**Lemma 3.7** $\vdash_{BV}\text{-}\Box\Box A \vee \Box\Box B \to \Box(\Box^+(\Box A < \Box B) \vee \Box^+(\Box B \leq \Box A))$.

*Proof:*

1. $\Box\Box A \vee \Box\Box B \to \Box(\Box A < \Box B \vee \Box B \leq \Box A)$
   $\to \Box((\Box A < \Box B \to \Box(\Box A < \Box B) \wedge (\Box B \leq \Box A \to \Box(\Box B \leq \Box A))) \to (\Box^+(\Box A < \Box B) \vee \Box^+(\Box B \leq \Box A)))$
   $\to \Box(\Box^+(\Box A < \Box B) \vee \Box^+(\Box B \leq \Box A))$ (by (V)).

**Corollary 3.8** $\vdash_{BV}\text{-}\Box A \vee \Box B \to \Box(\Box A < \Box B \to \Box(\Box A < \Box B))$.

*Proof:* Trivial.

We are now ready to prove Theorem 3.1:

*Proof of Theorem 3.1:*

1. $\vdash_{BVC}\text{-}\Box\Box B(c) \to \Box(\Box^+(\Box B(c) \leq \Box C(c)) \vee \Box^+(\Box C(c) < \Box B(c))$
   (by Lemma 3.7)
2. $\vdash_{BVC}\text{-}\Box\Box B(c) \to \Box(\Box^+ c \vee \Box^+\neg c)$ (by the fixed point equation of $c$)
3. $\vdash_{BVC}\text{-}\Box\Box B(c) \to \Box B(c)$ (by Lemmas 3.4 and 3.6)
4. $\vdash_{BVC}\text{-}\Box\Box B(c)$ (by formalized Löb)
5. $\vdash_{BVC}\text{-}\Box(\Box^+ c \vee \Box^+\neg c)$ (by 2 and 4)
6. $\vdash_{BVC}\text{-}\Box(\Box^{k+1} c \vee \Box^{k+1}\bot)$ (by Lemma 3.2)
7. $\vdash_{BVC}\text{-}\Box^{k+2} c$
8. $\vdash_{BVC} c$ (by $\Box E$).

The refinement that we were looking for is an immediate consequence of Theorem 3.1:

**Theorem 3.9** *Let $B(p)$ and $C(p)$ be L-formulas; then $A(p) \equiv \Box B(p) \leq \Box C(p)$ has only provable fixed points in $BV$ iff*
(i) $\vdash_L B(T)$
(ii) $\vdash_L \Box^+(\Box B(\bot) \to \Box C(\bot)) \to \Box^{k+1}\bot$, *for some* $k$.

*Proof:* ($\Rightarrow$) If $c$ is a fixed point for $A(p)$ then $\vdash_{BVC} c$, therefore $\vdash_{RC} c$ and by Lemma 2.3 in [4] $\vdash_{R^-} \Box^+(c \leftrightarrow A(c)) \to \Box^{k+1}c$ for some $k$. Now apply Theorem 3.3 in de Jongh–Montagna [4].
($\Leftarrow$) by Theorem 3.1.

By Theorems 3.10 and 3.3 (cf. [4]) it follows that the formulas of the form $A(p) \equiv \Box B(p) \leq \Box C(p)$ having only provable fixed points in $R$ are exactly the formulas having only provable fixed points in $BV$. In other words, to obtain the formulas having only provable fixed points we do not need the strong $\Sigma$-completeness schema (i.e., $A \to \Box A$, for every $\Sigma$-formula $A$) but we can replace it by the weaker (V).

Although Theorem 3.10 is formulated with *iff* one should note that, unlike with $R$ and PA, $A(p) \equiv \Box B(p) \leq \Box C(p)$ having only provable fixed points in $I\Delta_0 + \Omega_1$ for all arithmetical interpretations does not imply that $A(p)$ has only provable fixed points in $BV$, since arithmetic completeness even of $L$ is unknown for $I\Delta_0 + \Omega_1$ (see [10]). At the present, Theorem 3.10 does imply that each formula of $R$ having only provable fixed points in PA has only provable fixed points in $I\Delta_0 + \Omega_1$ when arithmetical interpretations are restricted to sentences. The restriction to sentences is essential; otherwise Visser's principle loses its validity (see [11]).

## *4 Independence of Visser's and Švejdar's schemas*

As already announced in the introduction, it can be shown that $I\Delta_0 + \Omega_1$ proves Švejdar's principle. Because the principle appears as a weak version of the $\Sigma$-completeness axiom it may be of some interest to study its relations with Visser's principle: in this section we will prove the independence of the two principles.

First of all we show that Švejdar's schema does not imply Visser's schema, i.e. $\nvdash_Z$ (V). To prove that, consider the formula $\Box^3 p < \Box^2 p$ having only provable fixed points in $R$, as proved in [4]. By Theorem 3.10 it follows that this formula has only provable fixed points in $BV$. On the other hand,[2] note that $\Box^3 p < \Box^2 p$ cannot have only provable fixed points in $Z$, because by Švejdar's essential reflexivity interpretation of $\Box A < \Box B$ as "there exists a proof of $A$ using axioms with smaller Gödel numbers than in any proof of $B$" (cf. [9]) that would mean that for the fixed point $c$ in PA, $\Box^2 c$ would have a proof in PA using axioms with smaller Gödel numbers than any proof of $\Box c$ would use. This is impossible because being a provable $\Sigma$-sentence, $\Box c$ would not need any but the axioms of $Q$ and we could take those as the zero base of our interpretation. This proves our claim.

At this point it may be of interest to remark that the formula $\Box^2 p < \Box p$ has only provable fixed points in $Z$.

The following argument is due to Visser: in $BC^-$ it is provable that $\Box^2 c \to \Box(\Box c \leq \Box^2 c \vee \Box^2 c < \Box c)$. Thus, in $ZC^-$, $\Box^2 c \to \Box c$ is provable, from which with Löb in $ZC$, immediately $c$ follows. Under the same arithmetical interpretation used in the previous argument, the result is not very surprising: it is well known that there are theorems provable in PA and not in $Q$. From these observations we can see that Švejdar's schema can by itself hardly be useful in studying formulas having only provable fixed points in $I\Delta_0 + \Omega_1$. Recall also that in the proof of Theorem 3.10, the schema (Šv) is not used.

To obtain our second claim, that Visser's schema does not imply Švejdar's schema (i.e., $\nvdash_{BV}$ (Šv)), it is enough to exhibit a countermodel of $BV$ to the formula $\Box p \to \Box(\Box q < \Box p \to q)$ (i.e., an instance of (Šv) where $p$ and $q$ are propositional variables).[3]

Let $A$ be the formula $\Box p \to \Box(\Box q < \Box p \to q)$ and consider the $A$-sound model $\langle\{1,2,3,\dots\}, R, \Vdash\rangle$ shown in Figure 1, where the forcing relation is restricted to subformulas of $A$, and where $E$ and $F$ stand for $\Box p \leq \Box q$ and $\Box q < \Box p$ respectively.

From the forcing relation indicated in the figure, note that: 2 does not force $p$ or $q$; 4 does not force $p$ or $E \to \Box E$, but does force $F \to \Box F$; for $kR5$ and $k = 5$, $k$ does not force $p, q, E, F$, but does force $E \to \Box E$ and $F \to \Box F$. In particular, note that 4 does not satisfy $\Box p \to \Box(\Box q < \Box p \to q)$.

Observe that the role of Node 1 is crucial to obtain a model forcing all the instances of Visser's principle; consider the formula $\Box\neg p$ and suppose that Node 1 did not exist. It is easy to check that $6 \nVdash \Box((E \to \Box E) \to \Box\neg p) \to \Box\Box\neg p$.

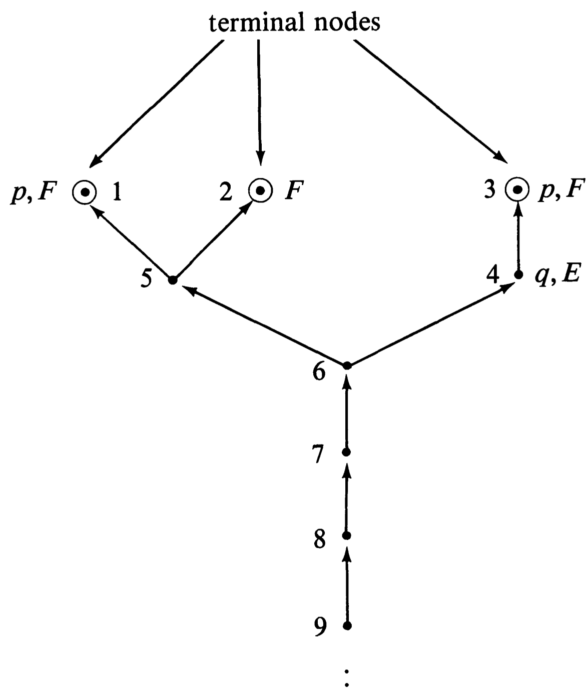


Figure 1.

We claim that under a suitable forcing extension given to the model, every instantiation of Visser's principle holds on the model. Before giving the procedure to define the appropriate forcing relation, let us fix some notation and definition that will be used in the sequel. We write $P$ to denote the set of all propositional variables except $p$ and $q$; $S^0$ for $\{p, q, \Box p, \Box q, \Box p \leq \Box q, \Box q < \Box p, \Box q \leq \Box p, \Box p < \Box q\}$; $S^{2m+1}$ to denote the closure of $S^{2m} \cup P$ under the propositional connectives and $\Box$ (obviously $P$ is effective only when $m = 0$); $S^{2m+2}$ for $S^{2m+1} \cup \{\Box A \leq \Box B, \Box A < \Box B \mid \Box A, \Box B \in S^{2m+1}\}$.

**Definition 4.1** Let $k, k'$ nodes of $\langle\{1,2,3,\dots\}, R, \Vdash\rangle$; we write:
$\Box A <_k \Box B$ iff $\exists k'$ (($k'Rk$ or $k' = k$) and $k' \Vdash \Box A$ and $k' \nVdash \Box B$)
$\Box A \leq_k \Box B$ iff $k \Vdash \Box A$ and $\forall k'$ (if ($k'Rk$ or $k' = k$) and $k' \Vdash \Box B$ then $k' \Vdash \Box A$).

Here is the procedure to construct the forcing relation:
*stage 0:* for all $r \in P$ fix $k \Vdash r$ iff $k \Vdash p$ for all nodes $k$
*stage 2m + 1:* automatically and uniquely define a forcing relation for all members of the closure $S^{2m+1}$
*stage 2m + 2:* call (as in de Jongh [2]) a boxed formula $\Box A$ *old* if $\Box A \in S^{2m}$ and *new* if $\Box A \in S^{2m+1} \backslash S^{2m}$. To give an extension of the forcing relation to $S^{2m+2}$, it is enough to define the forcing on witness comparison formulas $\Box A \leq \Box B$ and $\Box A < \Box B$ (belonging to $S^{2m+2}$) for $\Box A$ and $\Box B$ both new, $\Box A$ old and $\Box B$ new, and for $\Box A$ new and $\Box B$ old. Before giving the way to construct the forcing let us recall two definitions occurring in de Jongh [2]:

(i) $k \Vdash \Box A < \Box B$ iff $\Box A <_k \Box B$ or, $\Box A \leq_k \Box B$ and $\Box A$ old, $\Box B$ new
(ii) $k \Vdash \Box A \leq \Box B$ iff $\Box A <_k \Box B$ or, $\Box A \leq_k \Box B$ and $\Box B$ new.

We are now ready to present the procedure, to repeat for all nodes $k$. Here it is:

If $k \in \{1,2,3,4,5\}$ and $5 \Vdash \Box A$ and $5 \nVdash \Box B$, then let $k \Vdash \Box A < \Box B$ and $k \Vdash \Box A \leq \Box B$.
Otherwise, fix the forcing on $\Box A < \Box B$, $\Box B < \Box A$, $\Box A \leq \Box B$ and $\Box B \leq \Box A$ as defined in (i) and (ii), respectively.

Apply the procedure repeatedly (i.e. for all $m \in \mathbb{N}$) so as to cover all formulas, and call the resulting model **M**.

Note that Nodes 1 and 3 satisfy the same formulas since they are always treated alike by the construction.

**Claim 1** $\forall s \in \Sigma.\ 5 \Vdash s \Rightarrow k \Vdash s$ *where* $k \in \{1,2,4\}$

*Proof:* Suppose $s \in \Sigma$ and $5 \Vdash s$; by cases:
$s \equiv \Box B$: by the previous observation Nodes 1 and 3 force the same formulas, therefore the claim;
$s \equiv \Box B < \Box C$: by stage $2m + 2$ of construction and definition (i);
$s \equiv \Box B \leq \Box C$: by definition (ii) on stage $2m + 2$ of construction;
$s \equiv$ "Boolean combination of $\Sigma$-formulas": by the previous cases.

Using Claim 1 and Definition 4.1, it is easy to check that stage $2m + 2$ excludes the existence of two boxed formulas $\Box A, \Box B$ for which $\Box A < \Box B$ and $\Box B \leq \Box A$ are both forced at Node 4.

**Claim 2** *$5 \Vdash C(S)$ for all finite sets $S$ of $\Sigma$-formulas.*

*Proof:* Straightforward from Claim 1.

**Claim 3** (*Persistency property*) *Let $\Box A, \Box B$ be two boxed formulas such that at least one of them is new at some stage $m \geq 1$;*
*if ($k \Vdash \Box A < \Box B$ and $kRk'$) then $k' \Vdash \Box A < \Box B$ and*
*if ($k \Vdash \Box A \leq \Box B$ and $kRk'$) then $k' \Vdash \Box A \leq \Box B$.*

*Proof:* Immediate from the forcing procedure and the following consequences of Definition 4.1:
if ($\Box A <_k \Box B$ and $kRk'$) then $\Box A <_{k'} \Box B$ and
if ($\Box A \leq_k \Box B$ and $kRk'$) then $\Box A \leq_{k'} \Box B$.

Note that the only witness comparison formulas that do not satisfy the persistency property are $E$ and $F$ (see the definition of forcing at Nodes 3 and 4).

**Claim 4** *All instances of Visser's schema are forced in each node of* **M**.

*Proof:* Obviously Nodes 1, 2, 3 satisfy the claim. Moreover, notice that Visser's principle is always satisfied at level 1 in any Kripke model since each $C(S)$ is always satisfied at terminal nodes. Therefore, 4 and 5 satisfy the claim. By induction we check the tail of points $k$:

$k = 6$: suppose there exist $C(S)$ and $s'$ such that $6 \nVdash \Box s'$ and $\forall k$ (if $6Rk$ then $k \Vdash C(S) \rightarrow s'$); it follows that $\exists h$ ($6Rh$ and $h \nVdash s'$ and $h \nVdash C(S)$); but $k \Vdash C(S)$ for $k \in \{1,2,3,5\}$ therefore $h$ must be 4. By Claim 1 we get a contradiction.

$k + 1$: (with $k + 1 > 6$) assume the claim holding for all $h$ such that $k + 1Rh$ and suppose there exist $C(S)$ and $s'$ such that $k + 1 \nVdash \Box s'$ and $\forall h$ (if $k + 1Rh$ then $h \Vdash C(S) \rightarrow s'$); it follows that $\exists h$ ($k + 1Rh$ and $h \nVdash s'$ and $h \nVdash C(S)$); this node must be $k$ since, by induction hypothesis, every instance of Visser's schema holds at $k$, so $k \Vdash \Box s'$. Therefore $k \nVdash C(S)$, i.e. for some $s \in S$, $k \Vdash s$ but $k \nVdash \Box s$. By cases:

- $s \equiv \Box B : k \Vdash \Box s$, a contradiction;
- $s \equiv \Box B < \Box C : k \Vdash \Box B$ and $B$ can be neither $p$ nor $q$ since $\Box p$ and $\Box q$ are not forced at any node $kR6$. Therefore by Claim 3, the forcing on witness comparison formulas must be persistent and this gives a contradiction.
- $s \equiv \Box B \leq \Box C$: similar to the previous case;
- $s \equiv$ "Boolean combination of $\Sigma$-formulas": by the previous cases.

To show that **M** is a model for $BV$ it suffices to prove the following

**Claim 5** *For all formulas $A$,*
*if $\mathbf{M} \models \Box A$ then $\mathbf{M} \models A$,*

*Proof:* Trivial.

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## NOTES

1. Prerequisites: the reader should be familiar with Smoryński [8]; knowledge of [4] also will be helpful.
2. The argument was suggested to the author by F. Montagna.
3. Observe that $\vdash_{BV} - \Box A \to \Box\Box(\Box B < \Box A \to B)$, for all formulas $A, B$. The proof is an immediate consequence of Lemma 3.7.

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***Appendix: Some theorems proved by Visser's principle*** In [11] the following theorems, proved using the principle (V), are pointed out:

**(V1)** $\Box \bigvee S \to \Box \bigvee S^+$
**(V2)** $\Box(\Box A \to \bigvee S) \wedge \Box(\bigvee S^+ \to A) \to \Box A$
**(V3)** $\Box(C(S) \to (A \to s')) \to \Box A \to \Box s'$
**(V4)** $\Box(C(S) \to (\Box s' \to s')) \to \Box s'$,

where $S$ is a finite set of $S$-formulas, $C(S) = \bigwedge \{s \to \Box s : s \in S\}$, $S^+ = \{s \wedge \Box s : s \in S\}$ and $s'$ a $\Sigma$-formula.

We will give the proof of them in the modal system $BV^-$:

(V1):
1. $\Box \bigvee S \to \Box(C(S) \to \bigvee \Box^+ S)$
2. $\Box(C(S) \to \bigvee \Box^+ S) \to \Box(\bigvee \Box^+ S)$ (by (V))
3. $\Box \bigvee S \to \Box(\bigvee \Box^+ S)$ (by 1 and 2)

(V2):
1. $\Box(\Box A \to \bigvee S) \to \Box(\Box\Box A \to \Box \bigvee S)$
$\to \Box(\Box\Box A \to \Box(\bigvee \Box^+ S))$ ((by (V1))
2. $\Box(\bigvee \Box^+ S \to A) \to \Box(\Box \bigvee \Box^+ S \to \Box A)$
3. $\Box(\Box A \to \bigvee S) \wedge \Box(\bigvee \Box^+ S \to A) \to \Box(\Box\Box A \to \Box A)$ (by 1 and 2)
$\to \Box\Box A$ (by formalized Löb)
$\to \Box S$
$\to \Box(\bigvee \Box^+ S)$ (by (V1))
$\to \Box A$

(V3):
1. $\Box(C(S) \to (A \to s')) \to \Box(A \to (C(S) \to s'))$
$\to \Box A \to \Box(C(S) \to s')$
$\to \Box A \to \Box s'$ (by (V))

(V4):
1. $\Box(C(S) \to (\Box s' \to s')) \to \Box(\Box(C(S) \to (\Box s' \to s')))$
$\to \Box(\Box\Box s' \to \Box s')$ (by (V3))
$\to \Box\Box s'$ (by formalized Löb)
$\to \Box(C(S) \to \Box s')$
$\to \Box(C(S) \to s')$
$\to \Box s'$ (by (V)).