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End Extensions of Normal Models of Open Induction

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Abstract A domain is *normal* if it is integrally closed in its fraction field. We prove that every countable normal model of Open Induction is normal.

1 Introduction Our goal is to show that countable normal models of Open Induction have normal end extensions. We begin with background information on Open Induction.

Let L be the language of arithmetic where we have function symbols $+$ and $\cdot$, a relation symbol $<$, and constant symbols 0 and 1. We let *Open Induction* be the L-theory axiomatized by the axioms for discrete ordered rings and the following schema:

$$\forall \bar{x}[(\phi(0,\bar{x}) \wedge \forall y \geq 0(\phi(y,\bar{x}) \rightarrow \phi(y+1,\bar{x}))) \rightarrow \forall y \geq 0\phi(y,\bar{x})]$$

where $\phi(u,\bar{v})$ is a quantifier-free L-formula.

Although it is customary to consider only positive elements in a model of arithmetic, the algebraic nature of Open Induction makes it convenient to consider the entire ordered ring. The algebraic nature of Open Induction is highlighted by the following result of Shepherdson.

Theorem 1.1 (Shepherdson [6]) *Let R be a discrete ordered ring and let K be the real closure of the fraction field of R. R is a model of Open Induction if and only if for all $\alpha \in K$ there is $r \in R$ such that $|r-\alpha| < 1$.*

Using this criterion Shepherdson showed that Open Induction is indeed a very weak theory. He constructed recursive models of Open Induction in which $\sqrt{2}$ is rational. The main idea behind his construction is the following:

Let M be a model of Open Induction and let K be the real closure of the fraction field of M. Consider the field $K^* = \bigcup K((t^{1/n}))$ where $K((X))$ denotes the field of formal Laurent series over K in the indeterminant X. K^* is the field of Puiseux series over K. It is well-known that K^* is a real closed field (see for example Walker [8]) where t is infinitesimally small with respect to K. We let

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$$M^* = \left\{ \sum_{n=m}^{0} a_n t^{n/q} \in K^* : m \leq 0 \text{ and } q_0 \in M \right\}.$$

It is easy to see that M^* is a discretely ordered ring and every element of K^* is within distance 1 of an element of M^*. Thus M^* is a model of Open Induction. In fact, M^* is an end extension of M. Thus every model of Open Induction has an end extension which is a model of Open Induction.

We say that a domain R is *normal* if it is integrally closed in its fraction field. In [1], van den Dries suggested that some of the pathologies of Shepherdson's models could be avoided if one considered normal models. For example, in a normal model $\sqrt{2}$ cannot be rational. On the other hand, many other extreme pathologies of Open Induction hold in normal models as well. For example, in MacIntyre and Marker [3] we construct a normal model of Open Induction M with a nonstandard prime p such that $M/(p)$ is an algebraically closed field (recently Smith [7] has shown that this is still possible if we insist that M is a Bezout ring).

Building normal models of Open Induction requires more subtle techniques than the Shepherdson construction. The main idea is the following lemma of Wilkie. A discrete ordered ring R is said to be a *Z-ring* if for any natural number n, $R/(n) \cong Z/(n)$.

Lemma 1.2 (Wilkie [9]) *Let R be a discrete ordered Z-ring and let $K \supset R$ be a $|R|^+$-saturated real closed field. Let $\alpha \in K$ such that for all $r \in R$, $|\alpha - r| > 1$. Then there is $\beta \in K$ such that $|\alpha - \beta| < 1$ and for all nonconstant polynomials $p(X) \in R[X]$, $|p(\beta)| > n$ for all $n \in Z$.*

Once we have such a β, consider $N = \bigcup R[\beta/n]$. N is a discretely ordered Z-ring and N contains an element within distance open of α. By iterating Lemma 1.2 and applying Shepherdson's criterion, Wilkie proved that every discretely ordered Z-ring can be extended to a model of Open Induction. Our goal is to give a strengthening of Lemma 1.2 which allows us to build normal end extensions of countable models.

Our main result is the following:

Theorem 1.3 *If M is a countable normal model of Open Induction, then M has a proper end extension which is a normal model of Open Induction.*

2 *The construction*

We fix M a countable normal model of Open Induction.

Definition Suppose $M \subset N$. We say that N is an *M-ring* if for all $a \in N$ and all $b \in M$ there is $c \in M$ such that $a \equiv c (\text{mod}\, b)$.

Clearly if a model of Open Induction N is an end extension of M then N is an M-ring.

For any domain R we let R^{cl} denote the *real closure* of R.

Definition Suppose $M \subset N$. If $\alpha \in N^{\text{cl}}$ we say that α is *M-bounded* if there is a $\beta \in M$ such that $|\alpha| < \beta$. We say that α is *M-infinitesimal* if $0 < |\alpha| < 1/\beta$ for all $\beta > 0$ in M.

We say that N is a *harmless* extension of M if for all $\alpha \in N^{\text{cl}}$, if α is M-bounded then there is $\beta \in M$ such that $|\alpha - \beta| < 1$.

By Theorem 1.1 it is clear that if N is a model of Open Induction end-extending M, then N is a harmless extension of M.

If N is an end extension of M we write $M \subset_e N$. We can now prove the generalization of Lemma 1.2 that we need.

Lemma 2.1 *Suppose $M \subset_e N$, $|N| = \aleph_0$, N is an M-ring and N is a harmless extension of M. Let $K \supset N$ be an $\aleph_1$-saturated real closed field. Suppose $a \in N^{\mathrm{cl}}$ and for all $n \in N$, $|a - n| > 1$. Then there is $b \in K$ such that $|b - a| < 1$ and for all nonconstant polynomials $p(X) \in N[X]$, $p(b)$ is not M-bounded, and $N[b]$ is a harmless extension of M.*

Proof: Let $\{p_n : n \in \omega\}$ list all nonconstant polynomials in $N[X]$. Let $\{f_n \cdot n \in \omega\}$ list all definable partial functions on N^{cl}. We will build sequences $c_0 < c_1 < \dots c_n < \dots < d_n < \dots < d_1 < d_0$ in N^{cl} such that for all n:

(i) $c_0 = a$ and $d_0 = a + 1$.
(ii) $p_n(x)$ is not M-bounded for any $x \in (c_{n+1}, d_{n+1})$.
(iii) Either: (a) f_n is undefined on (c_{n+1}, d_{n+1}) or (b) $f_n(x)$ is not M-bounded for any $x \in (c_{n+1}, d_{n+1})$ or (c) there is $m \in M$ such that $|m - f_n(x)| < 1$ for all $x \in (c_{n+1}, d_{n+1})$.
(iv) There is $m \in M$ such that $d_n - c_n > 1/m$.

If we build such sequences then the desired b will be chosen so that for all n, $c_n < b < d_n$. This is possible since K is $\aleph_1$-saturated. Clearly for all nonconstant $p(X) \in N[X]$, $p(b)$ is not M-bounded and for all definable functions f on N^{cl} such that $f(b)$ is defined, either $f(b)$ is not M-bounded or $f(b)$ is within distance one of an element of M. Since every element of $N[b]^{\mathrm{cl}}$ is of the form $f(b)$ for some N-definable function f (as real closed fields have definable Skolem functions), $N[b]$ is a harmless extension of M. Thus we need only build these sequences.

We begin with $c_0 = a$ and $d_0 = a + 1$.

State $n + 1$:

We are given $c = c_n$ and $d = d_n$. We first must see how to shrink this interval to (c', d') so that $p(x) = p_n(x)$ is not M-bounded on (c', d') and $d' - c'$ is not M-infinitesimal.

Let s be the degree of p. We can find $\alpha \in N$ and $\theta_1, \dots, \theta_s$ in the algebraic closure of N such that $p(X) = \alpha(X - \theta_1) \dots (X - \theta_s)$. We choose c' and d' so that $d' - c'$ is not M-infinitesimal and no $x \in (c', d')$ is M-infinitesimally close to any θ_i.

Suppose $x \in (c', d')$ and, for some $m \in M$, $m > 0$ and $|p(x)| < m$. Since none of the $|x - \theta_i|$ is *M-infinitesimal, each is* M-bounded. Also α is M-bounded and hence in M (since N is an end extension of M). Let $l \in M$ be such that $\sum |x - \theta_i| < l$. Note that $\sum \theta_i = \alpha_1/\alpha$ where $\alpha_1 \in N$ (as α_1 is the X^{s-1} coefficient of $p(X)$).

Now:

$$0 \leq \left| x - \frac{\alpha_1}{s\alpha} \right| = \sum \left| \frac{x - \theta_i}{s} \right| \leq \frac{1}{s} \sum |x - \theta_i| < \frac{l}{s}.$$

Since N is an M-ring, there are $b \in N$ and $r \in M$ such that $\alpha_1 \le s\alpha b + r$. But then:

$$|x - b| = \left|x - \frac{\alpha_1}{s\alpha} + \frac{r}{s\alpha}\right| < \frac{l}{s} + \left|\frac{r}{s\alpha}\right|.$$

So $|x - b|$ is M-bounded. Since N is harmless there is $y \in M$ such that $|x - b - y| < 1$. But $b + y \in N$, contradicting our assumptions about a.

We now must show how to meet Condition (iii). Suppose we have an interval (c, d) such that $d - c$ is not M-infinitesimal and an N^{cl}-definable partial function f. We begin by breaking up the $\mathrm{dom}(f) \cap (c, d)$ into a finite union of points and intervals. If all of these intervals are of M-infinitesimal length then we can find a subinterval (c', d') of non-M-infinitesimal length such that f is undefined everywhere on this interval.

Thus, without loss of generality we may assume that f is total on (c, d). We can decompose (c, d) into a finite number of points and intervals such that on each interval f is differentiable and both f and f' are monotonic on that interval (see for example Pillay and Steinhorn [5]). One of these intervals will have non-M-infinitesimal length, and thus without loss of generality we may assume that f and f' are monotonic on (c, d). For definiteness we will assume that both are strictly increasing, all other cases are similar.

Subcase 1: $f(c)$ and $f(d)$ are M-bounded.

By the Mean Value Theorem, there is $\xi \in (c, d)$ such that $f'(\xi) = [f(d) - f(c)]/(d - c)$. Since f' is increasing on (c, d), $\xi - c > d - \xi$. In particular $\xi - c$ is not M-infinitesimal. Since $f(d)$ and $f(d)$ are M-bounded and $d - c$ is not M-infinitesimal, there is an $l \in M$ such that $|f'(\xi)| < l$ and $\xi - c > 1/l$. But then for all $x \in [c, c + (1/2l)]$, $0 < f(x) - f(c) < |x - c| f'(g) < \frac{1}{2}$. Since N is a harmless extension of M, there is an $m \in M$ such that $|m - f(c)| \le \frac{1}{2}$. Thus, for all $x \in [c, c + (1/2l)]$, $|f(x) - m| < 1$.

Subcase 2: $f(c) < f(d) < M$ or $M < f(c) < f(d)$.

In either case f is not M-bounded on (c, d).

Subcase 3: $f(c) < M$ and $f(d)$ is M-bounded.

Let $e = (c + d)/2$. If $f(e) < M$, then (c, e) is as in Subcase 2. Otherwise (e, d) is as in Subcase 1.

Subcase 4: $f(c)$ is M-bounded and $f(d) > M$.

Similar to Case 3.

Subcase 5: $f(c) < M < f(d)$.

Let e be as above. One of the previous cases applied to (c, e) or to (e, d).

This completes the proof of Lemma 2.1

We need one more lemma for the proof.

Lemma 2.2 *Suppose $N \supset M$ is a normal, discretely ordered M-ring, K is an ordered field extending N, and $\beta \in K$ is such that for all nonconstant polynomials $p(X) \in N[X]$, $p(\beta)$ is not M-bounded. Let*

$$R = \bigcup_{m \in M} N\left[\frac{\beta}{m}\right].$$

Then R is a normal, discretely ordered M-ring. Moreover, if N is an end extension of M, then so is R.

Proof: Suppose $0 < p(\beta/m)) < 1$, where

$$p(X) = \sum_{n=0}^{d} a_n X^n$$

and $d > 0$. Let

$$q(X) = \sum_{n=0}^{d} m^{d-n} a_n X^n.$$

Then $q(\beta) < m^d$, contradicting the fact that $q(\beta)$ is not M-bounded. A similar argument shows that if N is an end extension so is R.

Suppose $p(X)$ is a monic polynomial in $R[X]$. Then for some $m \in M$, $p(X) \in N[\beta/m][X]$. But $N[\beta/m]$ is a simple transcendental extension of N. Thus since N is normal, $N[\beta/m]$ is normal (see for example Jacobson [2]). R and $N[\beta/m]$ have the same fraction field. Hence if $p(X)$ has a zero in the fraction field of R, it already has a zero in $N[\beta/m]$. Thus R is normal.

If $p(X) \in N[X]$ has constant term a_0, then for all $b \in M$, $p(\beta/m)$ is congruent to a_0 (mod b). Thus since N is an M-ring, R is also an M-ring.

We can now prove Theorem 1.3.

Let M be a countable, normal model of Open Induction. Let $K \supset M$ be an $\aleph_1$-saturated real closed field. Our entire construction will take place inside K.

Let $\alpha \in K$ such that, for all $m \in M$, $m < \alpha$. Then for all $x \in M^{\text{cl}}$, $x < \alpha$. Thus for all nonconstant $p(X) \in M[X]$, $p(\alpha)$ is not M-bounded. In general, if E is a real closed field and F is obtained from E by adding an infinite element and taking the real closure, then every E-bounded element of F is infinitesimally close to an element of E (this is easy to check for real closed fields and is proved in a general setting in Marker [4]). Thus $M[\alpha]$ is a harmless extension of M.

Let

$$N_0 = \bigcup_{m \in M} M\left[\frac{\alpha}{m}\right].$$

By Lemma 2.2, N_0 is a normal, discretely ordered M-ring. We now iterate the constructions from Lemma 2.1 and Lemma 2.2 to build $M \subset_e N$ a normal, discretely ordered M-ring such that every element of N^{cl} is within distance one of an element of N. By Shepherdson's Theorem, N is a model of Open Induction.

3 *Open Questions*

Two obvious questions come to mind.

Question 1 Does every uncountable normal model of Open Induction have a normal end extension?

Definition A model M of Open Induction is said to be *Bezout* if any two elements $a, b \in M$ have a greatest common divisor d and there are λ and μ in M such that $\lambda a + \mu b = d$.

Question 2 Does every (countable) Bezout model of Open Induction have a Bezout end extension?

In [7] Smith gives several constructions of pathological Bezout models of Open Induction but his constructions do not seem amenable to starting with an arbitrary Bezout model as the base.

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