Notre Dame Journal of Formal Logic
Volume 32, Number 2, Spring 1991

Bounds in Weak Truth-Table Reducibility

KAROL HABART

Abstract A necessary and sufficient condition on a recursive function is given so that arbitrary sets can be truth-table reduced via this function as the bound. A corresponding hierarchy of recursive functions is introduced and some partial results and an open problem are formulated.

Weak truth-table reducibility, often called bounded Turing reducibility, is defined as follows: $A \subseteq \omega$ is weak-truth–table reducible to $B \subseteq \omega$ ($A \leq_{\text{wtt}} B$) if there is a recursive function f and an algorithm which answers questions of the form "$n \in A$?" when supplied answers to any questions it asks of the form "$m \in B$?" for $m \leq f(n)$. The function f is called the bound of the reduction.

The hierarchy of subsets of ω induced by the relation $\leq_{\text{wtt}}$ was extensively studied in the past (cf. [1]). In this paper, however, a hierarchy of the *bounds* (i.e. of recursive functions) is considered. We denote by $\mathcal{S}(f)$ the set of A such that there is a B such that $A \leq_{\text{wtt}} B$ via a reduction with bound f, and we write $f \ll g$ iff $\mathcal{S}(f) \subseteq \mathcal{S}(g)$. Of course, $\mathcal{R} \subseteq \mathcal{S}(f) \subseteq 2^\omega$ for all recursive functions f. We give necessary and sufficient conditions on f for $\mathcal{S}(f) = \mathcal{R}$ and for $\mathcal{S}(f) = 2^\omega$, i.e. for f being on the bottom and on the top of the hierarchy induced by $\ll$. We also give a necessary condition for $f \ll g$.

Our interest is focused to the bound of the wtt-reduction by the following phenomenon: A part of an (in general) nonrecursive set B can be given by a list. Having the set A Turing reduced to B, a part of A is given which corresponds to the list of a part of B, and which may be much larger than the list itself—depending mainly on the bound of the reduction.

A motivation for the study of our hierarchy of bounds comes also from the theory of nets of automata. Consider a chain of automata numbered by natural numbers. Suppose each automaton is in one of the two states 0 and 1. Then the state of the whole net is uniquely determined by a set $B \subseteq \omega$ in an obvious way. Now let the automata work, and after some time all of them may stop and the net may come into a state determined by a set A. In a fairly devised net we would have $A \leq_{\text{wtt}} B$. The bound f of this reduction depends on how the

Received August 7, 1989; revised October 13, 1989

communication between the automata is devised. Thus the relation $\ll$ is a generalization of the relation "less powerful" among the different kinds of communication between automata in a net.

We fix our notation first. We use ω^ω to denote the set of functions from ω to ω, $\mathcal{R}$ denotes the recursive functions; $f \upharpoonright A$ means the restriction of f to the domain A; $f[A]$ denotes the image of A under f, $f^{-1}[A] = \{x : f(x) \in A\}$; $|A|$ denotes the cardinality of A, 2^A the power set of the set A; $2^{<\omega}$ denotes finite sequences of 0's and 1's; if σ is a finite string (i.e. ranging over $2^{<\omega}$) we use the length function $\mathrm{lh}(\sigma) = \mu x[x \notin \mathrm{dom}\,\sigma] = |\mathrm{dom}\,\sigma|$. For a finite set F let ind F denote the canonical index of F, i.e. if $F = \{y_1, \ldots, y_n\}$, $y_1 < \cdots < y_n$, then $\mathrm{ind}\,F = 2^{y_1} + \cdots + 2^{y_n}$; $\mathrm{ind}\,\varnothing = 0$. We write $F = D_{\mathrm{ind}\,F}$.

We identify sets with their characteristic functions, the integer n with the set $\{0,1,\ldots,n-1\}$, and the integer 0 with the set $\varnothing$.

Let $\langle e\rangle^B$ denote the (possibly partial) recursive function with index e relative to the set B. For $\sigma \in 2^{<\omega}$ let us define $\sigma' \in 2^\omega$ by $\sigma'(x) = \sigma(x)$ $(x < \mathrm{lh}\,\sigma)$ and $\sigma'(x) = 0$ $(x \geq \mathrm{lh}\,\sigma)$ and define $\langle e\rangle^\sigma(x) = y$ iff $\langle e\rangle^{\sigma'}(x) = y$ and only numbers z with $z < \mathrm{lh}(\sigma)$ are used in the computation. We define $\langle e,f\rangle^A$, the Turing oracle function with index e, oracle A, and bound f as follows:

$$\langle e,f\rangle^A(x) = y \leftrightarrow (\exists\sigma \in 2^{<\omega})[\sigma \subseteq A \wedge \mathrm{lh}(\sigma) \leq f(x) + 1 \wedge \langle e\rangle^\sigma(x) = y].$$

Further let

$$\mathcal{S}(f) = \{S \subseteq \omega : (\exists A \subseteq \omega)(\exists e \in \omega)[S = \langle e,f\rangle^A]\},$$

i.e. $\mathcal{S}(f)$ denotes the set of all subsets of ω which are weak truth-table reducible to an oracle via bound f.

Obviously

$$2^\omega \supseteq \mathcal{S}(f) \supseteq \mathcal{R}.$$

Put

$$f \ll g \leftrightarrow \mathcal{S}(f) \subseteq \mathcal{S}(g)$$

and call f maximal iff $\mathcal{S}(f) = 2^\omega$ and minimal iff $\mathcal{S}(f) = \mathcal{R}$.

It is obvious that, e.g., $\lambda x(x)$ is maximal and $\lambda x(0)$ is minimal. A function f defined by

$$f(2x) = 0$$
$$f(2x+1) = 2x+1 \quad (x \in \omega)$$

is not maximal, because for each $A \in \mathcal{S}(f)$ the set $A \cap \{2x : x \in \omega\}$ must be recursive; and it is not minimal, because for some $A \in \mathcal{S}(f)$ the set $A \cap \{2x + 1 : x \in \omega\}$ need not to be recursive.

Theorem 1 *f is minimal iff f is bounded.*

Proof: If f is bounded then the minimality of f follows immediately.

So assume f is not bounded. Obviously there is $e \in \omega$ so that for all $A \subseteq \omega$, $x \in \omega$

$$\langle e,f\rangle^A(x) = A(f(x)).$$

Because f is unbounded there is a recursive subset $S \subseteq \omega$ so that $f[S]$ is recursive, infinite and f is one-one on S. Choosing A so that $f[S] \cap A$ is nonrecursive we get $S \cap \langle e,f\rangle^A$ nonrecursive whence $\mathcal{S}(f) \nsubseteq \mathcal{R}$.

Lemma 1 *If*

$$\text{(L1.1)} \qquad \sup_{x\in\omega}(|f^{-1}[x]| - x) < \infty$$

then f is maximal.

Proof: Let $C - 1$ be the supremum in (L1.1). We define a recursive function g by induction as follows:

$$g(0) = C + f(0)$$
$$g(x+1) = \max((1 + C + f(x+1))\backslash g[x+1]).$$

In order to have g well-defined we need to show for every $x \in \omega$:

$$(1 + C + f(x))\backslash g[x] \neq 0.$$

Assume $(1 + C + f(x))\backslash g[x] = 0$ for some x and let it be the least such x. Then $g[x] \cap (1 + C + f(x)) = 1 + C + f(x)$. Choose y maximal with $g[x] \cap y = y$. Obviously $y \geq 1 + C + f(x)$.

Claim *For each $z \in x$ if $g(z) < y$ then $C + f(z) < y$.*

Proof: Assume $g(z) < y$ and $C + f(z) \geq y$. Then obviously $z \neq 0$. Because $g(z) = \max((1 + C + f(z))\backslash g[z])$ we have $y \in g[z]$ whence $y \in g[x]$, too. But then $g[x] \cap (1 + y) = 1 + y$. This contradicts the choice of y.

Because for each $z \in x$ by definition $g(z) \notin g[z]$, g is injective on x and we have $|g^{-1}[y]| = y$, i.e. by our claim $g^{-1}[y] \subseteq f^{-1}[y - C]$ whence $|f^{-1}[y - C]| \geq y$, i.e. $|f^{-1}[y - C]| - (y - C) \geq C$. This contradicts the fact that $C - 1$ is the supremum in (L1.1). Thus g is well-defined and injective. Moreover,

$$\text{(L1.2)} \qquad g(x) \leq f(x) + C \qquad (x \in \omega)$$

is immediate.

Let $A \subseteq \omega$ be arbitrary. Choose $\sigma \in 2^{<\omega}$ so that $\text{lh}(\sigma) = C$ and $\sigma(g(x)) = 1$ iff $g(x) < C$ and $x \in A$. Choose $B \subseteq \omega$ so that $g(x) - C \in B$ iff $g(x) \geq C$ and $x \in A$. This is all possible because of the injectivity of g.

Finally, choose $e \in \omega$ so that

$$\langle e,f\rangle^S(x) = \begin{cases} \sigma(g(x)) & \text{if } g(x) < C \\ S(g(x) - C) & \text{if } g(x) \geq C. \end{cases}$$

This is possible because of (L1.2). Then obviously

$$A = \langle e,f\rangle^B.$$

Note: The construction of the recursive function g in the proof of Lemma 1 shows that (L1.1) implies the following: there is a one-one recursive function g and a constant C such that (L1.2) holds. This condition is even equivalent to (L1.1) and hence a condition on f for maximality.

In order to show the converse of Lemma 1, and thus to give a necessary and sufficient condition for maximality, we shall prove a more general result. We shall introduce a recursive functional Θ which will be of much use later on. It will be defined by an auxiliary functional Φ. For recursive f the function Φf is defined by:

$$\Phi f(x,0) = 0$$
$$\Phi f(x,y+1) = \min\{y+1, \Phi f(x,y) + |f^{-1}[\{y\}] \cap D_x|\}.$$

The function Θf is then:

$$\Theta f(x) = \Phi f(x, 1 + \max f[D_x]) \quad (\max 0 = 0).$$

The functional Θ was introduced to have a result like Lemma 6. Intuitively, for a finite set D_x $\Theta f(x)$ yields something like the cardinality of that part of the oracle B that will carry some information for $A \upharpoonright D_x$ in addition to the index e when $A = \langle e,f\rangle^B$. We give some properties of Θ in the following lemmas.

Lemma 2 $\quad (\forall y > \max f[D_x])\, [\Theta f(x) = \Phi f(x,y)]$.

Proof: Let $y > \max f[D_x] + 1$. Then $f^{-1}[\{y-1\}] \cap D_x = 0$ and so $\Phi f(x,y) = \Phi f(x,y-1)$.

Lemma 3 $\quad \Theta f(x) \le |D_x|$.

Proof: One shows easily that

$$\Phi f(x,y) \le \sum_{z<y} |f^{-1}[\{z\}] \cap D_x|$$

whence $\Theta f(x) \le |D_x|$.

Lemma 4 *If $D_x \subseteq D_y$ then $\Theta f(x) \le \Theta f(y)$.*

Proof: Obviously $\Phi f(x,z)$ grows in z for fixed x. Now it suffices to show that if $D_x \subseteq D_y$ then $\Phi f(x,z) \le \Phi f(y,z)$. This follows immediately by induction.

Lemma 5 *If $D_y = D_x \cup \{z\}$ then $\Theta f(y) - \Theta f(x) \le 1$ (i.e. for all x, y with $D_x \subseteq D_y$ we have $\Theta f(y) - \Theta f(x) \le |D_y \backslash D_x|$).*

Proof: Let $f(z) = u$. Then $\Phi f(x,v) = \Phi f(y,v)$ for $v \le u$. Obviously $\Phi f(y,u+1) - \Phi f(x,u+1) \le 1$ and then also $\Phi f(y,v) - \Phi f(x,v) \le 1$ for $v > u$. By Lemma 2 we get then $\Theta f(y) - \Theta f(x) \le 1$.

Lemma 6 *For every recursive f and each $e \in \omega$:*

$$|\{\langle e,f\rangle^A \upharpoonright D_x : A \subseteq \omega\}| \le 2^{\Theta f(x)}.$$

Proof: Put $S = \{\langle e,f\rangle^A \upharpoonright D_x : A \subseteq \omega\}$ and $S_y = \{\langle e,f\rangle^A \upharpoonright [f^{-1}[y] \cap D_x : A \subseteq \omega\}$. Obviously $S \subseteq 2^{\max D_x + 1}$ and $S = S_{1+\max f[D_x]}$. Thus it suffices to show

(L6.1) $$|S_y| \le 2^{\Phi f(x,y)}.$$

We show this by induction on y. Obviously for $y = 0$ (L6.1) is fulfilled. Now assume (L6.1) and consider $|S_{y+1}|$. For every $z \in f^{-1}[y+1] \cap D_x$ and $A \in 2^\omega$ we have

$$\langle e,f\rangle^A(z) = \langle e,f\rangle^{A \upharpoonright (y+1)}(z).$$

Thus

(L6.2) $$|S_{y+1}| \leq |\{A \upharpoonright (y+1) : A \subseteq \omega\}| \leq |2^{y+1}| = 2^{y+1}.$$

On the other hand

$$S_{y+1} = \{\langle e,f\rangle^A \upharpoonright (f^{-1}[y] \cap D_x) \cup (f^{-1}[\{y\}] \cap D_x) : A \subseteq \omega\}$$
$$\subseteq \{\alpha \cup \beta : \alpha \in S_y \wedge \beta \in 2^{f^{-1}[\{y\}] \cap D_x}\}$$

whence

$$|S_{y+1}| \leq |S_y| \cdot 2^{|f^{-1}[\{y\}] \cap D_x|},$$

i.e. together with (L6.2) we get finally

$$|S_{y+1}| \leq 2^{\Phi f(x, y+1)}.$$

Theorem 2 *For every f, g:*

(T2.1) $$f \ll g \rightarrow \sup_{x \in \omega} (\Theta f(x) - \Theta g(x)) < \infty.$$

Proof: Assume that the supremum in (T2.1) is infinite.

Claim 1 *One can construct a recursive sequence $(s_n)_{n=0}^{\infty}$ so that*

(T2.2) $$\max D_{s_n} < \min D_{s_m} \quad (n < m)$$

and

$$\Theta f(s_0) > \Theta g(s_0)$$

(T2.3) $$\Theta f(s_{n+1}) > \Theta g(s_{n+1}) + \Theta f(s_n).$$

Proof: The existence of a suitable s_0 is obvious. Now let $s_0, \ldots, s_n$ be found. Put $z = \max D_{s_n}$. Then there is a smallest u with $\Theta f(u) - \Theta g(u) > \Theta f(s_n) + z + 1$. Let s_{n+1} be the canonical index of $D_u \backslash (z+1)$. Then by Lemma 5 $\Theta f(s_{n+1}) \geq \Theta f(u) - (z+1)$ and by Lemma 4 $\Theta g(s_{n+1}) \leq \Theta g(u)$ so that $\Theta f(s_{n+1}) - \Theta g(s_{n+1}) \geq \Theta f(u) - \Theta g(u) - (z+1) > \Theta f(s_n)$.

Now define a function h as follows:

$$h(x) = \begin{cases} 0 & \text{if } x \notin \bigcup_{n \in \omega} D_{s_n} \\ \min\{f(x), |x \cap f^{-1}[\{f(x)\}] \cap D_{s_n}| \\ \qquad + \Theta f(\mathrm{ind}(f^{-1}[f(x)] \cap D_{s_n}))\} & \text{if } x \in D_{s_n}. \end{cases}$$

Because we can compute $\mathrm{ind}(f^{-1}[f(x)] \cap D_{s_n})$ and because under (T2.2 and 2.3) one can decide whether $x \in \bigcup_{n \in \omega} D_{s_n}$ and if so then compute n so that $x \in D_{s_n}$, h must be a recursive function.

Claim 2 *We have*

(T2.4) $$h[f^{-1}[y] \cap D_{s_n}] = \Theta f(\mathrm{ind}(f^{-1}[y] \cap D_{s_n})).$$

Proof: For $y = 0$ (T2.4) is obvious. Now assume (T2.4). We have

$$f^{-1}[y+1] \cap D_{s_n} = (f^{-1}[y] \cap D_{s_n}) \cup (f^{-1}[\{y\}] \cap D_{s_n}).$$

If $f^{-1}[\{y\}] \cap D_{s_n} = 0$ then obviously (T2.4) holds for $y + 1$, too. So let $f^{-1}[\{y\}] \cap D_{s_n} = \{z_0, \ldots, z_k\}$, $z_0 < \cdots < z_k$. By the definition of h we have

$$h(z_i) = \min\{y, i + \Theta f(\mathrm{ind}(f^{-1}[y] \cap D_{s_n}))\},$$

i.e. together with the induction assumption

$$\text{(T2.5)} \qquad h[f^{-1}[y+1] \cap D_{s_n}] = \min\{y+1, |f^{-1}[\{y\}] \cap D_{s_n}| + \Theta f(\mathrm{ind}(f^{-1}[y] \cap D_{s_n}))\}.$$

Further it is obvious that for $(0 \leq u \leq y)$

$$\Phi f(\mathrm{ind}(f^{-1}[y+1] \cap D_{s_n}), u) = \Phi f(\mathrm{ind}(f^{-1}[y] \cap D_{s_n}), u)$$

so that we have

$$\Theta f(\mathrm{ind}(f^{-1}[y] \cap D_{s_n})) = \Phi f(\mathrm{ind}(f^{-1}[y+1] \cap D_{s_n}), y)$$

and

$$\Theta f(\mathrm{ind}(f^{-1}[y+1] \cap D_{s_n})) = \Phi f(\mathrm{ind}(f^{-1}[y+1] \cap D_{s_n}), y+1)$$

whence

$$\Theta f(\mathrm{ind}(f^{-1}[y+1] \cap D_{s_n})) = \min\{y+1, \Theta f(\mathrm{ind}(f^{-1}[y] \cap D_{s_n})) + |f^{-1}[\{y\}] \cap (f^{-1}[y+1] \cap D_{s_n})|\}$$

so that by (T2.5)

$$h[f^{-1}[y+1] \cap D_{s_n}] = \Theta f(\mathrm{ind}(f^{-1}[y+1] \cap D_{s_n})).$$

This completes the proof of Claim 2.

Because $D_{s_n} = D_{s_n} \cap f^{-1}[1 + \max f[D_{s_n}]]$ we have by Claim 2:

$$\text{(T2.6)} \qquad h[D_{s_n}] = \Theta f(s_n).$$

Now let e be an index so that

$$\langle e, f\rangle^A(x) = \begin{cases} 1 & \text{if } h(x) \in A \\ 0 & \text{elsewhere.} \end{cases}$$

Then by (T2.6) for every $n \in \omega$ and $\rho \in 2^{<\omega}$ where $\mathrm{lh}(\rho) = r \leq \Theta f(s_n)$ we have:

$$\text{(T2.7)} \qquad |\{\langle e, f\rangle^\sigma \lceil D_{s_n} : \sigma \in 2^{\Theta f(s_n)} \wedge \sigma \lceil r = \rho\}| = 2^{\Theta f(s_n) - r}.$$

Now we define a set $U = \bigcup_{n \in \omega} \sigma_n$ where $\sigma_n \in 2^{\Theta f(s_n)}$ is defined as follows:

Choose $\sigma_0 \in 2^{\Theta f(s_0)}$ so that

$$\langle e, f\rangle^{\sigma_0} \lceil D_{s_0} \notin \{\langle 0, g\rangle^A \lceil D_{s_0} : A \subseteq \omega\}.$$

This is possible because of (T2.7), Lemma 6, and (T2.3). Now assume σ_n is chosen. Then choose σ_{n+1} so that $\sigma_{n+1} \lceil \Theta f(s_n) = \sigma_n$ and

$$\langle e, f\rangle^{\sigma_{n+1}} \lceil D_{s_{n+1}} \notin \{\langle n+1, g\rangle^A \lceil D_{s_{n+1}} : A \subseteq \omega\}.$$

This is again possible because of (T2.7), Lemma 6, and (T2.3).[1]

We have now for every $n \in \omega$ and $A \subseteq \omega$:

$$\langle e,f\rangle^U \lceil D_{s_n} = \langle e,f\rangle^{\sigma_n} \lceil D_{s_n} \neq \langle n,g\rangle^a \lceil D_{s_n},$$

i.e.

$$\langle e,f\rangle^U \neq \langle n,g\rangle^A$$

whence $f \not\ll g$. Contradiction.

Whether the converse of Theorem 2 holds or not is an open question. The author conjectures that condition (T2.1) is a necessary and sufficient condition for $f \ll g$ but he has managed to find neither a proof nor a counterexample yet.

Now we begin proving the converse of Lemma 1 using Theorem 2.

Lemma 7 $\quad \Theta(\lambda z(z))(x) = |D_x|$.

Proof: Put $f = \lambda z(z)$. We shall show

(L7.1) $$\Phi f(x,y) = |f^{-1}[y] \cap D_x|.$$

Lemma 7 follows then immediately from (L7.1).

For $y = 0$ (L7.1) is obvious. Now assume (L7.1) for some y. Now $|f^{-1}[\{y\}] \cap D_x| \leq 1$ and because $\Phi f(x,y) \leq y$ we have

$$\Phi f(x,y) + |f^{-1}[\{y\}] \cap D_x| \leq y + 1,$$

i.e.

$$\Phi f(x,y+1) = |f^{-1}[y] \cap D_x| + |f^{-1}[\{y\}] \cap D_x| = |f^{-1}[y+1] \cap D_x|.$$

Lemma 8 *If $f^{-1}[z]$ is finite then* $\Theta f\,(\mathrm{ind} f^{-1}[z]) \leq z$.

Proof: Obviously $1 + \max f[f^{-1}[z]] \leq z$ and because for all y

$$\Phi f(x,y) \leq y$$

Lemma 8 follows immediately.

Lemma 9 $\quad \Theta(\lambda z(0))(x) \leq 1 \ (x \in \omega)$.

Proof: Put $f = \lambda z(0)$. We shall show

(L9.1) $$\Phi f(x,y) \leq 1 \quad (x,y \in \omega).$$

For $y = 0$ and $y = 1$ (L9.1) is immediate. Now assume (L9.1) for some $y \geq 1$. Then $f^{-1}[\{y\}] = 0$, i.e.

$$\Phi f(x,y+1) = \Phi f(x,y) \leq 1.$$

Lemma 10 *If* $\sup_{x\in\omega}(\Theta(\lambda z(z))(x) - \Theta f(x)) < \infty$ *then for all x, $f^{-1}[x]$ is finite.*

Proof: Suppose that for some x, $f^{-1}[x]$ is infinite. Let $y_0, y_1, \ldots$ enumerate $f^{-1}[x]$ and for each $k \geq 0$, let $x_k = \mathrm{ind}\{y_0, \ldots, y_k\}$. Then for all $k \geq 0$, $\Theta f(x_k) = \Phi f(x_k, 1 + \max f[D_{x_k}]) \leq \Phi f(x_k, 1 + x) \leq x + 1$ while by Lemma 7 $\Theta(\lambda z(z))(x_k) = |D_{x_k}| = k + 1$. Contradiction.

Theorem 3 *f is maximal iff*

(T3.1) $$\sup_{x\in\omega}(|f^{-1}[x]| - x) < \infty.$$

Proof: According to Lemma 1 it suffices to show that if f is maximal then (T3.1) holds. If f is maximal then $f \gg \lambda z(z)$ and by Theorem 2

$$\sup_{x\in\omega}(\Theta(\lambda z(z))(x) - \Theta f(x)) < \infty,$$

so by Lemma 10 we have in particular

$$\sup_{x\in\omega}(\Theta(\lambda z(z))(\operatorname{ind} f^{-1}[x]) - \Theta f(\operatorname{ind} f^{-1}[x])) < \infty$$

whence (T3.1) follows immediately by Lemmas 7 and 8.

Theorem 3 yields several sufficient conditions for f not to be maximal.

Corollary 1 *If*

$$\limsup \frac{f(x)}{x} < 1$$

then f is not maximal.

Proof: Let $\limsup[f(x)/x] < C < 1$. Put $d = (1 - C)/C$. Then $d > 0$. Obviously, for some $M \geq 0$

$$|f^{-1}[x]| \geq \frac{1}{C} \cdot x - M$$

whence

$$\sup_{x\in\omega}(|f^{-1}[x]| - x) \geq \sup_{x\in\omega} d \cdot x - \frac{M}{C} = \infty.$$

Corollary 2 *If*

$$\lim_{x\to\infty} \frac{\max_{y\leq x} f(y)}{x} < 1$$

then f is not maximal.

Proof: Let $g(x) = \max_{y\leq x} f(y)$. Thus $g(x) \geq f(x)\,(x \in \omega)$ so that if f is maximal then g must be maximal, too. Corollary 2 follows then immediately by Corollary 1.

Finally we present two lemmas which serve as examples where the converse of Theorem 2 holds. They are just the cases where f is maximal and minimal.

Lemma 11 $\lambda z(z) \ll f$ *if and only if*

(L11.1) $$\sup_{x\in\omega}(\Theta(\lambda z(z))(x) - \Theta f(x)) < \infty.$$

Proof: By Theorem 2 it suffices to show that from (L11.1) the maximality of f follows. Now if (L11.1) then by Lemma 10 we have in particular

$$\sup_{x\in\omega}(\Theta(\lambda z(z))(\operatorname{ind} f^{-1}[x]) - \Theta f(\operatorname{ind} f^{-1}[x])) < \infty$$

whence by Lemmas 7 and 8

$$\sup_{x\in\omega}(|f^{-1}[x]| - x) < \infty$$

so that by Lemma 1 the maximality of f follows.

Corollary 3 *f is maximal iff*

$$\sup_{x\in\omega}|\Theta(\lambda z(z))(x) - \Theta f(x)| < \infty.$$

Lemma 12 *$f \ll \lambda z(0)$ if and only if*

(L12.1) $$\sup_{x\in\omega}(\Theta f(x) - \Theta(\lambda z(0))(x)) < \infty.$$

Proof: According to Theorem 2 we need only show that (L12.1) implies the minimality of f. Thus assume (L12.1). Then by Lemma 9 Θf is bounded. Choose x so that $\Theta f(x)$ is maximal and assume that f is not bounded. Then there is $u \in \omega \setminus D_x$ with $f(u) > 1 + \max f[D_x]$. Let $D_v = D_x \cup \{u\}$. Because $\Phi f(a,b) \leq b$ we have $\Theta f(x) < f(u)$ and obviously $\Phi f(v, f(u)) \geq \Theta f(x)$. Then

$$\Theta f(v) = \Phi f(v, f(u) + 1) = \min\{f(u) + 1, \Phi f(v, f(u)) + 1\} \geq \Theta f(x) + 1.$$

This contradicts the choice of x. Hence f is bounded, too, so that by Theorem 1 f must be minimal.

Corollary 4 *f is minimal iff*

$$\sup_{x\in\omega}|\Theta f(x) - \Theta(\lambda z(0))(x)| < \infty.$$

NOTE

1. Note that U is not recursively enumerable in general; though, by this construction, U is recursive in K (i.e. of degree $0'$).

REFERENCES

[1] Friedberg, R. and H. Rogers, Jr., "Reducibility and completeness for sets of integers," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 5 (1959), pp. 117–125.

[2] Rogers, H., Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, London, 1967.

[3] Soare, R., *Recursively enumerable sets and degrees*, Springer Verlag, Berlin and Heidelberg, 1987.

Mathematical Institute of the Slovak Academy of Sciences
Obrancov mieru 49
814 73 Bratislava, Czechoslovakia