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Book Review

David Lewis, *Parts of Classes* (with an appendix by John P. Burgess, A. P. Hazen, and David Lewis). Basil Blackwell, Oxford, 1991. 165 pages.

In *Parts of Classes* David Lewis starts from one simple suggestion, namely that a subclass is a part of a class. Using this, he provides us with a detailed account of classes and sets, as well as an interesting synthesis of set theory with arithmetic. It should come as no surprise that he manages to pack an immense amount into in a mere hundred and twenty pages, and that he does this in his usual accessible and lucid style, without unnecessary technicalities. In addition there is a fascinating appendix in which some crucial results due to Burgess and Hazen are discussed.

I shall first provide a brief exposition of *Parts of Classes*. Later I shall make a couple critical remarks. Let us begin, then, with Lewis' notion of a class, as that which has members. He contrasts a class with an individual—that which has no members. Given this contrast, not only are there the proper classes which are not sets, but there is also a set, the null set, which is not a class, because it has no members. With this clarification of the notion of a class, Lewis is able to state his First Thesis, namely that one class is part of another iff the first is a subclass of the second (p. 4). He then argues for the strengthening of this to his Main Thesis, namely that the parts of a class are all and only its subclasses (p. 7).

One of the things we need to think about is just how plausible is the First Thesis and hence the Main Thesis. But even if we have some misgivings about it, it is surely of considerable interest to see the development of set theory based on the Main Thesis. And the first issue which Lewis tackles here is the null set. He resists the suggestion that there is no such thing as the null set, largely because he endorses the orthodox set-theoretic construction of mathematical entities. Within that orthodoxy there are good reasons, which Lewis expounds, for relying on the null set. He suggests that *any* individual will do for the null set (p. 13). In fact he takes the null set to be the individual which is the fusion of all individuals, but he grants that this is an arbitrary choice. As he acknowledges, an arbitrary choice of null set hardly differs from not making a choice. And Lewis could have adopted an explicitly structuralist position, in which no choice is made, and the phrase 'the null set' is permissibly interpreted as referring to any individual whatever. In that case he would adopt van Fraassen's supervaluation

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procedure, and say that what is true of the null set is just what is true on all permissible interpretations of the phrase 'null set', and what is false is whatever is false on all permissible interpretations. That would have the advantage that intuitively infelicitous assertions such as that every cat is part of the null set would turn out to be neither true nor false, while on Lewis' account they are true. But that is a minor point. The more interesting point to note is that when it comes to the null set Lewis has already accepted something very like Structuralism by acknowledging the arbitrariness of his choice. And the issue of whether we should develop an account of sets and classes along structuralist lines is an important topic in Chapter 2.

The chief consequence of the Main Thesis is that every class is the fusion of its singletons. And Lewis points out that this can be used to isolate what is mysterious about classes. For we could sum up his account of classes in one 'equation': Mereology + plural quantification + the singleton function = class theory. Later, in Chapter 3, he shows that much of what is usually done with set theory can be done using plural quantification and mereology alone. (Especially interesting here is his discussion of distinctions of size, showing for instance how to characterize the finite/infinite distinction without recourse to class theory. This discussion continues into the appendix where such notions as accessibility and measurability are introduced.)

Granted the Main Thesis, then, Lewis traces the puzzling character of classes to the singleton function. In Chapter 2, he argues that the informal remarks used to introduce sets and classes to students in no way prepare them for the crucial distinction between a class of items and the fusion of those items. Once we do somehow grasp the distinction between something and its singleton, then every non-null set can indeed be treated as a fusion of singletons, which turn out to be mereological atoms. But what, we should ask, *is* the difference between a singleton and its member? In Chapter 2 Lewis considers but rejects a number of answers to that question, including various metaphysical speculations in which singletons are treated as properties or as states of affairs. He also expresses reservations about (without entirely rejecting) a structuralist account according to which the singleton function is permissibly interpreted as any relation which has the appropriate formal characteristics. The appendix establishes that there are likely to be several such relations, and using the method of supervaluations, we do not have to choose between them.

Why then is Lewis so reluctant to embrace this structuralist account? Largely because he considers it to be a change of topic. Current mathematics, he claims, is based on set-theory as intended by mathematicians, and mathematicians do not intend a structuralist interpretation of sets and classes. I shall return to the whole question of structuralist interpretations later, arguing that Lewis is unnecessarily reluctant to embrace them. As it is, he inclines towards the view that the singleton function is a primitive relation, which, using the resources of mereology and plural quantification, is enough to get the whole of set theory going.

In Chapter 3, as I have already indicated, Lewis shows what can be done using mereology and plural quantification alone. In that chapter he also defends his use of plural quantification without recourse to set-theory, and he defends his use of mereology, including his plausible suggestion that mereological composition is a species of identity.

In Chapter 4 Lewis shows how we can indeed regain versions of the standard

axioms of iterative set theory using the singleton function, along with mereology and plural quantification. He also presents an interesting comparison between Peano arithmetic and set-theory, arguing that the successor relation is the very same as the singleton function. This is not only itself of interest but it also allows for intermediate cases between set-theory and arithmetic.

I hope this survey gives some idea of the power of Lewis' approach, based on the Main Thesis, namely that the parts of a class are all and only its subclasses. But should we accept that thesis? If the singleton relation was less puzzling, then the Main Thesis would indeed be justified, as Lewis suggests, by its fruitfulness. But, as he concedes, the singleton relation is somewhat puzzling: we are introduced to it without being given much information as to what it is. We should, therefore, seriously consider what alternatives there are to the Main Thesis. One is to take class membership as primitive and define subclasses in terms of membership. Lewis would call that unhelpful. But is it any worse than taking the singleton relation as primitive? An alternative which I like is to treat the relation of monadic instantiation which holds between a property and an instance as a primitive. The properties are then characterized not as classes of possibilia, but as those items capable of monadic instantiation. We can then use something like Russell's No Class Theory to handle classes. This need not, however, be an eliminative theory of classes. Rather, we can go structuralist. A permissible interpretation will be any way of interpreting classes as properties so that set-membership is interpreted as instantiation. We then use van Fraassen's method of supervaluation. In this way the Main Thesis is false unless, counter-intuitively, the conjunctive property *being both F and G* is taken as a part of its conjunct *being F*. I think that Lewis' chief objection to this approach is that there is no guarantee that there are properties corresponding to all the classes we need. In particular there could, he submits (p. 55), be two qualitatively identical things. In that case there are no properties which take the place of the singleton, unless, problematically, we are realists about nonqualitative haecceities. This lack of properties corresponding to singletons is indeed disastrous if we are treating classes as fusions of singletons. But I cannot see what is wrong with it if we abandon Lewis' Main Thesis and hence do not give singletons a special role as mereological atoms. Suppose Tweedledum and Tweedledee share all the same qualitative properties. Then on the proposed account of classes the assertion: "There is a set consisting only of Tweedledee" is false, because it is false on any permissible interpretation of classes as qualitative properties. But why should we care? There will be enough sets to do mathematics, without the set consisting only of Tweedledee (provided we have a suitable hierarchy of properties of properties). It is worth mentioning, in passing, that if we allow unrestricted conjunction of properties there are vast numbers of properties with no instances, such as the property of being both square and round. Hence the null set can be treated uniformly with all other sets: the phrase 'the null set' may be interpreted as referring to any property which lacks instances.

I would suggest, then, that there are viable alternatives to the Main Thesis. So I question Lewis' claim that the Main Thesis is supported by its fruitfulness.

My other criticism of Lewis concerns his reluctance to accept a structuralist account of the singleton relation. Why? He seems to have two arguments against such structuralism. The first I call the "Running-Out-of-Atoms Objection". It is that there might not be enough reality for the structuralist account to work.

We might, he fears, run out of the atoms which are, on any given interpretation, to be the singletons. Indeed it seems that we have to posit a lot of reality of one sort or another to save set-theoretical mathematics. Is it worth saving? Assuming it is, it is surely preferable to propose the weaker hypothesis that there are enough atoms of some sort or another, rather than the stronger one that there are enough atoms which are already singletons. Hence I am not persuaded by the Running-Out-of-Atoms Objection.

The other objection which Lewis raises I call the "Mathematicians' Intention Objection". It is that the "structuralist redemption of set-theory is not retroactive. . . . It would be highhanded anachronism to claim that set-theory was structuralist all along" (p. 53). He goes onto say that Structuralism "does not fit present day mathematical practice, because that practice does not include any knowledge of how to Ramsify out the singleton function" (p. 54). What the Mathematicians' Intention Objection amounts to is the complaint that mathematicians did not intend set theory to be interpreted in a structuralist fashion, so set-theory as they intended it would be false unless we can do better than Structuralism. How, then, I ask, did they intend set-theory to be interpreted? They may not have had any explicit intentions. But we can foist on them some implicit intentions such as one which Lewis, rightly I think, relies on, namely that set theory not be "Skolemized" (p. 50). And we might also insist that mathematicians have implicitly intended their assertions that various sets exist to be taken literally. Let us suppose, then, that set-theory is to be interpreted realistically in that weak sense of realism which amounts only to the assertion that "There are sets" is literally true. This is, however, no reason for rejecting a structuralist interpretation. Realism about sets is the thesis that "There are sets" is literally true, namely (the structuralist can gloss) true under all permissible interpretations. An example might help to persuade us that structuralism need not be incompatible with realism (in the weak sense being considered). Consider the problem of the indeterminacy of the reference of "cat". A cat considered in abstraction from any one of its hairs is still a cat. So if there is a cat on the mat there are thousands of permissible interpretations for "the cat on the mat", but not, surely, thousands of cats on the mat. Using the method of supervaluation we find that "There is just one cat on the mat" is true because it is true on any permissible interpretation. That is, in its simple way, a structuralist theory of cats. And it is quite compatible with Realism about cats in the weak sense of literally asserting that there are cats. I conclude that the realist intentions of mathematicians in no way excludes a structuralist interpretation of set theory, so the Mathematicians Intention Objection, like the Running-Out-of-Atoms Objection, fails to persuade me.

I conclude that Lewis does not take seriously enough the structuralist account of sets which he, along with Burgess and Hazen, has done so much to develop.

Parts of Classes is not merely essential reading for anyone interested in knowing the nature of classes; it is also enjoyable reading. A truly fine book.

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